Indeterminacy in dynamic models:
life horizon does not matter

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Abstract: In this paper, we consider an aggregate overlapping generations model with endogenous labor, consumption in both periods of life and productive external effects coming from the average capital and labor. We show that local indeterminacy of equilibria is not generated by capital but by extremely small labor externalities and that under gross substitutability it requires a large enough elasticity of capital-labor substitution and a large enough elasticity of the labor supply. We also show that local indeterminacy easily occurs in a Cobb-Douglas economy. These results will allow us to prove that life horizon is not a crucial argument for the occurrence of multiple equilibria.

Keywords: Indeterminacy, endogenous cycles, overlapping generations, endogenous labor supply, externalities.

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1 Introduction

In this paper, we consider an aggregate overlapping generations model with endogenous labor, consumption in both periods of life and productive externalities coming from the average capital and labor. We show that local indeterminacy of equilibria is not generated by capital but by extremely small labor externalities and that under gross substitutability it requires a large enough elasticity of capital-labor substitution and a large enough elasticity of the labor supply. We also show that local indeterminacy easily occurs in a Cobb-Douglas economy. These results will allow us to prove that life horizon is not a crucial argument for the occurrence of multiple equilibria.

Models with a finite number of infinitely lived agents are often distinguished from models with an infinite number of finitely lived agents on the ground of the determinacy property of their perfect foresight equilibria. Indeed, under perfect competition, optimal growth models are generically characterized by locally unique equilibria,\(^1\) while many robust examples of overlapping generations models with a continuum of equilibria have been exhibited.\(^2\)

Under imperfect competition, such a distinction is no longer obvious. It is now a well-established fact that even infinitely lived agents models may then exhibit a continuum of competitive equilibrium paths. As a result, in the recent period, the Ramsey one-sector growth model augmented to include endogenous labor supply and external effects has become a standard framework for the analysis of local indeterminacy and expectations-driven fluctuations based on the existence of sunspot equilibria.\(^3\) While positive capital externalities do not provide any mechanism for the occurrence of a continuum of equilibria,\(^4\) Benhabib and Farmer [1] show that strong labor externalities which generate an increasing labor demand function with respect to wage may imply the existence of local indeterminacy in a Cobb-

\(^1\)See Kehoe, Levine and Romer [17].
\(^3\)See the recent survey of Benhabib and Farmer [2].
\(^4\)See Kehoe [15], Boldrin and Rustichini [3].
Douglas economy. More recently, Pintus [22], by considering a general formulation for preferences and technology, relaxes the conditions of Benhabib and Farmer. He shows that local indeterminacy may arise under small labor externalities (i.e. a decreasing labor demand function) provided the elasticity of capital-labor substitution is sufficiently larger than one, the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply are large enough.

Following Reichlin [23] and Grandmont [12], most of the contributions dealing with aggregate overlapping generations models have focussed on a formulation without first period consumption. In such a framework, Cazzavillan [4] then shows that local indeterminacy easily occurs for a large set of realistic parameters values as soon as small capital externalities are considered. While extended formulations in which agents consume over their whole life-cycle have shown that under perfect competition local indeterminacy becomes less likely when the share of first period consumption is large, a widespread conjecture was suggesting that a continuum of equilibria could again easily occur if small capital externalities were introduced.

We will prove in this paper that this conjecture is wrong and that the conditions for the existence of locally indeterminate equilibria in one-sector models with external effects do not depend on the life horizon of agents. From this point of view, we will show that models with a finite number of infinitely lived agents and models with an infinite number of finitely lived agents are actually very similar.

We consider a model with consumption in both periods of life and in which the share of first period consumption over the wage income is large enough to be compatible with standard estimates. Our main result is then to show that as soon as we impose such a restriction, the behavior of finite-lived agents becomes very close to that of infinite-lived agents.

We first prove that when a large share of first period consumption over the wage income is considered, local indeterminacy of equilibria cannot be generated by capital externalities but occurs with extremely small labor externalities provided the elasticity of capital-labor substitution and the elasticity of labor supply are large enough. Moreover, low market imper-

\[^{5}\text{The elasticity of intertemporal substitution in consumption and the elasticity of capital-labor substitution are equal to one.}\]

\[^{6}\text{See Cazzavillan and Pintus [8], Nourry and Venditti [21].}\]
fections imply that capital and labor are more substitutable than in the usual Cobb-Douglas specification. Secondly, we show that even with Cobb-Douglas preferences and technology, local indeterminacy may easily occur but with slightly larger labor externalities. Thirdly, we prove that under our restriction on first period consumption the locally indeterminate normalized steady state is always characterized by an under-accumulation of capital.

As this clearly appears from direct comparisons with Benhabib and Farmer [1] and Pintus [22], all these conclusions are also obtained within infinite horizon models. Our main conclusion is then: life horizon does not matter for the occurrence of indeterminacy in dynamic models.

This paper is organized as follows: The next section sets up the basic model. In section 3 we prove the existence of a normalized steady states. Section 4 contains the derivation of the characteristic polynomial and presents the geometrical method used for the local dynamic analysis. In section 5 we present our main results on local indeterminacy. Section 6 presents some detailed comparisons with infinite horizon models while section 7 contains some concluding comments. All the proofs are gathered in a final appendix.

2 The model

Consider a perfectly competitive world where economic activity is performed over infinite discrete time in which there are identical non altruistic agents. Each agent lives for two periods: he works during the first, supplying elastically an amount of labor \( l \) such that \( 0 \leq l \leq \ell \), with \( \ell > 0 \) (possibly infinite) his endowment of labor. He has preferences for his consumptions \( (c, \hat{c}) \), when he is young, and \( \hat{c} \), when he is old), and derives disutility from labor according to the following function

\[
u(c, \hat{c}) - v(l/B)\]

with \( B > 0 \) a scaling parameter.

Assumption 1 .

i) \( u(c, \hat{c}) \) is \( C^2 \), strictly increasing with respect to each argument \( u_1(c, \hat{c}) > 0, u_2(c, \hat{c}) > 0 \), concave and homogeneous of degree one. Moreover, for all \( c, \hat{c} > 0 \), \( \lim_{\hat{c}/c \to 0} u_1/u_2 = 0 \) and \( \lim_{\hat{c}/c \to +\infty} u_1/u_2 = +\infty \).

ii) \( v(l/B) \) is \( C^2 \), strictly increasing \( v'(l/B) > 0 \), strictly convex \( v''(l/B) > 0 \) and \( \lim_{l \to 0} v'(l/B) = +\infty \).
Remark: The homogeneity assumption is introduced in order to characterize the fundamentals in terms of standard elasticities and thus to provide simple economically interpretable conditions. Relaxing this assumption would not fundamentally modify our conclusions.

Each agent is assumed to have one child so that population is constant and normalized to one. Considering the wage rate $w_t$ and the expected interest factor $R_{t+1}^e$ as given, he maximizes his utility function over his life-cycle as follows:

$$\max_{c_t, \hat{c}_{t+1}, l_t, K_{t+1}} u(c_t, \hat{c}_{t+1}) - v(l_t/B)$$

s.t. $$w_t l_t = c_t + K_{t+1}$$
$$R_{t+1}^e K_{t+1} = \hat{c}_{t+1}$$
$$0 \leq l_t \leq \ell$$

Notice from the first budget constraint that all the savings of young agents is affected to productive capital and we assume total depreciation of capital. Consumers perfectly expect the interest factor $R_{t+1}^e = R_{t+1}$. Assumption 1 implies the existence and uniqueness of interior solutions for optimal saving, i.e. the amount of capital $K_{t+1}$, and labor supply $l_t$. The first order conditions can be written as follows:

$$\frac{u_1(c_t, \hat{c}_{t+1})}{u_2(c_t, \hat{c}_{t+1})} = R_{t+1}$$
$$u_1(c_t, \hat{c}_{t+1}) w_t = v'(l_t/B)/B$$
$$w_t l_t = c_t + \frac{\hat{c}_{t+1}}{R_{t+1}}$$
$$K_{t+1} = w_t l_t - c_t$$

From Assumption 1, $u(c_t, \hat{c}_{t+1})$ is homogeneous of degree one. It follows that the first partial derivatives are homogeneous of degree zero and the first order conditions (2) and (3) become

$$g(\hat{c}_{t+1}/c_t) = \frac{u_1(1, \hat{c}_{t+1}/c_t)}{u_2(1, \hat{c}_{t+1}/c_t)} = R_{t+1}$$
$$u_1(1, \hat{c}_{t+1}/c_t) w_t = v'(l_t/B)/B$$

Again from Assumption 1, we derive that the second derivatives of $u(c, \hat{c})$ are homogeneous of degree $-1$ and linked as follows

$$0 = c u_{11}(c, \hat{c}) + \hat{c} u_{12}(c, \hat{c}) = u_{11}(1, \hat{c}/c) + (\hat{c}/c) u_{12}(1, \hat{c}/c)$$
$$0 = c u_{12}(c, \hat{c}) + \hat{c} u_{22}(c, \hat{c}) = u_{12}(1, \hat{c}/c) + (\hat{c}/c) u_{22}(1, \hat{c}/c)$$
From this we conclude that \( g'(\hat{c}/c) > 0 \) and thus
\[
\frac{\hat{c}_{t+1}}{c_t} = g^{-1}(R_{t+1}) \equiv h(R_{t+1}) \tag{9}
\]
Moreover combining (2) and (4) we derive
\[
c_t + c_t \frac{u_2(c_t, \hat{c}_{t+1})}{u_1(c_t, \hat{c}_{t+1})c_t} = c_t \frac{u(1, \hat{c}_{t+1}/c_t)}{u_1(1, \hat{c}_{t+1}/c_t)} = w_t l_t
\]
This last equation allows therefore to define the propensity to consume of the young, or equivalently the share of first period consumption over the wage income, as follows
\[
\alpha(R) = \frac{u_1(1, h(R))}{u(1, h(R))} \in (0,1) \tag{10}
\]
and to get \( c_t = \alpha(R_{t+1})w_t l_t \). We also conclude that the first order condition (5) becomes
\[
K_{t+1} = (1 - \alpha(R_{t+1}))w_t l_t \tag{11}
\]
For future reference we may compute the elasticity of intertemporal substitution in consumption
\[
\gamma(R) = \frac{R}{g'(R)h(R)} = -\left( \frac{u_{11}(c, \hat{c})c}{u_1(c, \hat{c})} + \frac{u_{22}(c, \hat{c})\hat{c}}{u_2(c, \hat{c})} \right)^{-1} \tag{12}
\]
and the elasticity of labor supply with respect to the wage rate
\[
e_{l}(l/B) = \frac{v'(l/B)}{(l/B)v''(l/B)} \tag{13}
\]
Notice that the elasticity of intertemporal substitution in consumption is equal to the inverse of the sum of the two Arrow-Pratt indices that characterize the curvature of the utility function. Moreover, the elasticity of the propensity to consume of the young derives as
\[
\frac{d\alpha(R)}{dR} R \frac{R}{\alpha(R)} = (1 - \gamma(R))(1 - \alpha(R)) \tag{14}
\]
It follows that the saving function is increasing with the interest factor \( R \) if and only if \( \gamma(R) > 1 \).

Consider now the technological side of the model. The final output is produced using capital \( K \) and labor \( L \). Although production takes place under constant returns to scale, we assume that each of the many firms benefit from positive externalities due to the contributions of the average levels of capital and labor, respectively \( \bar{K} \) and \( \bar{L} \). Capital external effects
are usually interpreted as coming from learning by doing while labor externalities are associated with thick market effects. The production function of a representative firm is thus \( AF(K, L)e(\bar{K}, \bar{L}) \), with \( F(K, L) \) homogeneous of degree one, \( e(\bar{K}, \bar{L}) \) increasing in each argument and \( A > 0 \) a scaling parameter. Denoting, for any \( L \neq 0 \), \( x = K/L \) the capital stock per labor unit, we may define the production function in intensive form as \( Af(x)e(\bar{K}, \bar{L}) \).

Notice that our formulation is very close to the one considered in Cazzavillan and Pintus [6]. However although these authors define the model with capital and labor externalities, they only study the dynamic properties of equilibrium paths with capital external effects.

**Assumption 2**. \( f(x) \) is positively valued, \( C^2 \), strictly increasing and strictly concave over \( \mathbb{R}_{++} \).

All firms being identical, the competitive equilibrium conditions imply that \( \bar{K} = K \) and \( \bar{L} = L \). The interest factor \( R_t \) and the wage rate \( w_t \) then satisfy:

\[
R_t = Af'(x_t)e(K_t, L_t) \tag{15}
\]

\[
w_t = A[f(x_t) - x_t f'(x_t)]e(K_t, L_t) \equiv Aw(x_t)e(K_t, L_t) \tag{16}
\]

We may also compute the share of capital in total income

\[
s(x) = \frac{x f'(x)}{f(x)} \tag{17}
\]

the elasticity of capital-labor substitution

\[
\sigma(x) = -\frac{(1 - s(x)) f'(x)}{xf''(x)} \tag{18}
\]

and the elasticities of \( e(K, L) \) with respect to capital and labor

\[
\varepsilon_{e,K}(K, L) = \frac{e_1(K, L)K}{e(K, L)} \geq 0, \quad \varepsilon_{e,L}(K, L) = \frac{e_2(K, L)L}{e(K, L)} \geq 0 \tag{19}
\]

The equilibrium on the labor market implies \( l_t = L_t \). We then derive from (7) and (11) the dynamical system characterizing competitive equilibrium paths

\[
K_{t+1} = \left[ 1 - \alpha \left( Af'(x_{t+1})e(K_{t+1}, l_{t+1}) \right) \right] Aw(x_t)e(K_t, l_t)l_t \tag{20}
\]

\[
v^t(l_t/B)/B = u_1 \left( 1, h \left( Af'(x_{t+1})e(K_{t+1}, l_{t+1}) \right) \right) Aw(x_t)e(K_t, l_t)
\]

with \( x_t = K_t/l_t \) and \( K_0 \) given.
3 Steady state

A steady state is a pair \((\bar{K}, \bar{l})\) such that
\[
\bar{K} = \left[ 1 - \alpha \left( A f'(\bar{K}/\bar{l})e(\bar{K}, \bar{l}) \right) \right] Aw(\bar{K}/\bar{l})e(\bar{K}, \bar{l})\bar{l}
\]
\[
v'(\bar{l}/B)/B = u_1 \left( 1, h \left( A f'(\bar{K}/\bar{l})e(\bar{K}, \bar{l}) \right) \right) Aw(\bar{K}/\bar{l})e(\bar{K}, \bar{l})\bar{l}
\]

(21)

As in the overlapping generations model with exogenous labor supply, the existence of a non-trivial steady state is not guaranteed even with some strengthened Inada conditions.\(^7\) In order to simplify the analysis, we will follow the procedure introduced in Cazzavillan, Lloyd-Braga and Pintus [5] and use the scaling parameters \(A\) and \(B\) in order to give conditions for the existence of a normalized steady state \((\bar{K}, \bar{l}) = (1, 1)\)

**Proposition 1.** Under Assumptions 1-2, let \(V(B) = v'(1/B)/B\). Then \((\bar{K}, \bar{l}) = (1, 1)\) is a steady state of the dynamical system (20) if and only if
\[
\lim_{z \to +\infty} \left( 1 - \alpha(z) \right) z = f'(1)/w(1) \text{ and } A^* > 0, B^* > 0 \text{ are respectively the unique solutions of}
\]

\[
1 = \left[ 1 - \alpha(A^* f'(1)e(1, 1)) \right] A^* w(1)e(1, 1)
\]
\[
B^* = V^{-1} \left( u_1 \left( 1, h \left( A^* f'(1)e(1, 1) \right) \right) \right) A^* w(1)e(1, 1)
\]

Proposition 1 gives conditions for the existence of a steady state. However, as well-known in OLG economies, uniqueness cannot be guaranteed. In order to illustrate this fact, consider a standard example of CES preferences:
\[
u(c, \hat{c}) = [\delta c - \rho + (1 - \delta) \hat{c} - \rho]^{-1/\rho}\)
\[
v(l/B) = (l/B)^{1+\beta}/(1 + \beta)\) with \(\delta \in (0, 1), \rho > -1 \text{ and } \beta > 0. \) We easily get \(\hat{c}/c = h(R) = [(1 - \delta) R/\delta]^{1/(1+\rho)}, V(B) = (1/B)^{1+\beta} \text{ and equations (21) with } (\bar{K}, \bar{l}) = (1, 1) \text{ reduce to}
\]
\[
1 = \left( 1 - \delta \right) \left[ \frac{1-\delta}{\delta} f'(1)e(1, 1) \right]^{-\rho/\rho} A^{1/(1+\rho)} \frac{\bar{K}}{\bar{L}} A^{-1/(1+\rho)} w(1)e(1, 1)
\]
\[
B = \left( \delta + (1 - \delta) \left[ \frac{1-\delta}{\delta} A f'(1)e(1, 1) \right]^{-\rho/\rho} A^{-1/(1+\rho)} w(1)e(1, 1) \right)^{-\frac{1-\rho}{\rho}}
\]

It follows that for any \(\rho > -1, \lim_{z \to +\infty} (1 - \alpha(z)) z = +\infty \text{ and there exists a unique } A^* \text{ solution of the first equation. The corresponding value for } B^* \text{ is}
\]

\(^7\)See Galor and Ryder [10].
obtained substituting $A^*$ in the second equation. An exhaustive numerical analysis also shows that there exist at most two steady states.

From a general point of view, under the conditions of Proposition 1, the existence of one normalized steady state is ensured so that any loss of stability through a saddle-node bifurcation cannot occur. Since an odd or even number of steady states may exist, transcritical and pitchfork bifurcations are plausible. However, pitchfork bifurcations are ruled out with CES preferences since at most two steady states can co-exist.

4 Characteristic polynomial and geometrical method

In the rest of the paper we will assume that the condition of Proposition 1 is satisfied in order to guarantee the existence of the normalized steady state.

Assumption 3. $\lim_{z \to +\infty} (1 - \alpha(z))z > f'(1)/w(1)$

To study the local stability properties of the normalized steady state $(\bar{K}, \bar{l}) = (1, 1)$, we linearize the dynamical system (20) around $(1, 1)$. We will therefore evaluate all the shares and elasticities previously defined at $(1, 1)$. From (10), (12), (13), (17), (18) and (19), we consider indeed $\alpha(A^*f'(1)e(1, 1)) = \alpha$, $\gamma(A^*f'(1)e(1, 1)) = \gamma$, $\epsilon_l(1/B^*) = \epsilon_l$, $s(1) = s$, $\sigma(1) = \sigma$, $\varepsilon_{e,K}(1, 1) = \varepsilon_{e,K}$ and $\varepsilon_{e,L}(1, 1) = \varepsilon_{e,L}$.

Proposition 2. Under Assumptions 1-3, the characteristic polynomial is

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda T + D \quad (22)$$

$$D = \left(1 + \epsilon_l \right) \frac{s + \sigma \varepsilon_{e,K}}{\epsilon_l (1 - \alpha)(1 - s + \sigma \varepsilon_{e,L})}$$

$$T = \frac{1}{(1 - \alpha)(1 - s + \sigma \varepsilon_{e,L})} \left( \varepsilon_{e,K}(1 - \sigma)(1 - \alpha \gamma) + \varepsilon_{e,L}(1 - \sigma - \alpha \gamma) + 1 - \sigma - \alpha \gamma(1 - s) + \frac{1 + \epsilon_l}{\epsilon_l} [\sigma - \alpha(1 - \gamma)(1 - s - \sigma \varepsilon_{e,K})] \right)$$

Our aim is to discuss the local determinacy properties of equilibria, i.e. the existence of a continuum of equilibrium paths starting from the same initial capital stock and converging to the normalized steady state. Our model
consists in one predetermined variable, the capital stock, and one forward variable, the labor supply. Therefore, the normalized steady state is locally indeterminate if and only if the local stable manifold is two-dimensional.

As in Grandmont, Pintus and de Vilder [13], we will analyze the local stability of \((\bar{K}, \bar{l}) = (1, 1)\) by studying the variations of the trace \(T\) and the determinant \(D\) in the \((T, D)\) plane when some parameters of interest vary continuously. This methodology allows also to easily study the occurrence of local bifurcations. In the current framework, since \(T\) and \(D\) are linear with respect to a function of the elasticity of labor supply, namely \(\varepsilon_l \equiv (1 + \varepsilon_l)/\varepsilon_l\), we may easily define a relationship, say \(\Delta(T)\), linking \(T\) and \(D\) for different values of \(\varepsilon_l\). As \(\varepsilon_l \in (0, +\infty)\), we have \(\varepsilon_l \in (1, +\infty)\). From Proposition 2, the locus \((T(\varepsilon_l), D(\varepsilon_l))\) is defined through the following relationship

\[
D = \Delta(T) = \frac{s+\sigma\varepsilon_e,K}{\sigma-a(1-\gamma)(1-s-\sigma\varepsilon_e,K)} T - \frac{s+\sigma\varepsilon_e,K}{(1-a)(1-s+\sigma\varepsilon_e,L)} \varepsilon_e,K(1-\sigma)(1-\alpha) + \varepsilon_e,L(1-\sigma-a\gamma)(1-s)
\]

The slope of \(\Delta(T)\) is denoted

\[
S = \frac{s+\sigma\varepsilon_e,K}{\sigma-a(1-\gamma)(1-s-\sigma\varepsilon_e,K)},
\]

The following figure provides an illustration of the \(\Delta(T)\) line.
In this figure we also introduce three other relevant lines: line $AC$ ($D = T - 1$) along which one characteristic root is equal to 1, line $AB$ ($D = -T - 1$) along which one characteristic root is equal to $-1$ and segment $BC$ ($D = 1, |T| < 2$) along which the characteristic roots are complex conjugate with modulus equal to 1. These lines divide the space $(T, D)$ into three different types of regions according to the number of characteristic roots with modulus less than 1. If the $\Delta(T)$ line crosses segment $AB$ as $\varepsilon_l$ goes through $\varepsilon_{F} \in (1, +\infty)$ then a flip bifurcation is generically expected to occur. If $\Delta(T)$ crosses segment $BC$ in its interior as $\varepsilon_l$ goes through $\varepsilon_{H} \in (1, +\infty)$ then a Hopf bifurcation is generically expected to occur. Finally, if $\Delta(T)$ crosses segment $AC$ as $\varepsilon_l$ goes through $\varepsilon_{T} \in (1, +\infty)$ then one characteristic root crosses +1. Under Assumption 3, a saddle-node bifurcation cannot occur. Moreover, since the analysis of a CES economy clearly shows that at most two steady states may co-exist so that pitchfork bifurcations are ruled out, the critical value $\varepsilon_{T}^{T}$ will be associated with a transcritical bifurcation.

In order to locate the $\Delta(T)$ line we need to study its slope $S$. Moreover, as $\varepsilon_l \in (1, +\infty)$, only a part of $\Delta(T)$ is relevant. We need therefore to compute the starting and end points of the pair $(T(\varepsilon_l), D(\varepsilon_l))$.

**Lemma 1.** Under Assumptions 1-3, the following results hold:

1. There exist $\bar{\gamma} > 1$ and $\bar{\sigma}$ such that $S \geq 0$ if and only if one of the following conditions is satisfied:
   1) $\gamma \in [1, \bar{\gamma}]$,
   2) $\gamma < 1$ and $\sigma \geq \bar{\sigma}$,
   3) $\gamma > \bar{\gamma}$ and $\sigma \leq \bar{\sigma}$.

2. $\lim_{\varepsilon_l \to +\infty} T(\varepsilon_l) = +(-)\infty$ if and only if $S > (\leq)0$,
   $\lim_{\varepsilon_l \to +\infty} D(\varepsilon_l) = +\infty$ and
   $$\lim_{\varepsilon_l \to -1} D(\varepsilon_l) = D_1 = \frac{s + \sigma \varepsilon_{e,K}}{(1-\alpha)(1-s+\sigma \varepsilon_{e,L})},$$
   $$\lim_{\varepsilon_l \to -1} T(\varepsilon_l) = T_1 = \frac{\varepsilon_{e,K}(1-\sigma-\alpha(\gamma-\sigma)) + \varepsilon_{e,L}(1-\sigma-\alpha \gamma) + sa + 1 - \alpha}{(1-\alpha)(1-s+\sigma \varepsilon_{e,L})}.$$

Therefore in graphical terms, the relevant part of $\Delta(T)$ is a half-line beginning in $(T_1, D_1)$ and pointing upwards to the left when $S < 0$, or to the right when $S > 0$. Considering given values for $\alpha, \gamma, \varepsilon_{e,K}$ and $\varepsilon_{e,L}$, we will also study how our half-line $\Delta(T)$ moves when the elasticity of capital-labor substitution $\sigma$ is modified. This amounts to analyze how the slope $S$ and the starting point $(T_1, D_1)$ changes with $\sigma$. Proceeding in a similar way, we
start by defining a relationship linking the initial points $T_1$ and $D_1$ of $\Delta(T)$ for different values of $\sigma \in (0, +\infty)$:

$$D_1 = \Delta_1(T_1) = \frac{s\varepsilon_{e,L} - (1-s)\varepsilon_{e,K}}{\varepsilon_{e,L} + (\varepsilon_{e,K} + \varepsilon_{e,L})[\varepsilon_{e,L}(1-\alpha\gamma) + (1-\alpha)(1-s)]} T_1 + \frac{\varepsilon_{e,K}(1-\alpha) + (\varepsilon_{e,K} + \varepsilon_{e,L})[\varepsilon_{e,K}(1-\alpha\gamma) + s]}{(1-\alpha)[\varepsilon_{e,L} + (\varepsilon_{e,K} + \varepsilon_{e,L})[\varepsilon_{e,L}(1-\alpha\gamma) + (1-\alpha)(1-s)]}$$

The slope of $\Delta_1(T_1)$ is denoted

$$S_1 = \frac{s\varepsilon_{e,L} - (1-s)\varepsilon_{e,K}}{\varepsilon_{e,L} + (\varepsilon_{e,K} + \varepsilon_{e,L})[\varepsilon_{e,L}(1-\alpha\gamma) + (1-\alpha)(1-s)]}$$

Straightforward computations give:

**Lemma 2.** Under Assumptions 1-3, the following results hold:

1. $\partial S / \partial \sigma < 0$ if and only if $\varepsilon_{e,K} \alpha (\gamma - 1) - s < 0$,
2. $\partial D / \partial \sigma < 0$ if and only if $(1-s)\varepsilon_{e,K} < s\varepsilon_{e,L}$ with
   $$\lim_{\sigma \to 0} D_1 = D_0 = \frac{s}{(1-\alpha)(1-s)}, \quad \lim_{\sigma \to +\infty} D_1 = D_\infty = \frac{\varepsilon_{e,K}}{(1-\alpha)\varepsilon_{e,L}}$$
3. $\partial T / \partial \sigma < 0$ if and only if $\varepsilon_{e,L} + (\varepsilon_{e,K} + \varepsilon_{e,L})[\varepsilon_{e,L}(1-\alpha\gamma) + (1-\alpha)(1-s)] > 0$ with
   $$\lim_{\sigma \to 0} T_1 = T_0 = \frac{(1-\alpha\gamma)(\varepsilon_{e,K} + \varepsilon_{e,L}) + s\alpha + 1-\alpha}{(1-\alpha)(1-s)}$$
   $$\lim_{\sigma \to +\infty} T_1 = T_\infty = \frac{\varepsilon_{e,K}}{\varepsilon_{e,L}} - \frac{1}{1-\alpha}$$

5 Local indeterminacy with first period consumption

We know from Proposition 2 that $D$ is an increasing function of $\varepsilon_l$. Since we have proved that $\lim_{\varepsilon_l \to +\infty} D(\varepsilon_l) = +\infty$, a necessary condition for the occurrence of local indeterminacy is $D_1 < 1$. Consider first the case without externalities, i.e. $\varepsilon_{e,K} = \varepsilon_{e,L} = 0$. It follows that $D$ does not depend on $\sigma$ and $D_1 = D_0$. We then get

$$D_0^0 \leq 1 \iff \alpha \leq \frac{1-2s}{1-s} \equiv \alpha_3$$

Considering that the share of capital in total income satisfies $s \approx 1/3$, we get $\alpha_3 \approx 0.5$. But commonly accepted values for the share of first period consumption over the wage income imply $\alpha > 0.5$. Therefore, as also shown in Cazzavillan and Pintus [7] and Nourry and Venditti [21], local indeterminacy is not likely as soon as young agents consume a realistically large portion of their wage income. Notice however that, even if $\alpha$ assumes a low
value, the occurrence of multiple equilibria requires the elasticity of capital-labor substitution to be less than the share of capital over total income, an assumption which is not in accordance with standard estimations.  

Consider now the case with only capital externalities, i.e. $\varepsilon_{e,K} > 0$ and $\varepsilon_{e,L} = 0$. This configuration has been studied by Cazzavillan [4] under the assumption that the propensity to consume of young agents $\alpha$ is equal to zero. He then shows that with small capital externalities, local indeterminacy may occur while the elasticity of capital-labor substitution is larger than the share of capital. When $\alpha > 0$, we find that $D_1^\infty = +\infty$ and local indeterminacy still requires $D_1^0 < 1$. The same comment as in the previous case applies. Therefore, we show that even with capital externalities, local indeterminacy is not likely as soon as young agents consume a realistically large portion of their wage income.  

As a consequence, we will prove in the following that as in infinite horizon growth models, labor externalities play a major role for the occurrence of multiple equilibria as soon as we introduce realistic values for the share of first period consumption over the wage income. From this point of view, finitely lived and infinitely lived agents behave in a similar way.

Consider finally the general case with capital and labor externalities, i.e. $\varepsilon_{e,K}, \varepsilon_{e,L} > 0$. We know from Lemma 2 that for any given values of $s$, $\varepsilon_{e,K}$ and $\varepsilon_{e,L}$, $D_1$ is a monotone function of $\sigma$. Therefore, local indeterminacy requires that either $D_1^0$ or $D_1^\infty$ is less than 1. We easily get

$$D_1^\infty \leq 1 \iff \frac{\varepsilon_{e,K}}{\varepsilon_{e,L}} \leq 1 - \alpha$$

Since realistic values for the propensity to consume of young agents imply to assume $\alpha > \alpha_3$, we obtain from inequality (23)

$$\frac{\varepsilon_{e,K}}{\varepsilon_{e,L}} \leq 1 - \alpha < \frac{s}{1-s}$$

Considering again that $s \approx 1/3$, we get $\varepsilon_{e,K} < \varepsilon_{e,L}/2$. Since we want to focus on cases with low externalities, this condition implies that $\varepsilon_{e,K}$ needs to be very close to zero.  

This condition rules out the case of output externalities in which $\varepsilon_{e,K}/\varepsilon_{e,L} = s/(1-s)$.  

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8See also Reichlin [23] for the case $\alpha = 0$.

9A similar result has been independently shown by Cazzavillan and Pintus [8] in an OLG model with preferences characterized by total separability over $(c, \hat{c}, l)$.

10This condition rules out the case of output externalities in which $\varepsilon_{e,K}/\varepsilon_{e,L} = s/(1-s)$. 

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consumption over the wage income in order to focus on realistic values. Finally, we will assume gross substitutability in order to get an increasing saving function with respect to the interest factor $R$. To summarize we introduce the following assumption:

**Assumption 4**. \( \varepsilon_{e,K} = 0, \alpha > \frac{1-2s}{1-s} \equiv \alpha_3, s \leq 1/2, \gamma \geq 1 \)

In order to use the geometrical method described in Grandmont, Pintus and de Vilder [13], we need first to study the locus \( \Delta_1(T_1) \) containing the starting point \( (T_1, D_1) \) of the \( \Delta(T) \) line. The major difficulty lies in the fact that with labor externalities, the \( \Delta_1(T_1) \) line may have very different shapes depending on the values of the various parameters. The extremities of the half-line \( \Delta_1(T_1) \) are given by the limit values of \( (T_1, D_1) \) when the elasticity of capital-labor substitution \( \sigma \) goes from 0 to +\( \infty \). Lemma 2 implies \( D_1^\infty = 0, T_1^\infty = -1/(1 - \alpha) < 0 \) and \( D_1^0 > 1 \). We thus start by studying the slope \( S_1 \) of \( \Delta_1(T_1) \), the values of \( D_1 \) when \( T_1 = \pm 2 \) and the sign of \( T_0 \).

Our main objective is to give conditions for local indeterminacy of equilibria under small labor externalities. We will then consider in the following the configurations associated with the smallest values of \( \varepsilon_{e,L} \).

**Lemma 3**. Under Assumptions 1-4, the following results hold:

1. When \( 1 - \alpha \gamma \geq 0 \):
   - i) \( S_1 \in (0, 1) \),
   - ii) there exists \( \varepsilon_{e,L}^* > 0 \) such that \( \Delta_1(2) > 1 \) if and only if \( \varepsilon_{e,L} < \varepsilon_{e,L}^* \),
   - iii) there exists \( \alpha_2 \in (\alpha_3, 1) \) such that \( \Delta_1(-2) < 1 \) when \( \alpha \in (\alpha_3, \alpha_2) \),
   - iv) when \( \alpha \in (\alpha_2, 1) \), there exists \( \varepsilon_{e,L}^{**} \in (0, \varepsilon_{e,L}^*) \) such that \( \Delta_1(-2) > 1 \) if and only if \( \varepsilon_{e,L} < \varepsilon_{e,L}^{**} \),
   - v) \( T_1^0 > 0 \).

2. When \( 1 - \alpha \gamma < 0 \):
   - i) there exist \( \tilde{\varepsilon}_{e,L} > 0 \) such that \( S_1 \in (0, 1) \) if and only if \( \varepsilon_{e,L} < \tilde{\varepsilon}_{e,L} \),
   - ii) \( \Delta_1(2) > 1 \),
   - iii) when \( \alpha \in (\alpha_3, \alpha_2) \), \( \Delta_1(-2) < 1 \) if and only if \( \varepsilon_{e,L} < \varepsilon_{e,L}^{**} \),
   - iv) when \( \alpha \in (\alpha_2, 1) \), \( \Delta_1(-2) > 1 \),
   - v) if \( \varepsilon_{e,L} < \tilde{\varepsilon}_{e,L} \), \( T_1^0 > 0 \).

To complete the analysis we need now to examine the intersections of \( \Delta_1(T_1) \) with the lines \( AC, AB \) and \( BC \), to compare the slopes \( S_1 \) and \( S \), and to study the values of \( D \) when \( T = \pm 2 \).
Lemma 4. Under Assumptions 1-4, the following results hold:
  i) when $1 - \alpha \gamma \geq 0$, there exists $\sigma_1 > 0$ such that $D_1 \geq T_1 - 1$ if and only if $\sigma \geq \sigma_1$,
  ii) when $1 - \alpha \gamma < 0$, $D_1 \geq T_1 - 1$,
  iii) there exists $\sigma_2 > 0$ such that $D_1 \leq 1$ if and only if $\sigma \geq \sigma_2$,
  iv) there exists $\sigma_3 > 0$ such that $D_1 \geq -T_1 - 1$ if and only if $\sigma \leq \sigma_3$,
  v) there exists $\sigma_4 \geq 0$ such that $S \leq S_1$ if $\sigma \geq \sigma_4$,
  vi) there exists $\sigma_5 \geq 0$ such that $\Delta(1) = 1$ if and only if $\sigma \leq \sigma_5$,
  vii) there exists $\sigma_6 \geq 0$ such that $S \leq S_1$ if $\sigma \geq \sigma_6$,
  viii) when $\alpha \in (\alpha_2, 1)$, there exists $\sigma_H \in (\sigma_2, \sigma_3)$ such that $\Delta(2) = 1$ if and only if one of the following conditions is satisfied:
  * $1 - \alpha \gamma \geq 0$ and $\varepsilon_{e,L} < \varepsilon_{e,L}^{**}$,
  * $1 - \alpha \gamma < 0$.

A last step consists in ranking all the critical bounds on the elasticity of capital labor substitution.

Lemma 5. Under Assumptions 1-4, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon_{e,L} < \bar{\varepsilon}$ the following results hold:
  1- when $\alpha \in (\alpha_3, \alpha_2)$, $\sigma_3 > \sigma_2 > \sigma_4 > \sigma_5 > \sigma_1$,
  2- when $\alpha \in (\alpha_2, 1)$, $\bar{\sigma}_H > \sigma_H > \sigma_2 > \sigma_3 > \sigma_4 > \sigma_5 > \sigma_1$

Lemma 4 shows that $\Delta_1(T_1)$ cannot cross the line $AC$ such that $D = T - 1$ when $1 - \alpha \gamma < 0$. The basic locations of the $\Delta_1(T_1)$ line may thus be summarized in Figures 2 and 3.

![Figure 2: $\Delta_1$ line with $\varepsilon_L$ low and $1 - \alpha \gamma > 0$.](image-url)
Focussing on cases in which the share of first period consumption in wage income has realistic values, we impose through Assumption 4 the restriction \( \alpha \in (\alpha_3, 1) \). Lemmas 4 and 5 show however that within this interval, we need to distinguish two cases depending on whether \( \alpha \) is lower or greater than \( \alpha_2 \).

### 5.1 \( \alpha \in (\alpha_3, \alpha_2) \)

As this clearly appears from Figure 1, the local stability properties of the normalized steady state \((1, 1)\) are discussed by locating the pair \((\mathcal{T}, D)\) on the \(\Delta(\mathcal{T})\) line as the elasticity of the labour supply \(\varepsilon_l\) varies over \((0, +\infty)\). Three critical values for this parameter may then be exhibited. Each of them is associated with the corresponding critical values of \(\varepsilon_l\) mentioned in Figure 1. The first one, denoted \(\varepsilon_l^F\), concerns the intersection of \(\Delta(\mathcal{T})\) with the line generated by segment \(AB\) and which is associated with a Flip bifurcation. The second one, denoted \(\varepsilon_l^H\), concerns the intersection of \(\Delta(\mathcal{T})\) with the line generated by segment \(BC\) and which is associated with a Hopf bifurcation when the intersection occurs in the interior of \(BC\). The third one, denoted \(\varepsilon_l^T\), concerns the intersection of \(\Delta(\mathcal{T})\) with the line generated by segment \(AC\) and which is associated with a transcritical bifurcation. Straightforward computations give the expressions of these critical values:

$$
\varepsilon_l^F = \frac{2s + \sigma - \sigma_3}{\varepsilon_{e,L}\alpha(\sigma - \sigma_3)}, \quad \varepsilon_l^H = \frac{s}{\varepsilon_{e,L}(1 - \alpha)(\sigma - \sigma_2)}, \quad \varepsilon_l^T = \frac{\sigma - \sigma_5}{\varepsilon_{e,L}(2 - \alpha)(\sigma - \sigma_1)}
$$

(24)
When \( \alpha \in (\alpha_3, \alpha_2) \), Lemma 5 provides a ranking for the critical bounds on the elasticity of capital-labor substitution. Notice however that we cannot easily compare the bounds \( \bar{\sigma}_H \) and \( \sigma_3 \). We will then provide two different propositions.

Consider first the case \( \bar{\sigma}_H > \sigma_3 \). All the results detailed in the next Proposition are summarized in Figure 4. Notice that the \( \Delta \) lines associated with the critical values of \( \sigma \) are clearly mentioned as, for instance, \( \Delta(\bar{\sigma}_H) \).

![Figure 4: \( \alpha \in (\alpha_3, \alpha_2) \) with \( \bar{\sigma}_H > \sigma_3 \) and \( 1 - \alpha \gamma > 0 \).](image)

**Proposition 3.** Under Assumptions 1-4, let \( \alpha \in (\alpha_3, \alpha_2) \) and \( \bar{\sigma}_H > \sigma_3 \). Consider the critical bounds (24) for the elasticity of labor supply. There exists \( \bar{\varepsilon} > 0 \) such that for any \( \varepsilon_{c,L} < \bar{\varepsilon} \), the following results generically hold:

i) \( \sigma \in (\bar{\sigma}_H, +\infty) \). The normalized steady state \((1,1)\) is a saddle-point for \( \varepsilon_l \in (\varepsilon_l^F, +\infty) \), undergoes a flip bifurcation at \( \varepsilon_l = \varepsilon_l^F \), becomes locally indeterminate for \( \varepsilon_l \in (\varepsilon_l^T, \varepsilon_l^F) \), undergoes a transcritical bifurcation at \( \varepsilon_l = \varepsilon_l^T \) and becomes again a saddle-point for \( \varepsilon_l < \varepsilon_l^T \).

ii) \( \sigma \in (\sigma_3, \bar{\sigma}_H) \). The normalized steady state \((1,1)\) is a saddle-point for \( \varepsilon_l \in (\varepsilon_l^T, +\infty) \), undergoes a flip bifurcation at \( \varepsilon_l = \varepsilon_l^T \), becomes locally indeterminate for \( \varepsilon_l \in (\varepsilon_l^H, \varepsilon_l^T) \), undergoes a Hopf bifurcation at \( \varepsilon_l = \varepsilon_l^H \), becomes a source for \( \varepsilon_l \in (\varepsilon_l^T, \varepsilon_l^H) \), undergoes a transcritical bifurcation \( \varepsilon_l = \varepsilon_l^T \) and becomes again a saddle-point for \( \varepsilon_l < \varepsilon_l^T \).
iii) $\sigma \in (\sigma_2, \sigma_3)$. The normalized steady state $(1, 1)$ is locally indeterminate for $\epsilon_l \in (\epsilon_l^H, +\infty)$, undergoes a Hopf bifurcation at $\epsilon_l = \epsilon_l^H$, becomes a source for $\epsilon_l \in (\epsilon_l^T, \epsilon_l^H)$, undergoes a transcritical bifurcation $\epsilon_l = \epsilon_l^T$ and becomes a saddle-point for $\epsilon_l < \epsilon_l^T$.

iv) $\sigma \in (\sigma_5, \sigma_2)$. The normalized steady state $(1, 1)$ is a source for $\epsilon_l \in (\epsilon_l^T, +\infty)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes a saddle-point for $\epsilon_l < \epsilon_l^T$.

v) $\sigma \in (\max\{0, \sigma_1\}, \sigma_5)$. The normalized steady state $(1, 1)$ is a source for any $\epsilon_l \in (0, +\infty)$.

vi) $\sigma \in (0, \max\{0, \sigma_1\})$. The normalized steady state $(1, 1)$ is a saddle-point for $\epsilon_l \in (\epsilon_l^T, +\infty)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes a source for $\epsilon_l < \epsilon_l^T$.

Consider now the case $\sigma_3 > \bar{\sigma}^H$. All the results detailed in the next Proposition are summarized in Figure 5.

**Figure 5:** $\alpha \in (\alpha_3, \alpha_2)$ with $\sigma_3 > \bar{\sigma}^H$ and $1 - \alpha \gamma > 0$.

**Proposition 4.** Under Assumptions 1-4, let $\alpha \in (\alpha_3, \alpha_2)$ and $\sigma_3 > \bar{\sigma}^H$. Consider the critical bounds (24) for the elasticity of labor supply. There

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11 As shown in Lemma 4, this case only occurs if $\sigma_1 > 0$, i.e. if $1 - \alpha \gamma > 0$. 
exists $\varepsilon > 0$ such that for any $\varepsilon L < \varepsilon$, the following results generically hold:

i) $\sigma \in (\sigma_3, +\infty)$. The normalized steady state $(1, 1)$ is a saddle-point for $\epsilon_l \in (\epsilon_l^F, +\infty)$, undergoes a flip bifurcation at $\epsilon_l = \epsilon_l^F$, becomes locally indeterminate for $\epsilon_l \in (\epsilon_l^T, \epsilon_l^F)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes a saddle-point for $\epsilon_l < \epsilon_l^T$.

ii) $\sigma \in (\sigma_2, \sigma_3)$. The normalized steady state $(1, 1)$ is locally indeterminate for $\epsilon_l \in (\epsilon_l^T, +\infty)$, undergoes a transcritical bifurcation $\epsilon_l = \epsilon_l^T$ and becomes again a saddle-point for $\epsilon_l < \epsilon_l^T$.

iii) $\sigma \in (\sigma_2, \sigma_3)$. The normalized steady state $(1, 1)$ is locally indeterminate for $\epsilon_l \in (\epsilon_l^H, +\infty)$, undergoes a Hopf bifurcation at $\epsilon_l = \epsilon_l^H$, becomes a source for $\epsilon_l \in (\epsilon_l^T, \epsilon_l^H)$, undergoes a transcritical bifurcation $\epsilon_l = \epsilon_l^T$ and becomes a saddle-point for $\epsilon_l < \epsilon_l^T$.

iv) $\sigma \in (\sigma_5, \sigma_2)$. The normalized steady state $(1, 1)$ is a source for $\epsilon_l \in (\epsilon_l^T, +\infty)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes a saddle-point for $\epsilon_l < \epsilon_l^T$.

v) $\sigma \in (\max\{0, \sigma_1\}, \sigma_2)$. The normalized steady state $(1, 1)$ is a source for any $\epsilon_l \in (0, +\infty)$.

vi) $\sigma \in (0, \max\{0, \sigma_1\})$. The normalized steady state $(1, 1)$ is a saddle-point for $\epsilon_l \in (\epsilon_l^T, +\infty)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_l^T$ and becomes a source for $\epsilon_l < \epsilon_l^T$.

Propositions 3 and 4 clearly show that with small labor externalities and a share of first period consumption over the wage income compatible with empirical estimates, local indeterminacy of equilibria requires a large enough elasticity of capital-labor substitution ($\sigma > \sigma_2 > 1$) and a large enough elasticity of the labor supply. Notice that there is no particular restriction on the elasticity of intertemporal substitution in consumption which is only assumed to be greater than one in order to meet the gross substitutability axiom. The main distinction between these two Propositions, which is based on the ranking of the critical values on the elasticity of capital-labor substitution, concerns the possible co-existence of flip and Hopf bifurcations. Indeed when $\sigma^H > \sigma_3$ and $\sigma$ is chosen into the interval $(\sigma_3, \sigma^H)$, a flip bifurcation and a Hopf bifurcation may occur consecutively for different values of the elasticity of the labor supply. On the contrary, such a case never occurs when $\sigma_3 > \sigma^H$ and these two bifurcations require different values for the elasticity of capital-labor substitution.
5.2 $\alpha \in (\alpha_2, 1)$

We may now consider the cases with a higher share of first period consumption over the wage income. Consider the associated ranking for the elasticity of capital-labor substitution provided by Lemma 5. The results detailed in the next Proposition are summarized in Figure 6.

**Proposition 5**. Under Assumptions 1-4, let $\alpha \in (\alpha_2, 1)$ and consider the critical bounds (24) for the elasticity of labor supply. There exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon_{e,L} < \bar{\varepsilon}$, the following results generically hold:

i) $\sigma \in (\bar{\sigma}^H, +\infty)$. The normalized steady state $(1, 1)$ is a saddle-point for $\epsilon_1 \in (\epsilon_1^F, +\infty)$, undergoes a flip bifurcation at $\epsilon_1 = \epsilon_1^F$, becomes locally indeterminate for $\epsilon_1 \in (\epsilon_1^T, \epsilon_1^F)$, undergoes a transcritical bifurcation at $\epsilon_1 = \epsilon_1^T$ and becomes again a saddle-point for $\epsilon_1 < \epsilon_1^T$.

ii) $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^H)$. The normalized steady state $(1, 1)$ is a saddle-point for $\epsilon_1 \in (\epsilon_1^F, +\infty)$, undergoes a flip bifurcation at $\epsilon_1 = \epsilon_1^F$, becomes locally indeterminate for $\epsilon_1 \in (\epsilon_1^H, \epsilon_1^F)$, undergoes a Hopf bifurcation at $\epsilon_1 = \epsilon_1^H$. 

Figure 6: $\alpha \in (\alpha_2, 1)$
becomes a source for $\epsilon_l \in (\epsilon_1^T, \epsilon_1^H)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_1^T$, and becomes again a saddle-point for $\epsilon_l < \epsilon_1^T$.

iii) $\sigma \in (\sigma_3, \sigma_3^H)$. The normalized steady state $(1, 1)$ is a saddle point for $\epsilon_l \in (\epsilon_1^F, \epsilon_1^H)$, undergoes a flip bifurcation at $\epsilon_l = \epsilon_1^F$, becomes a source for $\epsilon_l \in (\epsilon_1^T, \epsilon_1^F)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_1^T$, and becomes again a saddle-point for $\epsilon_l < \epsilon_1^T$.

iv) $\sigma \in (\sigma_5, \sigma_3)$. The normalized steady state $(1, 1)$ is a source for $\epsilon_l \in (\epsilon_1^T, +\infty)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_1^T$, and becomes a saddle-point for $\epsilon_l < \epsilon_1^T$.

v) $\sigma \in (\max\{0, \sigma_1\}, \sigma_5)$. The normalized steady state $(1, 1)$ is a source for any $\epsilon_l \in (0, +\infty)$.

vi) $\sigma \in (0, \max\{0, \sigma_1\})$. The normalized steady state $(1, 1)$ is a saddle-point for $\epsilon_l \in (\epsilon_1^T, +\infty)$, undergoes a transcritical bifurcation at $\epsilon_l = \epsilon_1^T$, and becomes a source for $\epsilon_l < \epsilon_1^T$.

When the share of first period consumption over the wage income is even greater, the same conclusions are obtained. Notice however that the main distinction with Propositions 3 and 4 concerns that fact that now local indeterminacy requires a priori larger elasticities of capital-labor substitution ($\sigma > \sigma_1 > \sigma_2 > 1$). There is therefore a trade-off between the values of $\alpha$ and $\sigma$: local indeterminacy with larger shares of first period consumption over the wage income requires larger elasticities of capital-labor substitution.

Before giving more precise economic intuitions, it is worthwhile comparing our main results with those obtained in the literature. It is well-known from Reichlin [23] that local indeterminacy of equilibria is only possible if the elasticity of capital-labor substitution is lower than the share of capital in total income, a restriction which is not compatible with the recent estimates of Duffy and Papageorgiou [9]. This result, obtained in a model without first period consumption, has been extended to models with consumptions in both periods by Lloyd-Braga [18], Cazzavillan and Pintus [7] and Nourry and Venditti [21]. The last two papers also show that the occurrence of multiple equilibria requires first period consumption to be a small fraction of wage income. Again, local indeterminacy becomes less likely as soon as fundamental parameters are compatible with standard estimates.

When imperfect competition is introduced, some of the previous criticisms may be weakened. Lloyd-Braga proves for instance that local inde-
terminacy may occur while the elasticity of substitution is higher than the share of capital when increasing returns to scale internal to the firm are considered. The same result also holds in a model without first period consumption when increasing returns are generated by capital externalities as shown in Cazzavillan [4]. However, we show in Section 5 that, even in presence of capital externalities, local indeterminacy requires a low enough share of first period consumption over the wage income which remains at variance with standard empirical estimates.

By considering instead small labor externalities, we show with Propositions 3, 4 and 5 that local indeterminacy becomes compatible with a large enough share of first period consumption over the wage income and an elasticity of capital-labor substitution greater than the share of capital in total income. However, as in all the contributions available in the literature, the elasticity of the labor supply needs to be large enough.

5.3 Economic intuitions

In order to provide some economic intuitions for our results, it is convenient to compute the following derivatives: From equation (3) we easily get

\[
\frac{dl}{dw} \frac{w}{l} = \epsilon_l > 0, \quad \frac{dl}{dR} \frac{R}{l} = (1 - \alpha)\epsilon_l > 0
\]

(25)

Notice that from the linear homogeneity of \( u(c, \hat{c}) \), it is easy to prove that the labor supply always increases with \( R \).\(^{12}\) Considering now the general formulation for the interest factor (15) and the definition of the elasticity of capital-labor substitution (18), we obtain

\[
\frac{dR}{dK} \frac{K}{R} = -\frac{1-s}{\sigma} + \epsilon_{e,K}, \quad \frac{dR}{dl} \frac{l}{R} = \frac{1-s}{\sigma} + \epsilon_{e,L} > 0
\]

(26)

Notice that when capital externalities are small, \( dR/dK \) remains negative.

We will use the same kind of intuitive argument as in Cazzavillan and Pintus [8]. As shown previously, local indeterminacy is closely related to deterministic fluctuations, based on the existence of flip and Hopf bifurcations, and stochastic fluctuations, based on the existence of sunspot equilibria.\(^{13}\) We then have to find a mechanism generating cyclical equilibrium paths.

We start by considering the model without external effects, i.e. \( \epsilon_{e,K} = \epsilon_{e,L} = 0 \). Let us start at time \( t \) from the steady state and assume an

\(^{12}\)The computation of this derivative is based on equation (31) in Appendix 8.2.

\(^{13}\)See for instance Woodford [24].
instantaneous increase in the capital stock $K_t$. This generates an increase in wage $w_t$ which implies from equation (25) an increase in the labor supply $l_t$. Since $K_{t+1} = (1 - \alpha)w_l l_t$, a higher capital stock in the next period is reached. But a higher $K_{t+1}$ is followed by a decrease in the interest factor which implies, from equation (25), a decrease in the labor supply $l_{t+1}$. A cyclical path will then be obtained if such a mechanism is strong enough to generate a decrease in the wage income which would decrease savings at time $t+1$ and capital at time $t+2$. Equation (26) finally shows that a strong effect on the interest factor is obtained when the elasticity of capital-labor substitution $\sigma$ is low while equation (25) shows that a strong effect on the labor supply is obtained when the share of first period consumption over the wage income $\alpha$ is low.

Consider now the model with capital externalities only, i.e. $\varepsilon_{e,L} = 0$. Since $\varepsilon_{e,K} > 0$, a higher capital stock $K_t$ generates a stronger increase in wage $w_t$ which then implies a stronger increase in the labor supply $l_t$. When $\alpha = 0$ (or low enough), a large increase in $K_{t+1}$ follows. Although the interest factor is less sensitive to capital variations (see equation (26)), and even with a larger elasticity of capital-labor substitution, the decrease in $R_{t+1}$ generates a strong enough decrease in the labor supply $l_{t+1}$ to lower savings and the next period capital stock. On the contrary, when $\alpha$ is large, the “small” increase in $K_{t+1}$ does not generate a strong enough reduction in $R_{t+1}$ which is then not sufficient to significantly lower the labor supply $l_{t+1}$. The cyclical reversal of the capital stock is thus less likely.

Consider finally the model with labor externalities only. Since $\varepsilon_{e,K} = 0$, a higher capital stock $K_t$ generates increases in wage $w_t$ and thus in the labor supply $l_t$ which are similar to those obtained in the model without external effects. Even if the share $\alpha$ is large, the increase in $K_{t+1}$ implies a decrease in $R_{t+1}$ which is maximal since $\varepsilon_{e,K} = 0$. A decrease in the labor supply $l_{t+1}$ then follows. But due to the presence of labor externalities, a larger $l_{t+1}$ also generates a general equilibrium, or feedback, effect on the interest factor (see $dR/dl$ in (26)) which is stronger than in the previous cases. As a result, a strong enough decrease in the wage income of period $t + 1$ and thus a cyclical reversal of the capital stock become compatible with a large share of first period consumption over the wage income.

\[14\] At the same time first period consumption is also increased.
5.4 A CES illustration

We may illustrate in a CES economy our main findings in order to show that local indeterminacy occurs under plausible parameters values, in particular with extremely small labor externalities and a share of first period consumption over the wage income compatible with standard estimates. From that point of view, notice that over the period 1950 – 2000 for the U.S., the annual ratio of consumption expenditures over GDP averages at 67%.

In our framework, total consumption over GDP is given at the steady state by
\[ \frac{c^++c^-}{y} = \alpha(1-s) + s \]
Considering a standard value for the share of capital in total income, \( s = 1/3 \), we easily compute the bounds \( \alpha_3 = 1/2 \) and \( \alpha_2 \approx 0.81385 \). Therefore, assuming \( \alpha = 0.51 \) implies a ratio of consumption expenditures over GDP of 67.3% compatible with the previous estimate. We thus consider the theoretical results given in Section 5.1 with \( \alpha \in (\alpha_3, \alpha_2) \).

Recent papers have questioned the empirical relevance of Cobb-Douglas technologies and unitary elasticities of capital-labor substitution. Duffy and Papageorgiou [9] for instance consider a panel of 82 countries over a 28-year period to estimate a CES production function specification. They find that for the entire sample of countries the assumption of unitary elasticity of substitution is rejected. Moreover, dividing the sample of countries up into several subsamples, they find that capital and labor have an elasticity of substitution significantly greater than unity (i.e. contained in \([1.14, 3.24]\)) in the richest group of countries.

The elasticity of intertemporal substitution in consumption is fixed slightly greater than one at \( \gamma = 1.1 \) in order to guarantee gross substitutability. It follows that \( 1 - \alpha \gamma > 0 \) and the upper bound on the amount of labor externalities considered in Propositions 3 and 4 is the following: \( \bar{\varepsilon} = \bar{\varepsilon}_{c,L} \approx 1.048\% \). Considering a very small labor externality with \( \varepsilon_{c,L} = 1\% \), we may then derive all the critical values for the elasticity of capital-labor substitution: \( \sigma_1 \approx 0.29463, \sigma_2 \approx 1.36, \sigma_3 \approx 259.68, \sigma_4 \approx 1.297, \sigma_5 \approx 0.29933 \) and \( \bar{\sigma}^H \approx 2.376 \). Since \( \sigma_3 > \bar{\sigma}^H > \sigma_2 > \sigma_4 > \sigma_5 > \sigma_1 \), we refer to the results given in Proposition 4.

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15 This number has been computed using the Penn World Data set available at the following address: http://www.bized.ac.uk/datarserv/penudata/pennhome.htm.

16 See Appendix 8.7.
i) Consider first the case $\sigma \in (\bar{\sigma}^H, \sigma_3) = (2.376, 259.68)$. In order to be compatible with the estimates given by Duffy and Papageorgiou we assume more precisely that $\sigma \in (2.376, 3.24)$. The steady state is thus locally indeterminate for any $\epsilon_l > \epsilon^H_l \approx 67$.

ii) Consider now the case $\sigma \in (\sigma_2, \bar{\sigma}^H) = (1.36, 2.376)$. The steady state will be locally indeterminate for any $\epsilon_l > \epsilon^H_l$ with $\epsilon^H_l$ a Hopf bifurcation value. Assuming that $\sigma \in (1.37, 2.37)$ we get $\epsilon^H_l \in (67.35, 6802.72)$.

While labor externalities are restricted to be arbitrarily small, these numerical illustrations clearly show that local indeterminacy of equilibria and endogenous fluctuations rely on plausible values for the elasticity of capital-labor substitution and the elasticity of intertemporal substitution in consumption provided the labor supply is sufficiently elastic. Notice from that point of view that most of the calibrations used in the RBC literature are based on an infinite elasticity of the labor supply which is associated with indivisible labor.

6 Comparisons with infinite horizon models

6.1 Indeterminacy in dynamic models: life horizon does not matter

One-sector infinitely-lived agent models with inelastic labor, Romer-type capital externalities and increasing returns at the social level have been considered initially by Kehoe [15] and Boldrin and Rustichini [3]. As shown in these contributions, local indeterminacy requires very strong negative externalities which improves enough the private marginal productivity of capital to destroy concavity at the social level. Obviously these conditions cannot be met by usual Cobb-Douglas or CES technologies. When standard positive externalities are considered, it is shown on the contrary that the steady state is either saddle-point stable or totally unstable.

Under standard formulations for the fundamentals, Benhabib and Farmer [1] have shown that local indeterminacy in one-sector models actually requires the consideration of elastic labor supply and aggregate externalities on labor. They assume separable preferences over consumption and leisure with a unitary elasticity of intertemporal substitution in consumption and
an infinitely elastic labor supply.\footnote{Following Hansen [14], an infinitely elastic labor supply is interpreted as indivisible labor at the individual level.} They also consider a Cobb-Douglas aggregate technology. In such a framework, the main conclusion of Benhabib and Farmer is the following: in order to get local indeterminacy of equilibria, externalities and thus the degree of increasing returns to scale must be large enough to imply that the aggregate labor demand curve should be upward-sloping and steeper than the aggregate labor supply curve. This is obviously a non-standard configuration for the labor market.

More recently, Pintus [22], by considering a general separable utility function and a general technology with constant returns to scale at the private level and productive externalities, shows that the conditions of Benhabib and Farmer are not necessary. Local indeterminacy may indeed arise with a standard decreasing equilibrium labor demand function and small externalities on labor provided the elasticity of capital-labor substitution is significantly greater than one and the elasticity of the labor supply is large enough. It also clearly appears that capital externalities do not play any positive role for the occurrence of local indeterminacy. Pintus also provides the following numerical illustration: considering an infinitely elastic labor supply and some labor externalities of 5\%, local indeterminacy arises when the elasticity of intertemporal substitution in consumption is greater than 16.6 and the elasticity of capital-labor substitution belongs to (2.4, 6.65).

Propositions 3, 4 and 5 show that the same conclusions are obtained in an OLG model with elastic labor as soon as the share of first period consumption over the wage income is large enough to be consistent with standard estimates. Local indeterminacy is then compatible with extremely small externalities on labor, an elasticity of intertemporal substitution in consumption slightly greater than unity and an elasticity of capital-labor substitution larger than one provided that the elasticity of the labor supply is large enough. These theoretical results are confirmed by numerical illustrations based on CES preferences and technology. We may then easily conclude that, contrary to a widespread opinion, \textit{the conditions for the occurrence of local indeterminacy in dynamic models with externalities do not depend on the life horizon of agents.}

At that point one question remains open: It is possible to get local indeterminacy when preferences and technology are Cobb-Douglas? We know
from Benhabib and Farmer [1] that the answer is positive for infinitely lived agent models provided the labor externalities are strong enough. On the contrary, Nourry and Venditti [21] show that the answer is negative in an OLG model without external effects. All our previous results on local indeterminacy have been obtained under the assumption $\sigma > \sigma_2 > 1$ in order to consider extremely small labor externalities. We may then consider an economy in which preferences and the technology are given by Cobb-Douglas functions with unitary elasticities of substitution.

Assumption 5. $\sigma = \gamma = 1$

Using $u(c, \hat{c}) = c^{\alpha} \hat{c}^{1-\alpha}$ and $F(K, L) = K^s L^{1-s}$, the share of first period consumption over the wage income $\alpha$ and the share of capital in total income $s$ are constant. Since the interest factor is given by $R = A s x^{s-1} e(l)$ with $x = K/l$, straightforward computations easily show that the existence and uniqueness of the steady state $(\bar{K}, \bar{l})$ are derived from the existence and uniqueness of the solution $R$ in the following equation

$$s = (1 - \alpha) R$$

Moreover Assumption 3 holds and the normalization procedure implies that $(\bar{K}, \bar{l}) = (1, 1)$. From Proposition 2 the determinant and trace simplify as

$$D = \left(\frac{1+\epsilon_l}{\epsilon_l}\right) \frac{s}{(1-\alpha)(1-s+\epsilon_e, \epsilon_l)}, \quad T = \left(\frac{1+\epsilon_l}{\epsilon_l}\right) \frac{1}{(1-\alpha)(1-s+\epsilon_e, \epsilon_l)} - \frac{\alpha}{1-\alpha}$$

so that the $\Delta$ line becomes

$$D = \Delta(T) = sT + \frac{s\alpha}{1-\alpha}$$

As previously, the starting point $(T_1, D_1)$ is obtained when $\epsilon_l = +\infty$ and satisfies

$$D_1 = \frac{s}{(1-\alpha)(1-s+\epsilon_e, \epsilon_l)}, \quad T_1 = \frac{1}{(1-\alpha)(1-s+\epsilon_e, \epsilon_l)} - \frac{\alpha}{1-\alpha}$$

In order to simplify the analysis and to provide direct comparisons with Benhabib and Farmer [1], we also restrict the value of the share of capital in total income to a standard value:

Assumption 6. $\alpha = 1/3$

We easily show the following results:

$$D_1 < 1 \iff \epsilon_{e, L} > \epsilon_{e, L}^1 \equiv \frac{1}{3(1-\alpha)} - \frac{2}{3}$$

$$T_1 < 2 \iff \epsilon_{e, L} > \epsilon_{e, L}^2 \equiv \frac{1}{2-\alpha} - \frac{2}{3}$$

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with $\varepsilon_{e,L}^1 > \varepsilon_{e,L}^2$. We also easily get that $T_1 > -2$ for any $\varepsilon_{e,L} > 0$. Notice now that $D_1 - T_1 + 1 > 0$ for any $\varepsilon_{e,L} > 0$ and

$$D_1 + T_1 + 1 > 0 \iff \varepsilon_{e,L} < \varepsilon_{e,L}^3 \equiv \frac{4}{3(2\alpha - 1)} - \frac{2}{3}$$

with $\varepsilon_{e,L}^3 > \varepsilon_{e,L}^1$ if and only if $\alpha < \bar{\alpha} \equiv 5/6$. Since $\bar{\alpha} > \alpha_3 = 1/2$, we conclude that under $\alpha \in (1/2, 5/6)$ and $\varepsilon_{e,L} \in (\varepsilon_{e,L}^1, \varepsilon_{e,L}^3)$, the starting point $(T_1, D_1)$ is within the triangle $ABC$. Since the slope of the $\Delta$ line is less than 1 we may summarize all these results in Figure 7:

![Figure 7: Local indeterminacy in a Cobb-Douglas economy](image)

Straightforward computations easily give the values of the critical bounds $\varepsilon_{l_H}^H, \varepsilon_{l_T}^T$ on the elasticity of labor supply $\varepsilon_l$ which are respectively associated with Hopf and transcritical bifurcations:

$$\varepsilon_{l_H}^H = \frac{1}{3(1-\alpha)(\varepsilon_{e,L} - \varepsilon_{e,L}^1)}, \quad \varepsilon_{l_T}^T = \frac{2}{3\varepsilon_{e,L}}$$

(27)

Notice that since the steady state is unique, the transcritical bifurcation is necessarily degenerated. We may then summarize all the results in the following Proposition:

**Proposition 6**. Local indeterminacy in a Cobb-Douglas economy. Under Assumptions 1-6, let $\alpha \in (1/2, 5/6)$, $\varepsilon_{e,L} \in (\varepsilon_{e,L}^1, \varepsilon_{e,L}^3)$ and consider
the critical bounds (27) for the elasticity of labor supply. Then the normalized steady state \((1, 1)\) is locally indeterminate for \(\epsilon_l \in (\epsilon_l^H, +\infty)\), undergoes a Hopf bifurcation at \(\epsilon_l = \epsilon_l^H\), becomes a source for \(\epsilon_l \in (\epsilon_l^T, \epsilon_l^H)\), undergoes a (degenerate) transcritical bifurcation at \(\epsilon_l = \epsilon_l^T\) and becomes a saddle-point for \(\epsilon_l < \epsilon_l^T\).

Therefore, as in infinite horizon models with labor externalities, local indeterminacy easily occurs in OLG models with significant first period consumption and small labor externalities while preferences and technology are Cobb-Douglas. However, these results rely on parameters values that appear to be much more empirically plausible than in the contribution of Benhabib and Farmer [1].\(^{18}\) Considering again that \(\alpha = 0.51\), we derive the bounds \(\epsilon_{e,L}^{1} \approx 0.0136\) and \(\epsilon_{e,L}^{3} \approx 66\). It follows that with 1.4\% of labor externalities, the steady state is locally indeterminate for any \(\epsilon_l > \epsilon_l^H \approx 1722.2\) with \(\epsilon_l^H\) a Hopf bifurcation value. Notice that with a Cobb-Douglas technology, local indeterminacy requires a slightly larger value of \(\epsilon_{e,L}\) than in cases in which \(\sigma > 1\), but labor externalities remains extremely small. Of course, the elasticity of the labor supply needs to be strong enough.

### 6.2 Under- versus over-accumulation of capital

Finitely-lived agents models, such that the OLG model, and infinitely-lived agents models, such that the Ramsey model, are also often distinguished with respect to their efficiency properties. While the long-run equilibrium in a Ramsey economy is given by the modified golden rule which is dynamically efficient, in an OLG economy the steady state may be dynamically inefficient if there is an over-accumulation of capital with respect to the golden rule, i.e. if the stationary interest factor \(R^*\) is strictly less than 1.\(^{19}\)

In presence of productive externalities, such a distinction is not obvious since in both models the equilibrium is affected by these market imperfections. However, the analysis of under- or over-accumulation drastically differs depending on whether the external effects come from capital or not. In presence of capital externalities the criterium based on the interest factor cannot be used as previously since the definition of the golden rule is directly affected by the externalities. On the contrary, if there are only labor

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\(^{18}\)Externalities are indeed extremely large (more than 43%).

\(^{19}\)See Galor and Ryder [11] with constant population.
externalities, the golden rule capital stock is defined as in the standard model without market imperfection. In our framework, we may define total stationary consumption as

\[ c + \dot{c} = wl + RK - K = AF(K, l)e(l) - K \]

Maximizing total consumption with respect to the capital stock gives

\[ AF_1(K, l)e(l) = Af'(k)e(l) = R = 1 \]

In such a case, we find the same conclusions as in models without market imperfection: in presence of labor externalities only, the steady state of a Ramsey model is always given by the modified golden rule which implies under-accumulation, while the stationary interest factor \( R^* \) of an OLG model is not necessarily greater than 1. We may then wonder in which case the normalized steady state of our formulation is characterized by an under-accumulation of capital. We easily get the following Proposition:

**Proposition 7.** Under Assumptions 1-3, the normalized steady state \((\bar{K}, \bar{l}) = (1, 1)\) is characterized by an under-accumulation of capital if and only if \( \alpha \geq \alpha_3 \).

This Proposition clearly shows that under-accumulation of capital is obtained as soon as the share of first period consumption in total income is high enough, i.e. compatible with standard estimates. This conclusion then implies that all our previous results concerning local indeterminacy of equilibria are associated with under-accumulation of capital as in Ramsey models augmented to include labor externalities.

## 7 Concluding comments

In this paper we have studied an OLG model with consumption in both periods of life and in which the share of first period consumption over the wage income is large enough to be compatible with standard estimates. We have shown that under gross substitutability, local indeterminacy of equilibria cannot be generated by capital externalities but occurs with extremely small labor externalities provided the elasticity of capital-labor substitution and the elasticity of the labor supply are large enough.

We have then proved that when dealing with local indeterminacy, and contrary to what occurs under perfect competition, models with a finite
number of infinitely lived agents and models with an infinite number of
finitely lived agents are actually very similar. Our results show that life
horizon does not matter for the occurrence of multiple equilibria in dynamic
models.

8 Appendix

8.1 Proof of Proposition 1

Let \( V(B) = v'(1/B)/B \). Under Assumption 1, we easily derive that \( V'(B) < 0 \) so that \( V(B) \) is invertible. \((\bar{K}, \bar{l}) = (1, 1)\) is a steady state if and only if there exist values for \( A \) and \( B \) such that

\[
1 = \left[ 1 - \alpha \left( Af'(1)e(1, 1) \right) \right] Aw(1)e(1, 1) \equiv G(A) \tag{28}
\]

\[
B = V^{-1} \left( u_1 \left( 1, h \left( Af'(1)e(1, 1) \right) \right) Aw(1)e(1, 1) \right) \tag{29}
\]

From equations (6), (9), (10) and (12) we easily compute the elasticity of
the propensity to consume \( \alpha(R) \):

\[
\frac{d\alpha(R)}{dR} \frac{R}{\alpha(R)} = (1 - \gamma(R))(1 - \alpha(R)) \tag{30}
\]

Recall now that the elasticity of intertemporal substitution in consumption
\( \gamma(R) \) may be greater or less than 1. It follows that \( \alpha(R) \) may be increasing or decreasing depending on the value of \( \gamma(R) \). We easily derive from (28)
and (30) the elasticity of \( G(A) \) as

\[
\frac{G'(A)A}{G(A)} = 1 - \alpha(R)(1 - \gamma(R))
\]

We then conclude that \( G'(A)A/G(A) \) is positive and thus \( G(A) \) is a mono-
tone increasing function for any \( \gamma(R) > 0 \). Recall finally that for any \( R \geq 0, \alpha(R) \in (0, 1) \). It follows that \( \lim_{z \to 0}(1 - \alpha(z)) \leq 1 \) and thus

\[
\lim_{A \to 0} \left[ 1 - \alpha \left( Af'(1)e(1, 1) \right) \right] A = 0
\]

We then conclude that there exists a unique \( A^* > 0 \) solution of equation (28)
if and only if \( \lim_{z \to +\infty}(1 - \alpha(z))z > f'(1)/w(1) \). \( B^* \) is obtained considering
\( A^* \) into equation (29). \( \square \)
8.2 Proof of Proposition 2

Consider the first order condition (2)

\[ u_1(1, \hat{c}/c) = u_2(1, \hat{c}/c)R \]

Since from (9) \( \hat{c}/c \) is a function of \( R \), differentiating this equation with respect to \( R \) gives

\[ u_{12}(1, \hat{c}/c) \frac{d\hat{c}/c}{dR} = u_{22}(1, \hat{c}/c)R \frac{d\hat{c}/c}{dR} + \frac{u_1(1, \hat{c}/c)}{R} \]

Using the homogeneity of \( u(c, \hat{c}) \) and (8), this equation simplifies to

\[ u_{12}(1, \hat{c}/c) \frac{d\hat{c}/c}{dR} = \frac{u_1(1, \hat{c}/c)}{R} \]

Notice now that from the budget constraints and equation (10) we get \( \hat{c}/c = (1 - \alpha (1 - \gamma)) R/\alpha (1 - \gamma) \). We finally obtain from all this

\[ du_1(1, h(R)) dR = u_{12}(1, \hat{c}/c) \frac{d\hat{c}/c}{dR} R \]

The result follows after straightforward simplifications.

8.3 Proof of Lemma 1

1- The slope of \( \Delta(T) \) may be written as

\[ S = \frac{s + \sigma e_{e,K}}{\sigma(1 + \alpha(1 - \gamma)e_{e,K}) - \alpha(1 - \gamma)(1 - s)} \]

It follows that \( S \geq 0 \) if and only if one of the following conditions holds:

i) \( \gamma \in [1, \bar{\gamma}] \) with \( \bar{\gamma} \equiv (1 + \alpha e_{e,K})/\alpha e_{e,K} > 1 \),

ii) \( \gamma < 1 \) and \( \sigma \geq \bar{\sigma} \) with \( \bar{\sigma} \equiv \alpha(1 - \gamma)(1 - s)/(1 + \alpha(1 - \gamma)e_{e,K}) > 0 \),

iii) \( \gamma > \bar{\gamma} \) and \( \sigma \leq \bar{\sigma} \).

2- The results follow from obvious computations.
8.4 Proof of Lemma 3

1- When $1 - \alpha \gamma \geq 0$ straightforward computations show that:

i) $S_1 \in (0, 1)$,

ii) $\Delta_1(2) > 1$ if and only if $\varepsilon_{e,L} < -\frac{\alpha^2(1-s) + \alpha(3-4s) + 2(2s-1)}{(1-\alpha)(1-\alpha\gamma)} \equiv \varepsilon^*_e,L$;

iii) there exists $\alpha_2 \equiv \frac{\alpha}{\gamma} \in (1/2, 1)$, and satisfying $\alpha_2 > \alpha_3$, such that when $\alpha \in (\alpha_3, \alpha_2)$, $\Delta_1(-2) < 1$,

iv) on the contrary, when $\alpha \in (\alpha_2, 1)$, $\Delta_1(-2) > 1$ if and only if $\varepsilon_{e,L} < -\frac{\alpha^2(1-s) + 3\alpha - 2}{(1-\alpha)(1-\alpha\gamma)} \equiv \tilde{\varepsilon}^{**}_{e,L}$.

2- When $1 - \alpha \gamma < 0$ we similarly get:

i) $S_1 > 0$ if and only if $\varepsilon_{e,L} < \frac{1+(1-\alpha)(1-s)}{\alpha\gamma} \equiv \tilde{\varepsilon}_{e,L}$, and $S_1 < 1$ if and only if $\varepsilon_{e,L} < \frac{(1-s)(2-\alpha)}{\alpha\gamma-1} \equiv \tilde{\varepsilon}_{e,L}$. By construction we have $\tilde{\varepsilon}_{e,L} < \varepsilon_{e,L} + \varepsilon^*_e,L$.

ii) $\Delta_1(2) > 1$;

iii) when $\alpha \in (\alpha_3, \alpha_2)$, $\Delta_1(-2) < 1$ if and only if $\varepsilon_{e,L} < \varepsilon^*_{e,L}$,

iv) when $\alpha \in (\alpha_2, 1)$, $\Delta_1(-2) > 1$.

8.5 Proof of Lemma 4

We easily derive from direct computations:

i) and ii): $D_1 \geq T_1 - 1$ if and only if $\sigma \geq \frac{1-\alpha\gamma}{2-\alpha} \equiv \sigma_1$. It follows that $\sigma_1 \geq 0$ if and only if $1 - \alpha \gamma \geq 0$.

iii) $D_1 \leq 1$ if and only if $\sigma \geq \frac{(1-s)(\alpha-\alpha_3)}{(1-\alpha)s_{e,L}} \equiv \sigma_2$.

iv) $D_1 \geq -T_1 - 1$ if and only if $\sigma \leq \frac{(1-s)(\alpha-\alpha_3)}{\alpha s_{e,L}} + \frac{1-\alpha\gamma}{\alpha} \equiv \sigma_3$.

v) $S \leq S_1$ if and only if $\sigma \geq 1 + (1 - \alpha\gamma)(1 - s + \varepsilon_{e,L}) \equiv \sigma_4$. Notice that depending on the value of $1 - \alpha \gamma$ this bound may be negative, in which case there is no restriction on the value of $\sigma$.

vi) $S \leq 1$ if and only if $\sigma \geq (1 - s)(\alpha - \alpha_3 + 1 - \alpha\gamma) \equiv \sigma_5$. Notice that depending on the value of $1 - \alpha \gamma$ this bound may be negative, in which case there is no restriction on the value of $\sigma$.

vi) obvious computations show that there exists $\sigma^H$ such that $\Delta(2) = 1$. We get from simple geometrical considerations $\bar{\sigma}^H > \sigma_2, \sigma_4$.

vii) when $\Delta_1(-2) > 1$, we may also define $\sigma^H$ as the value of $\sigma$ such that $\Delta(-2) = 1$ which obviously satisfies $\bar{\sigma}^H > \tilde{\sigma}^H$.

\[ \square \]
8.6 Proof of Lemma 5

Consider the bounds on $\sigma$ introduced in Lemma 4. Direct computations prove the following facts:
i) Under Assumption 4, obvious geometrical arguments show that $\sigma_3 > \sigma_2 > \sigma_1$ when $\alpha \in (\alpha_3, \alpha_2)$ and $\sigma_2 > \sigma_3 > \sigma_1$ when $\alpha \in (\alpha_3, \alpha_2)$. Moreover we have $\sigma_5 > \sigma_1$.

$\sigma_4 > \sigma_5$ without any restriction when $1 - \alpha \gamma \geq 0$ but when $1 - \alpha \gamma < 0$, this inequality holds if and only if $\varepsilon_{e,L} < \bar{\varepsilon}_{e,L}$.

$\sigma_3 > \sigma_4$ if $\varepsilon_{e,L} < s$.

$\sigma_2 > \sigma_4$ for any given $\varepsilon_{e,L}$ when $\gamma$ is large enough to imply $1 - \alpha \gamma < -(1 - s + \varepsilon_{e,L})^{-1}$. On the contrary, when $1 - \alpha \gamma > -(1 - s + \varepsilon_{e,L})^{-1}$, there exists $\varepsilon_{e,L} > 0$ such that $\sigma_2 > \sigma_4$ if $\varepsilon_{e,L} < \bar{\varepsilon}_{e,L}$.

v) let us finally recall from Lemma 4 that $\bar{\sigma}^H > \sigma_2, \sigma_4$ and $\bar{\sigma}^H > \bar{\sigma}_H > \sigma_2$.

The rest of the proof is obtained by choosing the value of $\bar{\varepsilon}$ as follows:
- if $1 - \alpha \gamma \geq 0$, $\bar{\varepsilon} = \min\{s, \varepsilon_{e,L}\}$,
- if $0 > 1 - \alpha \gamma > -(1 - s + \varepsilon_{e,L})^{-1}$, $\bar{\varepsilon} = \min\{s, \varepsilon_{e,L}, \bar{\varepsilon}_{e,L}\}$,
- if $1 - \alpha \gamma < -(1 - s + \varepsilon_{e,L})^{-1}$, $\bar{\varepsilon} = \min\{s, \bar{\varepsilon}_{e,L}\}$. \hfill \qed

8.7 Proof of Propositions 3 and 4

These two Propositions only differ with respect to the ranking of $\bar{\sigma}^H$ and $\sigma_3$. All the local stability results are derived from Lemmas 1-5 and Figures 4 – 5. Considering the bound $\bar{\varepsilon}$ introduced in Lemma 5 (see its proof in Appendix 8.6), the critical value $\bar{\varepsilon}$ is chosen as follows:
- if $1 - \alpha \gamma > 0$, $\bar{\varepsilon} = \min\{\bar{\varepsilon}, \varepsilon_{e,L}^{**}\}$,
- if $0 > 1 - \alpha \gamma > -(1 - s + \varepsilon_{e,L})^{-1}$, $\bar{\varepsilon} = \min\{\bar{\varepsilon}, \varepsilon_{e,L}^{**}\}$,
- if $1 - \alpha \gamma < -(1 - s + \varepsilon_{e,L})^{-1}$, $\bar{\varepsilon} = \min\{\bar{\varepsilon}, \varepsilon_{e,L}^{**}\}$. \hfill \qed

8.8 Proof of Proposition 5

All the local stability results are derived from Lemmas 1-5 and Figure 6. Considering the bound $\bar{\varepsilon}$ introduced in Lemma 5, the critical value $\bar{\varepsilon}$ is chosen as follows:

$\bar{\varepsilon}_{e,L}$ is the lower positive root of the polynomial $\sigma_2 - \sigma_4 = -\varepsilon_{e,L}^{2}(1 - \alpha)(1 - \alpha \gamma) - \varepsilon_{e,L}(1 - \alpha)[1 + (1 - \alpha \gamma)(1 - s)] + (1 - s)(\alpha - \alpha_3) = 0$.  

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- if $1 - \alpha \gamma > 0$, $\bar{\varepsilon} = min\{\tilde{\varepsilon}, \varepsilon^{**}_{e,L}\}$,
- if $0 > 1 - \alpha \gamma > -(1 - s + \varepsilon_{e,L})^{-1}$, $\bar{\varepsilon} = \tilde{\varepsilon}$,
- if $1 - \alpha \gamma < -(1 - s + \varepsilon_{e,L})^{-1}$, $\bar{\varepsilon} = \tilde{\varepsilon}$.

8.9 Proof of Proposition 7

Since $\varepsilon_{e,K} = 0$, the stationary interest factor corresponding to the normalized steady state is given by $R^* = A^* f'(1)e(1)$. Proposition 1 gives the following value for the scaling parameter $A^*$:

$$A^* = \frac{1}{(1 - \alpha)w(1)\varepsilon(1)}$$

It follows that the stationary interest factor satisfies

$$R^* = \frac{f'(1)e(1)}{(1 - \alpha)w(1)e(1)} = \left[(1 - \alpha)\frac{f'(1) - f''(1)}{f'(1)}\right]^{-1} = \frac{1}{(1 - \alpha)\left(\frac{f''(1)}{f'(1)}\right) - 1}$$

$$= \frac{s}{(1 - \alpha)(1 - s)}$$

It follows easily that $R^* \geq 1$ if and only if $\alpha \geq \alpha_3$.

References


