Wealth inequality and macroeconomic volatility in two-sector economies

Christian GHIGLINO
Dept. of Economics, Queen Mary, University of London, UK

and

Alain VENDITTI
CNRS - GREQAM, Marseille, France

Abstract: We explore the link between wealth inequality and macroeconomic volatility in a two-sector neoclassical growth model with heterogeneous agents. The model has a unique aggregate steady state which is independent of the level of inequality. However, whenever preferences do not exhibit hyperbolic absolute risk aversion, wealth heterogeneity may affect the volatility of output. In particular, we show that when consumers have identical preferences and absolute risk tolerance is a strictly convex function, inequality is a factor that favors macroeconomic volatility. In the opposite case, equality favors macroeconomic volatility. Weak empirical evidence suggests that risk tolerance is concave in which case our finding would imply that inequality tends to remove fluctuations.

Keywords: Economic growth, Heterogeneity, Wealth and Income Inequality, Macroeconomic Volatility, Fluctuations.

Journal of Economic Literature Classification Numbers: D30, D50, D90, O41.

1E-mail: c.ghiglino@qmul.ac.uk
2E-mail: venditti@ehess.univ-mrs.fr
1 Introduction

The relationship between income and wealth inequality and growth has been explored in a large number of empirical studies. Inequality has a negative effect on the growth rate of GDP in several studies but the link is never very strong and other studies give non-significant or opposed results (see Benhabib (2003)). Several other economic variables have also been considered and the results are sometimes promising. In particular, there is some evidence that wealth and income inequality generates macroeconomic volatility. The reverse causality has also been found (see Breen and Garcia-Penalosa (2004)). Note that the existence of a link between inequality and instability would be of great importance in particular in relation to issues of public policy. Indeed, since Musgrave (1959), it is a common practice to dissociate macroeconomic stabilization policies and redistribution policies, their instruments being a priori different. However, if volatility and wealth inequality are correlated, it becomes fundamental to know how they interact and when these two dimensions of public intervention need to be considered simultaneously.

The theoretical literature on the link between inequality and fluctuations is very scarce. In fact, most of the recent literature adopts stochastic intertemporal models in which agents face uninsurable shocks. The analysis then focuses on the stationary makov equilibria, so that inequality may affect the economy only if it affects this specific type of equilibrium (see, e.g. Krusell and Smith (1998)). Similarly, in deterministic models the standard analysis focuses on the balanced growth paths and the effect of inequality is investigated performing “comparative statics”. However, heterogeneity may have a much larger impact if the dynamic properties of the paths are taken into account. Inequality may affect the economy because it affects the stability properties of the stationary quilibria and generate periodical solutions while leaving the value of the stationary solutions unchanged. We will focus on this channel.

We adopt a neoclassical two-sector growth model with a purely consumption good and a purely capital good. Labor is provided inelastically. Agents are heterogeneous in respect to the share of the initial stock of capital and in labor endowments. We assume that markets are complete and agents face no borrowing constraints. Then without lack of generality we may suppose

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3Recent surveys are Aghion, Caroli and Garcia-Penalosa (1999) and Benabou (2000).
4See also Hausmann and Gavin (1996).
that there is no uncertainty. Due to the structure of the model individual characteristics and heterogeneity do not affect the steady state values of the aggregate variables. Note that in its representative agent version, for some standard technologies and preferences the model exhibits instability and fluctuations.\(^5\)

We focus on how wealth and income inequality affects the dynamics. An economy may be unequal in several dimensions. However, agents show far less variations in taste than in income or wealth. Furthermore, wealth or income inequality is relevant for economic policy because of the possibility of redistributions (while the government can do little about taste heterogeneity). When agents have identical preferences, comparing wealth or income distributions may be done using a variety of criteria. It is usual to assume that inequality at least does not increase when income is transferred from a rich to a poor individual. In this case distributions are ranked using their Lorenz curves or equivalently using second-order stochastic dominance, i.e. the expected value of increasing concave functions decreases with inequality. In fact, we follow Rothschild and Stiglitz [12] and consider a similar notion in which attention is restricted to continuous and concave functions. A nice property is that then a risk averter consumer would always choose the more equal distribution.

Our first result is that provided preferences do not have hyperbolic absolute risk aversion, wealth and income inequality is correlated to macroeconomic volatility. The main result is that the relation is positive or negative depending on the concavity of the inverse of the agents’ absolute risk aversion, called absolute risk tolerance. In particular, we find that when absolute risk tolerance is a convex function, inequality favors instability. In the opposite case, inequality favors stability and smoothness of the optimal path.

A weakness our findings is that the usual axioms on preferences do not restrict the concavity properties of absolute risk aversion or its inverse. Furthermore, as direct empirical investigation of the properties of preferences is usually impossible only indirect model dependent evidence can be collected. Asset theory provides most of the indirect empirical evidence on the properties of absolute risk aversion. As reviewed by Gollier (2001), this evidence suggests that absolute risk tolerance is not linear. In fact, there is some weak evidence in support of the concavity of absolute risk tolerance (see Guiso and Paiella (2003)). According to the present paper, this evidence would suggest that agent’s heterogeneity favors stability. The other set of indications is provided by household intertemporal behavior. Evidence there suggests

\(^5\)See Benhabib and Nishimura (1985).
that households do not behave as single consumers. Within the framework provided by the models of collective behavior, this fact implies that either the efficiency assumption of the collective model or the hyperbolic absolute risk aversion specification is violated (see, e.g. Vermeulen (2002) or Mazzocco (2003)). Consequently, the non-linearity of absolute risk aversion is not refuted by this data.

The analysis is standard for economies with heterogeneous agents (see Ghiglino (2002)). Wealth and income heterogeneity affects the “social” utility function even in the absence of heterogeneity in preferences. The first welfare theorem allows to focus on the properties of the Pareto optimal allocations. These are obtained as solutions to a planner’s problem characterized by a social utility function depending on the welfare weights. In the model, these weights are continuous functions of the initial conditions. Consequently, the local dynamic properties of the general equilibrium model with heterogeneous agents and those of the planner’s problem with the welfare weights fixed at their steady state values are identical. Decentralization of these equilibria only occurs at a second stage of our analysis where we characterize the effect of agent’s heterogeneity on dynamics.

The present paper extends Ghiglino and Olszak (2001) and Ghiglino (2003) in two directions. First, these papers consider a model drawn from Boldrin and Deneckere (1990) in which technology belongs to a very small class. Particularly disturbing is that the investment good sector is characterized by a Leontief technology (while it is CES in the consumption good sector). Furthermore, only relatively small subset of these economies are investigated. Second, only the local stability of the steady state is investigated in these papers without any consideration on the existence of periodic equilibria and on global dynamics. Clearly, these problems preclude the cited papers to be used in the general analysis of the relation between volatility and inequality. A few papers deal with the impact of heterogeneity on the occurrence of local indeterminacy in models with externalities. Ghiglino and Olszak-Duquenne (2002) is based on Example 2.1 in Boldrin and Rustichini (1994) in which a labor augmenting externality is introduced in a two-sector growth model with a Leontief investment sector. The result that local indeterminacy is related to heterogeneity suffers the same weaknesses as in the papers cited above. Herrendorf et al. (2000) analyze an overlapping generation model with heterogeneously productive agents. Inequality is shown to reduce indeterminacy but general equilibrium effects are eliminated by the assumption that prices are “fixed”. Finally, Ghiglino and Sorger (2002) looks at a continuous time growth model with a productive externality, endogenous labour supply, logarithmic preferences and two
types of agents. They show that the initial distribution in individual wealth may have dramatic effects as driving the economy to a poverty trap or generate indeterminacy. However, no results pertaining to the effect of wealth inequality on indeterminacy of the steady state is obtained because of the lack of continuity of the welfare weights.

The paper is organized as follows: In section 2 the model is introduced while the equilibria are defined in Section 3. Section 4 focuses on the relationship between endowment distribution and the occurrence of endogenous fluctuations. In section 5 the occurrence of macroeconomic volatility is related to heterogeneity. Section 6 presents a CES example and section 7 concludes.

2 The model

In the present paper we consider a competitive two-sector economy with different types of agents.

2.1 Producers

The technological side is formalized as in standard optimal growth models but we introduce heterogeneity across agents. There are two produced goods, one consumption good \( y_0 \) and one capital good \( y \). The consumption good cannot be used as capital so it is entirely consumed. The capital good cannot be consumed and partially depreciates in each period at a constant rate \( \mu \in [0, 1] \). There are two inputs, capital and labor. We also assume that labor is inelastically supplied and that its total amount is normalized to 1. Each good is produced with a standard constant returns to scale technology:

\[
y_0 = f^0(k^0, l^0), \quad y = f^1(k^1, l^1)
\]

with \( k^0 + k^1 \leq k \), \( k \) being the total stock of capital, and \( l^0 + l^1 \leq 1 \).

**Assumption 1.** Each production function \( f^j : \mathbb{R}_+^2 \to \mathbb{R}_+ \), \( j = 0, 1 \), is \( C^2 \), increasing in each argument, concave, homogeneous of degree one and such that for any \( x > 0 \), \( f^1_1(0, x) = f^2_2(x, 0) = +\infty \), \( f^1_1(+\infty, x) = f^2_2(x, +\infty) = 0 \).

There are two representative firms, one for each sector. For any \( t \geq 0 \), we denote by \( w_t \) the labor wage rate, \( r_t \) the gross rental rate of capital and \( p_t \) the price of investment good. They are expressed in units of the consumption good which has the price fixed at one in all periods. In a
decentralized economy, the firms take the price sequence \( \{w_t, r_t, p_t\} \) as given. Their optimal decision problems reduce to the choice of factors demand sequences \( (k^0_t, t^0_t) \) which maximize the present discounted value of a stream of future profits. Therefore the consumption good sector solves
\[
\max \quad y_{0t} - r_t k^0_t - w_t l^0_t \\
s.t. \quad y_{0t} \leq f^0(k^0_t, l^0_t) \quad \text{for all } t
\]
and the capital good sector solves
\[
\max \quad p_t y_t - r^1_t k_t - w_t l^1_t \\
s.t. \quad y_t \leq f^1(k^1_t, l^1_t) \quad \text{for all } t
\]
These profit maximization can be equivalently formulated as follows: For any given \( (k, y) \), solving the following problem of optimal allocation of productive factors between the two sectors:
\[
T(k, y) = \max_{k^0, k^1, l^0, l^1} f^0(k^0_t, l^0_t) \\
s.t. \quad y_j \leq f^1(k^1_t, l^1_t) \\
k^0 + k^1 \leq k \\
l^0 + l^1 \leq 1 \\
k^0, k^1, l^0, l^1 \geq 0 \tag{1}
\]
gives the social production function \( T(k, y) \) which describes the frontier of the production possibility set. It follows that \( T(k, y) \) gives the maximal output of the consumption good. It is also easy to show that the first derivatives of the social production function give the rental rate of capital and the price of the investment good:
\[
T_1(k, y) = r(k, y), \quad T_2(k, y) = -p(k, y), \tag{2}
\]
From constant returns to scale we also get \( w(k, y) = T(k, y) - r(k, y)k + p(k, y)y \).

2.2 Consumers
There are \( n \) agents. In each period consumers provide inelastically a constant amount of labor \( \omega_i, i = 1, ..., n \), with \( \sum_{i=1}^n \omega_i = 1 \).

A model in which the amount of labor provided is endogenously determined could be analyzed but at a much higher cost.
\[ U^i(x^i) = \sum_{t=0}^{\infty} \delta^t u_i(x_{it}) \]

where \( x_{it} \) is the consumption of agent \( i \) at time \( t \) and \( x^i \) is its intertemporal consumption stream. Each instantaneous utility function satisfies the following basic restrictions:

**Assumption 2**. \( u_i(x_i) \) is \( C^2 \), such that \( u'_i(x_i) > 0 \), \( u''_i(x_i) < 0 \) for any \( x_i > 0 \), and satisfies the Inada condition \( \lim_{x_i \to 0} u'_i(x_i) = +\infty \).

In a decentralized economy, an agent \( i \) maximizes his intertemporal utility function subject to a single budget constraint

\[ \sum_{t=0}^{\infty} R_t x_{it} = \sum_{t=0}^{\infty} R_t w_i \omega_i + \theta_i p_0 k_0 \quad \text{with} \quad i = 1, \ldots, n. \]

where the price of the consumption good has been normalized to one at each period \( t \geq 0 \), and the discount factors \( R_t \) are defined as:

\[ R_t = \prod_{r=0}^{t} \frac{1}{1 + d_r} \]

with \( d_t = (r_t + p_t - p_{t-1})/p_{t-1} \) the common interest rate.

### 3 Competitive equilibria

From the first welfare theorem, we know that every competitive equilibrium obtained in the decentralized economy is a Pareto optimum. In the current section we first define competitive equilibria and then characterize the set of Pareto optima.

**Definition 1**. A competitive equilibrium is a sequence of prices \( \{w_t, r_t, p_t\}_{t=0}^{\infty} \) such that markets clear for every \( t \geq 0 \)

\[ \bullet \quad l_t^0 + l_t^1 = \sum_{i=1}^{n} \omega_i = 1 \]

\[ \bullet \quad \sum_{i=1}^{n} x_{it} = f^0(k_t^0, l_t) \]

\[ \bullet \quad k_{t+1} = f^1(k_t^1, l_t^1) + (1 - \mu)k_t, \]

\[ \bullet \quad k_0^0 + k_0^1 = k_0 \quad \text{with} \quad k_0 \quad \text{given} \]

where
• \( (x_{it}) \) is a solution to the individual maximization program of agent \( i \), \( i = 1, \ldots, n \), for \( \{w_t, r_t, p_t\}_{t=0}^{\infty} \).

• \( (k_j, l_j^t) \) is a solution to profit maximization for firm \( j \), \( j = 0, 1 \), for \( \{w_t, r_t, p_t\}_{t=0}^{\infty} \).

Every competitive equilibrium is a Pareto optimal allocation. A Pareto optimal allocation is a solution to the planner’s problem for a given vector of welfare weights \( \eta \in [0, 1]^{n-1} \):

\[
\max_{\{x_t, y_t\}_{t \geq 0}} \sum_{i=1}^{n-1} \eta_i \delta^i u_i(x_{it}) + (1 - \sum_{i=1}^{n-1} \eta_i) \delta^i u_n(x_{nt})
\]

s.t. \( \sum_{i=1}^{n} x_{it} = T(k_t, y_t) \)

\( k_{t+1} = y_t + (1 - \mu)k_t \)

\( k_0 \) given,

The solution to the above program depends on the vector \( \eta \) and on \( k_0 \). The set of Pareto optima is obtained when \( \eta \) spans \([0, 1]^{n-1}\) with \( \sum_{i=1}^{n-1} \eta_i \leq 1 \).

A given competitive equilibrium is obtained for a \( \eta \) such that the associated allocations saturate the budget constraint of all the consumers. Note also that the solutions are interior as soon as \( \omega_i \neq 0 \) or \( \theta_i \neq 0 \) for \( i = 1, \ldots, n \).

Let \( u \) be a social utility function such that

\[
u(x) = \max_{\{x_t\}_{t \geq 0}} \sum_{i=1}^{n-1} \eta_i u_i(x_{it}) + (1 - \sum_{i=1}^{n-1} \eta_i) u_n(x_{nt})
\]

s.t. \( \sum_{i=1}^{n} x_{it} = x \) \hspace{1cm} (3)

We may then define the indirect utility function as follows

\[ V(k_t, k_{t+1}) = u(T(k_t, k_{t+1} - (1 - \mu)k_t)) \]

Consider now the social production function \( T(k_t, y_t) \) with \( y_t = k_{t+1} - (1 - \mu)k_t \). We easily derive that \( T(k_t, y_t) = 0 \) if and only if \( k_{t+1} = f^1(k_t, 1) + (1 - \mu)k_t \equiv g(k_t) \). Moreover from Assumption 1 we get \( g'(0) = +\infty \) and \( g'(+\infty) = 1 - \mu \) so that there exists \( \bar{k} > 0 \) such that \( y_t + (1 - \mu)k_t > k_t \) when \( k_t < \bar{k} \) while \( y_t + (1 - \mu)k_t < k_t \) when \( k_t > \bar{k} \). It follows that it is not possible to maintain stocks over \( \bar{k} \). We may therefore define the set of admissible paths as

\[ D = \{ (k_t, k_{t+1}) \in \mathbb{R}_+^2 | 0 \leq k_t \leq \bar{k}, (1 - \mu)k_t \leq k_{t+1} \leq f^1(k_t, 1) + (1 - \mu)k_t \} \]

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It is easy to show that $D$ is a compact, convex set. It follows that the planner’s problem is equivalent to

$$\max_{\{k_t\}_{t \geq 0}} \sum_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1})$$

s.t. $(k_t, k_{t+1}) \in D$

$$k_0 \text{ given}$$

Note that the solution depends on $k_0$. In the present framework it is a standard result that the set of interior Pareto optima is the set of $\{k_t\}_{t \geq 0}$ that are solutions to the following system of Euler equations

$$V_2(k_t, k_{t+1}) + \delta V_1(k_{t+1}, k_{t+2}) = 0$$

and that satisfy the transversality condition

$$\lim_{t \to +\infty} \delta^t k_t V_1(k_t, k_{t+1}) = 0$$

Notice that using equations (2), the Euler equation becomes:

$$u'(x_t) T_2(k_t, y_t) + \delta u'(x_{t+1}) [T_1(k_{t+1}, y_{t+1}) - (1 - \mu) T_2(k_{t+1}, y_{t+1})] = 0$$

An interior aggregate steady state is a sequence $k_t = k^*$, $\forall t \geq 0$, that solves the Euler equation.

**Lemma 1.** Under Assumption 1, for any $\delta \in (0, 1]$, there exists a unique steady state $k^*$ solution of the Euler equation (5).

**Proof:** See Becker and Tsyganov (2002).

The aggregate consumption $x^*$ can also be obtained as

$$x^* = T(k^*, \mu k^*) \equiv T^*$$

Note that at the steady state, aggregate capital and consumption depend only on the characteristics of technologies through the social production function $T(k, y)$.

Near the steady state the behavior of the dynamic system is equivalent to the behavior of the linearized system. The dynamic properties of the steady state are then related to the eigenvalues of the matrix associated to the linearized system. Denote $T_{ij}(k^*, \mu k^*)$, $V_{ij}^* = V_{ij}(k^*, k^*)$, $i, j = 1, 2$, the second derivatives of the social production function and the indirect utility function evaluated at the steady state. We easily derive the following characteristic polynomial

$$\chi(z) = z^2 + \beta_1 z + \beta_2$$
\[ P(\lambda) = \lambda^2 \delta V_{12}^* + \lambda (V_{11}^* + \delta V_{22}^*) + V_{12}^* = 0 \] (7)

As shown in Benhabib and Nishimura (1985), and denoting \( a_{00} = l_0 / y_0, \ a_{10} = k_0 / y_0, \ a_{01} = l_1 / y, \ a_{11} = k_1 / y \) the capital and labor coefficients in each sector, it is easy to get

\[ T_{12}(k, y) = -T_{11}(k, y)b(k, y) \] (8)

where

\[ b(k, y) = a_{01} / a_{00} - a_{11} / a_{10} \] (9)

is the relative capital intensity difference across sectors. We get \( b(k, y) > 0 \) if and only if the investment (consumption) good is capital intensive. Similarly, we have

\[ T_{22}(k, y) = T_{11}(k, y)b(k, y)^2 \]

It follows therefore that the stability property of the steady state will depend on the capital intensity difference across sectors \( b = b(k^*, \mu k^*) \), the second derivative with respect to \( k \) of the social production function \( T_{11}^* \), and the first and second order derivatives of the instantaneous utility function, all evaluated at the steady state.

**Definition 2.** Let \( u \) be the social utility function, \( u : R_+ \to R \). Let \( \rho(x) = -u'(x)/u''(x) \) be the inverse of the social absolute risk aversion, also called social absolute risk tolerance.

Based on all these expressions, and denoting \( \vartheta = [1 - \delta(1 - \mu)]^{-1} \), we easily compute from the definition of the indirect utility function\(^7\)

\[ V_{11}^* = u''(x^*)\vartheta T_{11}^* + u'(x^*)\vartheta T_{11}^* [1 + (1 - \mu)b]^2 \]
\[ V_{12}^* = -u''(x^*)\vartheta T_{11}^* - u'(x^*)\vartheta T_{11}^* b [1 + (1 - \mu)b] \]
\[ V_{22}^* = u''(x^*)\vartheta^2 (\vartheta T_{11}^*)^2 + u'(x^*)\vartheta T_{11}^* b^2 \]

The characteristic polynomial then becomes:

\[ P(\lambda) = \lambda^2 \delta \left\{ \vartheta (\vartheta T_{11}^*)^2 - \rho(x^*) T_{11}^* [1 + (1 - \mu)b] \right\} \]
\[ -\lambda \left\{ \delta(1 + \vartheta)(\vartheta T_{11}^*)^2 - \rho(x^*) T_{11}^* \left[ b^2 + \delta [1 + (1 - \mu)b]^2 \right] \right\} \]
\[ + \delta (\vartheta T_{11}^*)^2 - \rho(x^*) T_{11}^* b [1 + (1 - \mu)b] = 0 \]

For given discount factor and technology parameters, the eigenvalues depend on \( \rho \). Note that \( \rho \) is positive and that \( \rho \) close to zero indicates a high degree of curvature of the utility function.

\(^7\)At the steady state, the Euler equation (6) becomes \( T_{12}^* + \delta [T_{11}^* - (1 - \mu)T_{22}^*] = 0 \) and may be equivalently rewritten as \( -T_{12}^* = \delta \vartheta T_{11}^* \). It follows that \( T_{11}^* - (1 - \mu)T_{22}^* = \vartheta T_{11}^* \).
4 Endogenous fluctuations and volatility

The steady state value of individual consumption depends on the individual characteristics because the return function depends on the welfare weights. The exact relationship is provided by the following Lemma.

**Lemma 2.** Under Assumptions 1-2, the individual allocations evaluated at the steady state $k^*$ are

$$x_i^*(\theta_i, \omega_i) = \omega_i \left[ x^* - (T_1^* + \mu T_2^*) k^* \right] - (1 - \delta) \theta_i T_2^* k^*$$

where $x^* = T(k^*, \mu k^*) \equiv T^*$.

**Proof:** The first order conditions associated to the individual program are

$$\delta^t u_i'(x_{it}) = \pi_t R_t \quad \forall t \geq 0 \text{ and } i = 1, \ldots, n$$

$$\sum_{t=0}^{\infty} R_t x_{it} = \sum_{t=0}^{\infty} R_t w_t \omega_i + \theta_i p_0 k_0$$

where $\pi_t$ is the Lagrange multiplier associated to the constrained maximization problem. From equations (2) and the Euler equation (6) we conclude that the common interest rate satisfies

$$1 + d_t = \frac{r_t + (1 - \mu) p_t}{p_{t-1}} = -\frac{T_1(k_t, y_t) - (1 - \mu) T_2(k_t, y_t)}{T_2(k_{t-1}, y_{t-1})}$$

The Euler equation (6) evaluated at a steady state $x_{it} = x_i^*$ gives

$$1 + d^* = \frac{T_1^*}{T_2^*} + 1 - \mu = \delta$$

and thus $R_t = \delta^t$. From the budget constraint we get

$$x_i^* = (1 - \delta) \left[ \frac{w^* \omega_i}{1 - \delta} + \theta_i p_0 k_0 \right]$$

Recall now that from equations (2) we get $T_1(k, y) = r(k, y)$, $T_2(k, y) = -p(k, y)$ and $w(k, y) = T(k, y) - r(k, y) k + p(k, y) y$. Substituting into the previous equation these expressions evaluated at the steady state with $y^* = \mu k^*$, $k_0 = k^*$ and $p_0 = p^*$ gives the result.

The curvature of the social utility function can now be expressed as a function of the individual consumptions and therefore of the individual shares of capital and labor endowments.

**Lemma 3.** Under Assumption 2, the absolute risk tolerance of the social utility function computed at the steady state is given by

$$\rho(x^*) = -\sum_{i=1}^{n} \frac{u_i'(x_i^*(\theta_i, \omega_i))}{y_i'}$$
Proof: See appendix.

In the present general equilibrium model, the social utility function depends on the welfare weights. Furthermore, these depend on the equilibrium allocations which in turn depend on the initial conditions and on the distribution of individual endowments. When the welfare weights are continuous functions of the initial capital stock and labor endowments the dynamic properties of the competitive equilibrium with endogenous welfare weights can be analyzed from the planner’s problem defined in terms of the social utility function with fixed welfare weights. The following Proposition gives sufficient conditions.

**Lemma 4.** Under Assumptions 1-2, the stability properties of the steady state of the general equilibrium model with endogenous weights and of the growth model with fixed welfare weights are equivalent in the following cases:

i) If for any \((k_t, k_{t+1}) \in \text{int}D\), \(b(k_t, k_{t+1} - (1 - \mu)k_t) \leq 0\);

ii) If for any \((k_t, k_{t+1}) \in \text{int}D\), \(T_2(k_t, k_{t+1} - (1 - \mu)k_t) + b(k_t, k_{t+1} - (1 - \mu)k_t)T_1(k_t, k_{t+1} - (1 - \mu)k_t) \neq 0\).

**Proof:** In a one-sector economy with heterogeneous agents, Kehoe, Levine and Romer (1990) show that the welfare weights are continuous functions of the initial capital stock. This continuity property happens to be satisfied because the value function of the planner’s problem (4) is \(C^2\). However, in a multisector economy such a property is much more difficult to obtain. Santos (1992) shows that one of the sufficient conditions is to assume strong concavity for the indirect utility function \(V(k_t, k_{t+1})\) (see Assumption B and Theorem 2.2 in Santos (1992)). Strong concavity implies that the hessian matrix of \(V(k_t, k_{t+1})\) is always non-singular and negative-definite. In other words, the smallest eigenvalues in absolute value of the Hessian matrix needs to be strictly positive over the domain of definition of \(V(k_t, k_{t+1})\). In our two-sector model, recall that the indirect utility function 

\[V(k_t, k_{t+1}) = u(T(k_t, k_{t+1} - (1 - \mu)k_t))\]

is defined over the compact, convex set \(D\). We know that \(T\) is concave non-strictly so that its Hessian matrix 

\[H_T = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix}\]

is singular, which means \(|H_T| = T_{11}T_{22} - T_{12}^2 = 0\) for any admissible \((k_t, k_{t+1})\). The partial derivatives of \(V\) are:

\[V_1(k_t, k_{t+1}) = u'(x_t) [T_1(k_t, y_t) - (1 - \mu)T_2(k_t, y_t)]\]

\[V_2(k_t, k_{t+1}) = u'(x_t)T_2(k_t, y_t)\]
Now consider the Hessian of $V$. We easily get

$$H_V(k_t, k_{t+1}) = u'(1 - (1 - \mu)) \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + u'' \begin{pmatrix} T_1 - (1 - \mu)T_2 \\ T_2 \end{pmatrix} \begin{pmatrix} T_1 - (1 - \mu)T_2 & T_2 \end{pmatrix}$$

The determinant of $H_V(k_t, k_{t+1})$ is

$$|H_V| = (u')^2 |H_T| + (u'')^2 \begin{vmatrix} T_1 - (1 - \mu)T_2 & 0 \\ 0 & T_1 - (1 - \mu)T_2 \end{vmatrix}$$

It clearly appears that strict concavity of the social utility function $u(x)$ is a necessary condition for the Hessian matrix of $V$ to be non singular. Such a property easily follows from Assumption 2. Moreover, if over the interior of the set $D$ we have $b(k_t, k_{t+1} - (1 - \mu)k_t) \leq 0$ or $T_2(k_t, k_{t+1} - (1 - \mu)k_t) + b(k_t, k_{t+1} - (1 - \mu)k_t)T_1(k_t, k_{t+1} - (1 - \mu)k_t) \neq 0$ then $|H_V| > 0$ and the value function of the planner’s problem (4) is $C^2$.

Lemma 4 shows that some simple conditions on the capital intensity difference ensure the continuity property of the welfare weights and therefore the equivalence between the decentralized and centralized formulations. In the sequel we assume that these conditions are satisfied and pursue our analysis of the equilibrium path when the welfare weights are fixed at their steady state values. We may first give conditions on the capital intensity difference $b$ for which stability does not depend on heterogeneity.

**Proposition 1.** Under Assumptions 1-2, for any $\rho(x^*) > 0$, the steady state is saddle-point stable with monotone convergence in the following cases:

1. when the investment good is capital intensive ($b > 0$);
2. when the consumption good is capital intensive with $b \leq -1/(1 - \mu)$.

Therefore, the distribution of shares and/or labor endowments do not affect stability.

**Proof:** It is easy to derive from the characteristic polynomial that the discriminant is equal to:

$$12$$
\[ \Delta = \left\{ \delta (\vartheta T^*_1)^2 (1 + \sqrt{\delta})^2 - \rho (x^*) T^*_1 \left[ b + \sqrt{\delta} [1 + (1 - \mu) b] \right]^2 \right\} \times \left\{ \delta (\vartheta T^*_1)^2 (1 - \sqrt{\delta})^2 - \rho (x^*) T^*_1 \left[ b - \sqrt{\delta} [1 + (1 - \mu) b] \right]^2 \right\} \geq 0 \]

Therefore the characteristic roots are real. Moreover we get

\[ P(0) = \delta \vartheta \rho \left[ 1 + (2 - \mu) b \right] T^*_1 > 0 \]

\[ P(1) = \rho (x^*)^2 (1 - \mu b) (\delta - \vartheta - 1 b) T^*_1 > 0 \]

The equality between prices and costs at the steady state gives

\[ (w^r \quad r) \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = (1 \quad p) \]

and thus \( w a_{01} + r a_{11} = p \). When evaluated at the steady state, the Euler equation (6) rewrites as \( p = \delta \vartheta r \). We then obtain after substitution in the previous equation

\[ \delta r (\vartheta - \delta^{-1} a_{11}) = w a_{01} > 0 \]

Prices positivity implies \( \vartheta - \delta^{-1} a_{11} > 0 \). From equation (9), we observe that

\[ \delta - \vartheta^{-1} b = \frac{a_{00}(\delta - \vartheta^{-1} a_{11}) + \vartheta^{-1} a_{10} a_{01}}{a_{00}} > 0 \]

Then we have \( b < \delta \vartheta \) which entails \( b < 1/\mu \) and thus \( P(1) < 0 \) for any \( \rho (x^*) > 0 \). Moreover we get \( P(0) > 0 \) for any \( \rho (x^*) > 0 \) if and only if \( b > 0 \) or \( b \in (-\infty, -1/(1 - \mu)] \). We conclude that when \( b > 0 \) or \( b < -1/(1 - \mu) \), we have \( P(1) P(0) < 0 \) so that there exists one characteristic root into \((0, 1)\) while the other is greater than 1 since the product of characteristic roots is equal to \(1/\delta \). Therefore the optimal path monotonically converges to the steady state.

Remark: Proposition 1 shows that when the investment good is capital intensive, the steady state in the growth model with social utility function is always saddle-point stable. The local equivalence with the general equilibrium with heterogeneous agents is thus obtained if condition \( ii) \) in Proposition 4 is satisfied at the steady state, i.e. \( T_2(k^*, \mu k^*) + b T_1(k^*, \mu k^*) = T_1(k^*, \mu k^*)(b - \delta \vartheta) \neq 0 \). We have shown in the proof of Proposition 1 that \( b - \delta \vartheta > 0 \), so that the equivalence holds.

In order to deal with the case in which heterogeneity matters, we need to define the interval of admissible values for the absolute risk tolerance which depends on the distribution of shares and/or labor endowments in the economy. Let indeed
\[ \rho = \min_{(\theta, \omega)} \rho(x^*), \quad \bar{\rho} = \max_{(\theta, \omega)} \rho(x^*) \]

We also need to introduce two critical values for the absolute risk tolerance of the social utility function:

\[ \rho_c = \frac{\delta^2 T^2}{b(1-(1-\mu)b)T_1^2}, \quad \rho_f = \frac{2\delta(1+\delta)\sigma^2 T^2}{[1+(2-\mu)b][\delta+(1+1-\mu)b]T_1^2} \]

which allows to determine respectively the sign of \( P(0) \) and \( P(-1) \). The following Propositions are the main results of this section. The first one deals with the existence of damped fluctuations. It provides in case 1. a first link between the distribution of shares and/or labor endowments and macroeconomic volatility.

**Proposition 2.** Under Assumptions 1-2, let the consumption good be capital intensive with \( b \in (-1/(1-\mu), -1/(2-\mu)) \cup [-\delta/(1+\delta(1-\mu)), 0) \).

1. If \( \rho_c \in (\rho, \bar{\rho}) \), then for \( \rho(x^*) < \rho_c \) the steady state is saddle-point stable with monotone convergence while it is saddle-point stable with oscillations for \( \rho(x^*) > \rho_c \).

2. If \( \rho_c \notin (\rho, \bar{\rho}) \), then for any \( \rho(x^*) \in [\rho, \bar{\rho}] \), the steady state is either saddle-point stable with monotone convergence if \( \rho_c > \bar{\rho} \) or saddle-point stable with oscillations if \( \rho_c < \rho \).

**Proof:** Recall that \( P(1) < 0 \) for any \( \rho(x^*) > 0 \). It is easy to see that when \( b > -1/(1-\mu) \), \( P(0) < 0 \) if and only if \( \rho(x^*) > \rho_c \). Moreover, \( P(-1) > 0 \) for any \( \rho(x^*) > 0 \) when \( b \in (-\infty, -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0) \). Since \( -1/(1-\mu) < -1/(2-\mu) \), let us assume that \( b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0) \). We conclude that when \( \rho(x^*) < \rho_c \), \( P(0)P(1) < 0 \) so that there exists one characteristic root into \((0, 1)\) while the other is greater than 1. It follows that the optimal path monotonically converges to the steady state. On the contrary, when \( \rho(x^*) > \rho_c \), \( P(0)P(-1) < 0 \) so that there exists one characteristic root into \((-1, 0)\) while the other is less than \(-1\). It follows that the optimal path converges to the steady state with oscillations. The final conclusion depends on whether \( \rho_c \) belongs or not to the interval \((\rho, \bar{\rho})\).

The following Proposition focuses on the existence of persistent oscillations. It provides a link between the distribution of shares and/or labor endowments and macroeconomic volatility based on endogenous fluctuations. In case 1. the distribution of wealth directly affects the stability of the steady state while in case 2. it only affects the dynamic properties of the optimal path.
Proposition 3. Under Assumptions 1-2, let the consumption good be capital intensive with \( b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu))) \).

1. If \( \rho f \in (\rho, \bar{\rho}) \), then for \( \rho(x^*) < \rho_f \) the steady state is saddle-point stable while it is unstable for \( \rho(x^*) > \rho_f \). Moreover, \( \rho_f \) is a flip bifurcation value and there generically exist period-two cycles, in a right (or left) neighborhood of \( \rho_f \), which are saddle-point stable (or unstable).

2. If \( \rho_f > \bar{\rho} \) but \( \rho_c \in (\rho, \bar{\rho}) \), then for \( \rho(x^*) < \rho_c \) the steady state is saddle-point stable with monotone convergence while it is saddle-point stable with oscillations for \( \rho(x^*) > \rho_c \).

3. If \( \rho_c, \rho_f \notin (\rho, \bar{\rho}) \), then for any \( \rho(x^*) \in [\rho, \bar{\rho}] \), the steady state is saddle-point stable with monotone convergence if \( \rho_c > \bar{\rho} \), saddle-point stable with oscillations if \( \rho_c < \bar{\rho} \) but \( \rho_f > \bar{\rho} \), or unstable if \( \rho_f < \bar{\rho} \).

Proof: When \( b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu))) \), we have already proved that \( P(0) < 0 \) if and only if \( \rho(x^*) > \rho_c \). Moreover, \( P(-1) > 0 \) if and only if \( \rho(x^*) < \rho_f \). Straightforward computations clearly show that since \( b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu))) \), we have \( \rho_c < \rho_f \). The final conclusion depends on whether \( \rho_f \) and \( \rho_c \) belong or not to the interval \( (\rho, \bar{\rho}) \).

If \( \rho_f \in (\rho, \bar{\rho}) \), the steady state is saddle-point stable when \( \rho(x^*) < \rho_f \), while it is unstable when \( \rho(x^*) > \rho_f \). It follows that when \( \rho(x^*) \) crosses \( \rho_f \) a flip bifurcation occurs. It is indeed easy to show that one eigenvalue crosses \(-1\) with positive speed when \( \rho(x^*) \) crosses \( \rho_f \) since \( P(-1) \) is linear in \( \rho(x^*) \) and

\[
\frac{\partial P(-1)}{\partial \rho(x^*)} = -[1 + (2-\mu)b][\delta + (1 - \mu)\delta b]T^{*}_{11} < 0
\]

If \( \rho_f > \bar{\rho} \) but \( \rho_c \in (\rho, \bar{\rho}) \), the steady state is saddle-point stable with monotone convergence when \( \rho(x^*) < \rho_c \), while it is saddle-point stable with oscillations when \( \rho(x^*) > \rho_c \).

If \( \rho_c, \rho_f \notin (\rho, \bar{\rho}) \), the steady state is saddle-point stable with monotone convergence if \( \rho_c > \bar{\rho} \), saddle-point stable with oscillations if \( \rho_c < \bar{\rho} \) but \( \rho_f > \bar{\rho} \), or unstable if \( \rho_f < \bar{\rho} \).

Remark: In case 3. of Proposition 2, if \( \rho_f < \rho \) the steady state is locally unstable but period-two cycles may still exist. Indeed it has been shown by Mitra and Nishimura (2001) that in a two-sector optimal growth model with linear preferences, period-two cycles may occur if the discount factor crosses some bifurcation value and may persist for a wide range of values of the parameter “far” from the bifurcation value.\(^8\) The same result may hold

\(^8\)As initially proved in Benhabib and Nishimura (1985), when utility is linear, period-two cycles may occur if the discount factor is less than 1 and crosses some bifurcation value.
in our framework with non-linear preferences and the absolute risk tolerance as the bifurcation parameter.

The previous result gives conditions for which wealth heterogeneity matters for stability and the occurrence of macroeconomic volatility. In these, as we will see, preferences play a crucial role.

5 On the effects of inequality on volatility

In this section we establish the link between macroeconomic volatility and agents heterogeneity. When agents have identical preferences, the spread in individual wealth, i.e. in shares of capital and/or labor endowments, characterizes the level of heterogeneity of the economy. Indeed, in this case the agents can be distributed on the real line according to their wealth.\textsuperscript{9} There are several possible formal definitions. We chose the following.

**Definition 3.** Let the $N$ types of consumers be ordered according to the steady state allocation, i.e. $x_i \leq x_j$ for $i < j$. Let $n_i(J)$ be the number of consumers of type $i$ in economy $J$ and let $n(J)$ be the corresponding distribution. Furthermore, assume that the mean of the distribution $\sum_{i=1}^{N} n_i(J) x_i$ is independent of $J$. Then Economy $A$ is said to be no more heterogenous, or unequal, than Economy $B$ if $\sum_{i=1}^{2n} n_i(A) f(x_i) \geq (\leq) \sum_{i=1}^{2n} n_i(B) f(x_i)$ for all continuous concave (convex) functions $f$. This case is denoted $n(A) \preceq_I n(B)$.

The intuition behind Definition 3 is that a spread in the distribution of consumer’s type decreases or increases the expected value of a function $f(x)$ depending on whether $f$ is concave or convex. Rothschild and Stiglitz (1970) have shown the equivalence of $\preceq_I$ with a class of intuitive notions of spread. In particular, they show that $n(A) \preceq_I n(B)$ implies that $n(B)$ has more weight in the tails than $n(A)$. Note that when considering the effect of a redistribution at most $N = 2n$ types need to be considered as there are $n$ types in the initial configuration and $n$ types in the final configuration. Finally, in equilibrium, $\sum_{i=1}^{N} n_i(J) x_i$ is equal to $x^\star$ regardless of the distribution because of market clearing. Therefore, assuming that $\sum_{i=1}^{N} n_i(J) x_i$ is independent of $J$ is not restrictive.

In the present paper we focus on the comparison of economies with the same total resources, i.e. comparing distributions with the same mean. However, when risk aversion is an increasing or a decreasing function economies

\textsuperscript{9}In a purely homogeneous economy all consumers have the same wealth.
with different total resources can be compared (see Cowell (2000)). Note that ordering distributions according to their variance is problematic because a risk averse consumer may chose the distribution with the largest variance. This is one of the reasons why the variance is not often used although it induces a complete ordering (see e.g. Moyes (1999)). Unless the distributions only differ by a translation and a stretching (or a shrinking), results obtained using Definition 3 do not allow to predict the effect of an increase in the variance of the distribution on stability (except if the Lorenz curves are assumed not to cross).

In order to analyze the effect of heterogeneity on the occurrence of volatility associated with persistent fluctuations, we introduce the following assumption:

**Assumption 3.** In case 1. of Proposition 3, the flip bifurcation generates saddle-point stable period-two cycles in a right neighborhood of $\rho_f$.

This assumption actually concerns technical restrictions on the nonlinear part of the Euler equation. Even if it remains difficult to get simple conditions, a number of robust examples of saddle-point stable period-two cycles have been provided by Boldrin and Deneckere (1990) and Mitra and Nishimura (2001).

The fact that heterogeneity may have an effect on stability and macroeconomic volatility is a consequence of Propositions 2 and 3.

**Proposition 4.** Let Assumptions 1-3 hold.

1. **When equality favors macroeconomic volatility.** Assume that the individual absolute risk tolerance is a strictly concave function. There exists a distribution $n(0)$ such that one of the following cases holds:
   i) If $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0]$ with $\rho_c \in (\rho_c, \bar{\rho})$, or $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$ with $\rho_f > \bar{\rho} > \rho_c$, the steady state is saddle-point stable with monotone convergence for any economy $J$ with $n(0) \preceq_J n(J)$ and is saddle-point stable with oscillations otherwise.
   ii) If $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$ with $\rho_f \in (\rho_c, \bar{\rho})$, the steady state is saddle-point stable for any economy $J$ with $n(0) \preceq_J n(J)$ and is unstable otherwise. Moreover, there generically exist period-two cycles, in a neighborhood of $n(0)$, which are saddle-point stable.

2. **When inequality favors macroeconomic volatility.** Assume that the individual absolute risk tolerance is a strictly convex function. There exists a distribution $n(0)$ such that one of the following cases holds:
   i) If $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0]$ with $\rho_c \in$
or \( b \in (-1/(2-\mu), \delta/(1+\delta(1-\mu))) \) with \( \rho_f > \bar{\rho} > \rho_c > \rho \), the steady state is saddle-point stable with monotone convergence for any economy \( J \) with \( n(J) \preceq n(0) \) and is saddle-point stable with oscillations otherwise.

ii) If \( b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu))) \) with \( \rho_f \in (\rho, \bar{\rho}) \), the steady state is saddle-point stable for any economy \( J \) with \( n(J) \preceq n(0) \) and is unstable otherwise. Moreover, there generically exist period-two cycles, in a neighborhood of \( n(0) \), which are saddle-point stable.

**Proof:** Let \( u_i(x) = v(x) \). Let \( i = 1, ..., N \) be the subscript indicating the type. Lemma 3 gives

\[
\rho(x^*) = -\sum_{i=1}^{N} \frac{n_i v(x_i^*)}{v''(x_i^*)}
\]

Provided \( W(x) = -v'(x)/v''(x) \) is a concave function, Definition 3 implies that B is more heterogeneous than A if and only if \( \sum_{i=1}^{N} n_i(A)W(x_i) \geq \sum_{i=1}^{N} n_i(B)W(x_i) \). If we define \( \rho(J) \) as the value of \( \rho(x^*) \) associated to the distribution \( n_i(J) \), the previous condition becomes \( \rho(A) \geq \rho(B) \). On the other hand, according to Propositions 2 and 3, an increase in \( \rho \) favors macroeconomic volatility. Therefore, when individual absolute risk tolerance \( W(x) \) is a concave function, homogeneity and equality favor volatility. The different subcases are derived from Propositions 2 and 3. The second part of the result is proven similarly.

The usual fundamental axioms on preferences do not limit the sign of the effect of inequality on instability. Indeed, the relevant quantity involves the third and fourth order derivatives of the utility functions which are not limited by the standard assumptions on preferences. Furthermore, direct empirical investigation of the properties of preferences is usually impossible so that only indirect, model dependent, evidence can be collected. Asset theory provides some evidence on the properties of absolute risk aversion. The traditional theory of precautionary saving requires the third derivative to be positive while the fourth derivative is unconstrained. Furthermore, recent research suggests that a positive third order derivative is not sufficient for the expected wealth accumulation to be increasing with the earning risks.\(^{10}\) A sufficient condition is that \( v'(x)v'''(x)/(v''(x))^2 \) is a constant \( k \) with \( k > 0 \), implying that the utility function belongs to a subset of the HARA class which includes the CARA and CRRA specifications (see Caroll and Kimball (1996)). The following result states that in this case inequality is neutral.

\(^{10}\)See Huggett and Vidon (2002).
Corollary 1. **HARA preferences.** Assume that individual preferences can be represented by a utility function of the HARA class, i.e.
\[
v(x) = \frac{1 - \gamma}{\gamma} \left( \frac{ax}{1 - \gamma} + b \right)^\gamma
\]
with \(a, b\) and \(\gamma\) as parameters. Then wealth inequality plays no role on the occurrence of macroeconomic volatility.

There is no evidence that absolute risk aversion and absolute risk tolerance are linear. In fact there are some good reasons to believe that absolute risk aversion is convex (see Gollier (2001)). Notice that the concavity of absolute risk tolerance can be related to the concavity of absolute risk aversion:
\[-v'(x)/v''(x)\] is strictly concave if
\[R''(x) > 2(R'(x)/R(x))^2\]
where \(R(x)\) is the individual absolute risk aversion. This condition may be fulfilled if and only if absolute risk aversion is sufficiently convex. Recently some week indirect evidence in support of the concavity of absolute risk tolerance have been obtained (see Gollier (2001) and Guiso and Paiella (2003)). According to the present paper, this evidence would suggest that agent’s heterogeneity favors stability.

6 A CES economy

Following Nishimura, Takahashi and Venditti (2003), we consider the example of a two-sector economy with CES technologies such that
\[
y_0 = \left( \alpha_0 l_0^{1-1/\gamma} + \alpha_1 k_0^{1-1/\gamma} \right)^{\gamma/(\gamma-1)}
\]
\[
y_1 = \left( \beta_0 l_1^{1-1/\gamma} + \beta_1 k_1^{1-1/\gamma} \right)^{\gamma/(\gamma-1)}
\]
Both sectors are thus characterized by the same elasticity of capital-labor substitution \(\gamma > 0\). Moreover, we assume that the capital stock fully depreciates within one period, i.e. \(\mu = 1\), and that \(\alpha_0 + \alpha_1 = \beta_0 + \beta_1 = 1\) in order for the production functions to collapse to Cobb-Douglas technologies in the particular case \(\gamma = 1\).

We will also assume the following restriction on parameters’ values:

**Assumption 4.** \(\beta_1 < \delta^{(1-\gamma)/\gamma}\)

For some given \(\beta_1\) and \(\delta\), Assumption 4 provides a lower bound \(\hat{\gamma} \in (0, 1)\). We have indeed
\[
\gamma > \left( 1 + \frac{\ln(\beta_1)}{\ln(\delta)} \right)^{-1} \equiv \hat{\gamma}
\]

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Such a restriction is quite standard when CES technologies are considered. It is well-known indeed that when the elasticity of capital/labor substitution is less than 1, the Inada conditions are not satisfied and corner solutions cannot be a priori ruled out. Throughout this section we will therefore assume that \( \gamma > \hat{\gamma} \).

**Remark:** When \( \gamma = 1 \), the technologies are Cobb-Douglas, the Inada conditions are satisfied and Assumption 4 becomes \( \beta_1 < 1 \) which always holds.

Under this restriction, existence and uniqueness of the steady state for capital \( k^* \) and consumption \( x^* \) are obtained:

\[
k^* = \left( \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1} \right) \gamma \left( \frac{\delta \beta_1}{\beta_0} \right)^{1-\gamma} \left( \frac{1}{1-(\delta \beta_1)^\gamma} \right) \left( 1-\frac{\alpha_1 \beta_0}{\alpha_0 \beta_1} \right) \]

\[
x^* = \left( \frac{1}{1-(\delta \beta_1)^\gamma} \right) \left( \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1} \right)^{1-\gamma} \left[ \alpha_0 \left( \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1} \right) 1-\gamma + \alpha_1 \left( \frac{\beta_0}{(\delta \beta_1)^{1-\gamma} \beta_1} \right) \right]^{\gamma-1}
\]

From Lemma 2, we obtain the individual stationary consumption levels as

\[
x_i^*(\theta_i, \omega_i) = \omega_i x^* + (1 - \delta) k^* T_1^* (\delta \theta_i - \omega_i)
\]

with 

\[
T_1^* = \alpha_1 \left[ \alpha_0 \left( \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1} \right) 1-\gamma \left( \delta \beta_1 \right)^{1-\gamma} \beta_1 + \alpha_1 \right]^{\gamma-1}
\]

We also easily derive that at the steady state the capital intensity difference across sectors is

\[
b = \left( \delta \beta_1 \right)^\gamma \left[ 1 - \left( \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1} \right) \right]^{\gamma}
\]

Therefore, the investment (consumption) good sector is capital intensive if and only if \( \beta_1 / \beta_0 > (\spadesuit) \alpha_1 / \alpha_0 \) and thus \( b > (\spadesuit)0 \).

As shown in Nishimura, Takahashi and Venditti (2003), tedious computations give the following characteristic polynomial

\[
P(\lambda) = (\lambda - \frac{1}{\delta}) (\delta \lambda - b) + \frac{\delta}{\rho(x^*)} T_{12}^2 (\delta \lambda - 1) (\lambda - 1) = 0
\]

with

\[
T_{12}^2 = \frac{\alpha_1 \beta_0}{\alpha_0} \left( \delta \beta_1 \right)^{1-\gamma} \left( \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1} \right)^{\gamma-1} \left[ 1 - (\delta \beta_1)^\gamma \right] \left[ (\delta \beta_1)^{1-\gamma} - \beta_1 \right]
\]

We may now introduce a specification of preferences that includes several special cases such that the CRRA formulation. As in Gollier (2001), we

\[11\]
assume that the instantaneous utility function of agent $i$ is such that its first derivative satisfies
\[ u'_i(x_i) = \exp\left(-r \frac{x_i^{1-\sigma}}{1-\sigma}\right) \]
with $r, \sigma > 0$. The individual absolute risk tolerance is then
\[ \rho_i(x_i) = \frac{x_i^\sigma}{r} \]
(12)
It follows that $r$ measures the degree of relative risk aversion evaluated at consumption level $x_i = 1$. The empirical evidence favors a value of risk aversion close to 2, so following Gollier (2001) we let $r = 2$. Moreover, the absolute risk tolerance is strictly concave if $\sigma \in (0, 1)$ and strictly convex if $\sigma > 1$. Notice that the CRRA formulation is obtained when $\sigma = 1$. Considering Lemma 2 and the expression of $x_i$ previously given, the individual and social absolute risk tolerances evaluated at the steady state become
\[ \rho_i(x_i^*) = \frac{[\omega_i x^* + (1 - \delta) k^* T_i^* (\delta \theta_i - \omega_i)]^\sigma}{2} \]
\[ \rho(x^*) = \frac{1}{2} \sum_{i=1}^n [\omega_i x^* + (1 - \delta) k^* T_i^* (\delta \theta_i - \omega_i)]^\sigma \]
In such a framework, we start by the case in which the distribution of shares and/or labor endowments does not affect stability. Proposition 1 becomes indeed:

**Corollary 2.** Under Assumptions 1, 2 and 4, if $\beta_1/\beta_0 > \alpha_1/\alpha_0$, then $k^*$ is saddle-point stable with monotone convergence for any $\rho(x^*) > 0$.

In order to deal with the case in which heterogeneity matters, we first define the interval of admissible values for the absolute risk tolerance. In our framework, the most unequal configuration is obtained if one individual, say $i = 1$, owns the total wealth of the economy while all the others do not own anything. In such a case, $\omega_1 = \theta_1 = 1$, $\omega_i = \theta_i = 0$ for any $i \neq 1$, and agent 1 will consume at the steady state
\[ x_i^* = x^* - (1 - \delta)^2 k^* T_i^* \equiv \bar{x} \]
On the contrary, the most equal configuration is obtained if every individual owns the same proportion of the total wealth, i.e. if $\omega_i = \theta_i = 1/n$. In this case, each agent $i$ consumes at the steady state
\[ x_i^* = \bar{x}/n \]
Since $\rho(x_i)$ is a monotone increasing function, we get
\[ \underline{\rho} = \frac{n}{2} \left(\frac{\bar{x}}{n}\right)^\sigma, \quad \bar{\rho} = \frac{\bar{x}^\sigma}{2} \]
We now introduce the following critical values for the absolute risk tolerance of the social utility function:

\[
\rho_c = -\frac{\delta T^2}{T_{12}^2}, \quad \rho_f = -\frac{b}{(1+b)(\delta+b)} \frac{2\delta(1+\delta)T^2}{T_{12}^2}
\]

It is easy to show that provided the consumption good is capital intensive, i.e. \( \beta_1/\beta_0 < \alpha_1/\alpha_0 \), for any given elasticity of capital-labor substitution \( \gamma > 0 \), we have \( \lim_{\sigma \to 0} \rho(x^*) = 1/2 \) and \( \lim_{\sigma \to +\infty} \rho(x^*) = 0 \). Since \( \rho(x^*) \) is a monotone function of \( \sigma \), we get \( \rho(x^*) \in (0, 1/2) \) for any \( \sigma \geq 0 \). Therefore, heterogeneity will affect the dynamics of the equilibrium path only if the following restriction is introduced:

**Assumption 5.** \( \rho_c, \rho_f \in (0, 1/2) \)

The following Corollary concerns Proposition 2 and deals with volatility associated with damped oscillations. Notice that the corresponding interval of values for the capital intensity difference across sectors is in this case \( b \in (-\infty, -1] \cup [-\delta, 0) \).

**Corollary 3.** Under Assumptions 1, 2, 4 and 5, let \( \beta_1/\beta_0 < \alpha_1/\alpha_0 \) with

\[
\frac{\alpha_1\beta_0}{\alpha_0\beta_1} \in \left(0, \left[\delta + (\delta(\gamma-1)/\gamma\beta_1)^{-\gamma}1/\gamma\right] \cup \left[1 + (\delta\beta_1)^{-1/\gamma}, +\infty\right) \right)
\]

and \( \rho_c \in (\overline{\rho}, \overline{\rho}) \). Then there exists \( \sigma_c \) such that the steady state is saddle-point stable with monotone convergence if \( \sigma > \sigma_c \) while it is saddle-point stable with oscillations if \( \sigma < \sigma_c \).

The next and last Corollary concerns Proposition 3 and deals with persistent volatility associated with endogenous fluctuations. Notice here that the corresponding interval of values for the capital intensity difference across sectors is \( b \in (-1, -\delta) \).

**Corollary 4.** Under Assumptions 1, 2, 4 and 5, let \( \beta_1/\beta_0 < \alpha_1/\alpha_0 \) with

\[
\frac{\alpha_1\beta_0}{\alpha_0\beta_1} \in \left(\left[\delta + (\delta(\gamma-1)/\gamma\beta_1)^{-\gamma}1/\gamma\right], +\infty\right)
\]

Then:

1. If \( \rho_f \in (\overline{\rho}, \overline{\rho}) \), there exists \( \sigma_f \) such that the steady state is saddle-point stable if \( \sigma > \sigma_f \) while it is unstable with oscillations if \( \sigma < \sigma_f \). Moreover, \( \sigma_f \) is a flip bifurcation value and there generically exist period-two cycles, in a left (or right) neighborhood of \( \sigma_f \), which are saddle-point stable (or unstable).

2. If \( \rho_f > \overline{\rho} > \rho_c > \overline{\rho} \), there exists \( \sigma_c \) such that the steady state is saddle-point stable with monotone convergence if \( \sigma > \sigma_c \) while it is saddle-point stable with oscillations if \( \sigma < \sigma_c \).
In Proposition 4, we have proved that the effect of inequality on macroeconomic volatility depends on whether the individual absolute risk tolerance is a concave or convex function. In the current example, since the parameter $r$ has been fixed at 2, equation (12) implies that for given technology parameters and therefore given steady state $x^*$, $\rho(x^*)$ varies with the curvature parameter $\sigma$. The previous Corollaries show that the effect of inequality on macroeconomic volatility depends on whether the critical bounds $\sigma_c$ and $\sigma_f$ are greater or less than 1. If $\sigma_c, \sigma_f < 1$, then the inverse of the absolute risk aversion is concave and equality will favor the occurrence of fluctuations. On the contrary, if $\sigma_c, \sigma_f > 1$, then the inverse of the absolute risk aversion is convex and inequality will favor macroeconomic volatility.

7 Conclusion

The present paper identifies the conditions on consumer’s preferences such that wealth inequality favors macroeconomic volatility and those that favor smoothness in a general two-sector growth model. The paper also shows that economies in which absolute risk aversion is linear are such that heterogeneity is neutral. The impact of heterogeneity depends on the concavity of the inverse of absolute risk aversion. The properties of the absolute risk tolerance are difficult to obtain but they play a crucial role also in asset pricing theory and some effort is being devoted to find direct empirical evidence and indirect, model dependent, evidence. These findings, as well as indirect evidence obtained from the collective household model, seem to suggest that absolute risk tolerance is not linear and is likely to be concave. According to the present paper, this evidence would suggest that agent’s heterogeneity matters in the existence of fluctuations.

It would be interesting to generalize the present paper to include models with several balanced growth paths. In this case it would be possible to relate the differences in the observed growth rate across countries to the level of inequality. Indeed, in case of models with several balanced growth paths characterized by different growth rates, wealth and income inequality may affect the stability of these and therefore affect the effective growth rate. However, endogenous growth models usually lack the required continuity property of the welfare weights, the reason why we failed in our attempts.
8 Appendix: Proof of Lemma 3

Without loss of generality assume there are three types of consumers. Then the social utility function is defined by

\[ u(x) = \max \mu_a n_a u_a(x_a) + \mu_b n_b u_b(x_b) + (1 - \mu_a - \mu_b) n_c u_c((x - n_a x_a + n_b x_b)/n_c) \]

The first and second order derivatives of the social utility function can be related to the derivatives of the individual utility function of the agents. Indeed, the first order conditions associated to the maximization problem that define the social utility function give

\[ \Psi^1(x_a, x_b, x; \mu_a, \mu_b) = \mu_a n_a u'_a(x_a) - (1 - \mu_a - \mu_b) n_a u'_c \left( \frac{x - n_a x_a + n_b x_b}{n_c} \right) = 0 \]
\[ \Psi^2(x_a, x_b, x; \mu_a, \mu_b) = \mu_b n_b u'_b(x_b) - (1 - \mu_a - \mu_b) n_b u'_c \left( \frac{x - n_a x_a + n_b x_b}{n_c} \right) = 0 \]

Then the following expressions are easily obtained

\[ u'(x) = (1 - \mu_a - \mu_b) u'_c \left( \frac{x - n_a x_a + n_b x_b}{n_c} \right) = \mu_a n_a u'_a(x_a) \]
\[ u''(x) = \mu_a n_a u''_a(x_a) \frac{\partial x_a}{\partial x} \]

where \( x \) represents the aggregate consumption. The implicit function theorem applied to \( \Psi \) allows to express \( x_a \) as a function of \( x \) near the steady state \( (x^*_a, x^*_b, x^*_c) \). In matrix form we can write,

\[
\begin{pmatrix}
\frac{\partial x_a}{\partial x} \\
\frac{\partial x_b}{\partial x} \\
\frac{\partial x_c}{\partial x}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial \Psi^1}{\partial x_a} & \frac{\partial \Psi^1}{\partial x_b} \\
\frac{\partial \Psi^2}{\partial x_a} & \frac{\partial \Psi^2}{\partial x_b}
\end{pmatrix}^{-1}
\begin{pmatrix}
\frac{\partial \Psi^1}{\partial x} \\
\frac{\partial \Psi^2}{\partial x}
\end{pmatrix}
\]

Some straightforward computations give

\[ x'_a(x^*) = \frac{\partial x^*_a}{\partial x} = \frac{\mu_c \mu_b n_c u'_a(x^*_a) u'_b(x^*_b)}{\mu_a \mu_b n_a u'_a(x^*_a) u'_b(x^*_b) + \mu_c \mu_a n_a u'_a(x^*_a) u'_c(x^*_c) + \mu_c \mu_b n_b u'_b(x^*_b) u'_c(x^*_c)} \]

where \( \mu_c = 1 - \mu_a - \mu_b \). The result then follows from the definition of \( \rho \).

9 References


