Money Matters
An Axiomatic Exploration of the Endowment Effect and the Preference Reversal Phenomenon

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1 Introduction

1.1 Motivation and Aims of the Paper

1.1.1 Reference-dependent Preferences, the Status Quo Bias and the Endowment Effect

One of the key ideas of Prospect Theory (PT) (Kahneman and Tversky, 1979), and, perhaps, the less controversial, is that, contrary to what is usually assumed in traditional microeconomics, preferences, and not only choice, may vary with the agent’s current endowment. They are reference dependent. This idea, although only relatively recently introduced in mainstream economics, meets the informal psychosociological intuition that the rich do not behave like the poor. To be sure, this intuition can be formally captured to some extent without introducing an explicit dependence of the utility function on the current endowment. For instance, to account for the fact that rich people are less risk averse than poor people, it suffices, in the context of expected utility, to choose a von Neumann-Morgenstern utility function that is less concave for high wealth than for low wealth. However, this does not capture more radical differences in tastes between the rich and the poor, for instance the changes in tastes that are dictated by the snobbish desire to appear to be a legitimate member of the high society, like the taste for abstract modern art as opposed to more classical forms of art\(^1\).

In behavioral economics, this idea is related to three more precise concepts: loss aversion, the status quo bias and the endowment effect.

Loss aversion is a psychological trait that is generally characterized by the oft-quoted phrase of Kahneman and Tversky: “losses loom larger than gains”. In other words, a loss of a certain size is perceived as more painful than a gain of the same size is perceived as enjoyable. Formally, if preferences for monetary gains or losses

\(^1\)This phenomenon has been thoroughly studied in sociology, for instance in Bourdieu (1987).
are represented by a function $v$, with $v(x) < 0$ iff $x < 0$, then loss aversion is usually captured by the equation

$$v(x) < -v(-x), \quad \forall x > 0.$$  

However, this is but a cardinal definition of loss aversion, since this property is not preserved by addition of sufficient large constant $b$ to the function $v$. It is interesting, therefore, to look for the ordinal counterpart of this cardinal concept, just in the same way as convexity of preferences is an ordinal counterpart of decreasing marginal utility that was intended in earlier times as an adequate formalization of another psychological law, the first Gossen Law. Schmidt and Zank (2005) deal with that problem in the context of decision under risk and Tversky and Kahneman (1991) in the context of commodity bundles (with two goods). This, however, is not completely satisfactory in the sense that it is not desirable that the ordinal counterpart of loss aversion depend on the mathematical or decision-theoretic context. One of the aims of the present paper is to deal with this issue in a context that is arguably more general.

The status quo bias is the general tendency to prefer sticking to the current position, only because it is the current position. Samuelson and Zeckhauser (1988) provided convincing evidence for this tendency. The status quo bias is generally associated with the endowment effect (Kahneman, Knetsch, and Thaler, 1991) which is the fact that people value more an object when they own it than when they don’t, all things being equal. The usual consequence of the endowment effect is the willingness to accept (WTA)/willingness to pay (WTP) gap, that is the fact that the WTA is greater than the WTP. After having for long been considered a very robust stylized fact, the reality of the WTA/WTP gap has recently been questioned (Plott and Zeiler, 2005, 2007). Even when it is observed, its interpretation as an endowment effect (i.e. as a consequence of loss aversion) is not warranted, since it could come from classical income or substitution effects rather than from the application of an intrinsic psychological law.

Let us note that the relationship between the three above-mentioned concepts and the idea of reference-dependence, on the one hand, and between them, on the other hand, is not completely clear. One could consider them both as the evidence for, the consequence or the cause of reference-dependence. Similarly, it is not completely clear, for instance, if loss aversion is the cause of the status-quo bias or if it is the other way round, and similarly for the endowment effect. In the context of Prospect Theory, however, it is generally taken for granted that loss aversion is the rationale for the status quo bias and the endowment effect. The first aim of this paper is to address this question axiomatically in a formal framework as general as possible and yet sufficiently structured for these different concepts to be unambiguously defined.

### 1.1.2 Attitudes vs Preferences

When considering reference-dependent preferences, one interesting question one may want to investigate is to what extent they can be explained by reference-independent factors. One idea is that people have a certain number of values or tastes that
maybe are not completely articulated (Fischhoff, 1991) and that, once a status quo is defined, or, more generally, when a decision context is specified, can be used to yield well-defined preferences that maybe reference-dependent. Some psychologists use the notion of attitude rather than the notion of preference (Kahneman, Ritov, and Schkade, 1999; McFadden, 1999). Attitudes are defined as “a psychological tendency that is expressed by evaluating a particular entity with some degree of favor or disfavor.” According to Kahneman et al. (1999), the WTP and WTA valuation reflect more attitudes than preferences.

It is generally held that attitudes do not lend themselves to a formal representation as preferences do (in addition to the very debatable claim that preferences are only about commodity bundles (Kahneman et al., 1999)). The second aim of this paper is to challenge this claim and to address the question of how attitudes and preferences can be formally related to one another in the mentioned above formal framework) (thus going beyond the psychological characterization of their relationship), in order also to make precise the idea that WTA and WTP are about attitudes and not preferences.

1.2 Main contributions

The first contribution of this paper is to propose a formal framework that we claim to be the adequate one to deal with the problems mentioned above. Specifically it consists in endowing an arbitrary set with a monetary structure, that is allowing for combination of objects of choice and money to be unambiguously defined. In other words, we intend to show how money can be introduced into decision theory in a fruitful way.

Using this structure, we are able to define various notions of status quo bias: the usual notion of status quo bias itself and strengthenings of it: the strong monetary status quo bias (SMSQB) and the monetary status quo bias (MSQB), as well as WTP and WTA functionals that can be applied in different contexts. Thanks to these definitions, we prove the following:

- Loss aversion is essentially equivalent to Strong Monetary Status quo Bias.
- The existence of a WTA/WTP gap is essentially equivalent to the Monetary Status quo Bias.

Along the way, and as a significant by-product of our formal framework, we formally define and characterize the notion of preference for liquidity and prove that it can be used to explain the preference reversal phenomenon, in the sense that, for agents that have the same preferences except for their preference for liquidity, a larger preference for liquidity implies more preference reversals. This finding, that is related to the psychological explanation of preference reversals by scale compatibility, constitutes a testable implication of our model.

Finally, we show that the WTP and WTA concepts that we use for these characterizations are all defined with respect to a revealed binary relation, that is less structured than the reference-dependent preferences from which it is derived. This less structured binary relation can be interpreted as a formalization of the basic
values of the decision maker that generate well defined preferences and that can be used also to model attitudes.

1.3 Related literature

1.3.1 Theoretical Literature on Reference-Dependent Preferences

If by theoretical literature on reference dependent preferences, one means a literature that deals with decision situations where the reference point is explicit and variable (as opposed for instance to the standard literature on Prospect theory or sign-dependent models where the reference point is fixed\(^2\)), then this literature is both not very extensive at the moment compared to the considerable empirical literature on the phenomenon of reference-dependence, and rapidly growing. Kahneman and Tversky started it by isolating the loss aversion concept from the context of decision under risk (Tversky and Kahneman, 1991), and proposing an axiomatic characterization of the additive constant loss aversion. Axiomatizing reference-dependent expected utility in a Savage framework, Sugden (2003) can be seen as generalizing this result\(^3\). The extension to uncertainty allows to apply this model to the preference reversal phenomenon (see also Schmidt, Starmer, and Sugden (2005)). Also, precise conditions are given for the WTA/WTP gap to hold in a strict manner. Sagi (2006) and Giraud (2006) study reference-dependence in the mathematical context of mixture spaces or vector spaces, therefore in a context more suitable for decision under risk. These papers present and/or axiomatize\(^4\), related reference-dependent decision making models that involve the use of a set of implicit utility functions or subjective attributes in the following manner: to be chosen over the status quo an alternative must beat it for a subset of these attributes (for all in Sagi’s model). These models are therefore very different from the additive model and its variants.

All the papers mentioned above, start, like Tversky and Kahneman (1991), with a family of preference relations \(\succ_r\) defined over a set \(X\) of alternatives, and derive axiomatically a representation of these preference relations. Masatlioglu and Ok (2005) and Masatlioglu and Ok (2006) take a different route by sticking to the revealed preference paradigm considering choice functions instead of preference relations. They provide axioms characterizing the existence of a status quo bias in this context. They also come up in Masatlioglu and Ok (2005) with a relationship between the status quo bias and a family of implicit attributes, but generalize this concept in Masatlioglu and Ok (2006) introducing a notion of correspondence for assessing alternatives. Finally, Koszegi and Rabin (2007) investigate, in the context of a simple model of reference-dependent preferences based on the separation of reference-dependent utility into a gain-loss utility component and a consumption

\(^2\)A noticeable exception is Schmidt (2003) that studies axiomatically reference-dependence in the context of cumulative prospect theory (CPT), explicitly starting from a family of CPT preferences depending on the endowment and providing nice axiomatic characterizations (in terms of tradeoff consistency) of properties of CPT like loss aversion.


\(^4\)In fact the axiomatization of Sagi (2006)’s model is to be found in the longer working paper version, Sagi (2001).
utility component, the consequences of assuming that the reference point is based on expectations of future consumption and not on the status quo. They do not discuss the axiomatic foundations of their model.

1.3.2 Literature on the Benefit Function

The mathematical concept we use to define the willingness to pay is directly inspired from the notion of benefit function introduced by Luenberger (1992). A number of authors have studied the properties of this function (Courtault, Crettez, and Hayek, 2004a,b), generalized it (Briec and Gardères, 2004) and used it in various contexts: consumer theory (Courtault, Crettez, and Hayek, 2005), as was intended originally by Luenberger, welfare theory (Luenberger, 1992, 1994, 1996; Courtault, Crettez, and Hayek, 2007), production theory (Chambers, Chung, and Färe, 1995, 1998), decision under uncertainty (Quiggin and Chambers, 1998).

1.3.3 Literature Related to the Technical Contribution

From a mathematical point of view, we use in this paper an algebraic structure seldom used in decision theory, namely $G$-sets, i.e pairs $(X,G)$ where $X$ is a set and $G$ is a group acting on $X$. Specifically, in this paper we concentrate on the case where $G$ is the additive group $\mathbb{R}$. As far as we know, the only papers using a similar mathematical structure (specifically the generalization of the concept of $G$-sets for the case where $G$ is a semi-group) are Lemaire and Le Menestrel (2004, 2006a,b). There is also a discussion of the related concept of vertically invariant functionals (which are morphisms for a particular $G$-set) in the working paper version of Maccheroni, Marinacci, and Rustichini (2006).

1.4 Structure of the Paper

In section 2, we describe the formal framework and define the general concept of WTP and WTA functions. In section 3, we introduce and state and prove the main results of the paper. An appendix presents the (rather involved) relationship between our definition of the benefit function and the one of Briec and Gardères (2004).

2 A fairly general framework for the willingness-to-pay and the willingness-to-accept concepts.

2.1 The framework

2.1.1 Intuition

The objective in this section is to precisely define the mathematical framework that is relevant when one wants to deal with the endowment effect, or, put differently, with the discrepancy between the willingness to pay and the willingness to accept. The idea is to define a mathematical structure that allows for the circulation of money
between different endowments of the decision maker. In other words, one wants to be able to differentiate between the object \( x \) and the object \( x + \lambda \), what we will denote \( x \oplus \lambda \), and still to be able to define preferences between these two objects. This will be done by the introduction of what we call **real monetary spaces (r.m.s.)**. The leading intuition that motivates this structure is the idea that the wealth of a person comprises both real assets and monetary assets, so that, if one denotes \( x = (a, w) \) the combination of the real asset \( a \) (say, a house) and the monetary asset \( w \), then one can also talk of the combination \( (a, w + \lambda) \) where \( \lambda \) is a certain amount of money. Then one can denote \( x \oplus \lambda \) this combination. The notion of r.m.s. generalizes this idea.

### 2.1.2 Real monetary spaces

**Definition 1** (Real monetary space). A pair \( (X, \oplus) \) is a **real monetary space** \(^5\) if \( X \) is a set and \( \oplus : X \times \mathbb{R} \to X \) is a mapping such that (denoting \( \oplus(x, \lambda) \) by \( x \oplus \lambda \)):

(i) For all \( x \in X \), \( x \oplus 0 = x \);

(ii) For all \( \lambda, \mu \in \mathbb{R} \), for all \( x \in X \),

\[
(x \oplus \lambda) \oplus \mu = x \oplus (\lambda + \mu).
\]

In mathematics a mapping \( \oplus \) satisfying (i) and (ii) is called an **action** of the group \((\mathbb{R}, +)\) on \( X \).

We shall denote \( x \oplus (-\lambda) \) by \( x \ominus \lambda \).

### 2.1.3 Examples

Let us give some examples of such a structure.

**Example 1.** \( \mathbb{R} \) is obviously a r.m.s. with its natural additive operation.

**Example 2.** Take \( V \) a vector space and \( g \in V \), \( g \neq 0 \). Assume \( X \) is a subset of \( V \) such that, for all \( x \in X \), for all \( \lambda \in \mathbb{R} \), \( x + \lambda g \in X \). Then one can define a r.m.s. by letting \( x \oplus \lambda = x + \lambda g \).

**Example 3.** Let \( X \) be a r.m.s. and \( S \) be a non-empty set. Then the set \( X^S \) of all mappings from \( S \) to \( X \) is a r.m.s. defining \( f \oplus \lambda \) by \( (f \oplus \lambda)(s) = f(s) \oplus \lambda \).

Similarly, the set \( \Delta_0(X) \) of all simple probability measures on \( X \) is a r.m.s. letting \( (p \oplus \lambda)(x) = p(x \ominus \lambda) \). If \( \{x_1, \ldots, x_n\} \) is the support of \( p \), then \( \{x_1 \ominus \lambda, \ldots, x_n \ominus \lambda\} \) is the support of \( p \oplus \lambda \), as \((p \oplus \lambda)(x_i \ominus \lambda) = p((x_i \ominus \lambda) \ominus \lambda) = p(x_i \ominus (\lambda - \lambda)) = p(x_i) > 0 \).

**Example 4** (Canonical r.m.s.). Let \( A \) be a set. Then \( X = A \times \mathbb{R} \) is a r.m.s. letting: \((a, w) \oplus \lambda := (a, w + \lambda)\). We shall call this r.m.s. the **canonical r.m.s. on** \( A \).

\(^5\)The proper mathematical name for this structure is \( \mathbb{R} \)-set. However we use another name to emphasize the intended interpretation and for the sake of possible generalizations.
This last example is interesting as it justifies the term “monetary” in this structure. Indeed, if we interpret \(a\) as the non-monetary assets of an agent and \(w\) as wealth, then the operation of adding \(\lambda\) amounts to increasing wealth by the same amount, without affecting non-monetary assets.

How general is this example, however? The next proposition will answer this question. Let us first introduce some very natural definitions.

**Definition 2.** Let \((X, \oplus)\) and \((X', \oplus')\) be two r.m.s. A morphism from \(X\) to \(X'\) is a mapping \(\varphi : X \rightarrow X'\) such that for all \(x \in X\) and \(\lambda \in \mathbb{R}\),

\[
\varphi(x \oplus \lambda) = \varphi(x) \oplus' \lambda.
\]

\(X\) and \(X'\) are said to be isomorphic as r.m.s if there exists a bijective morphism from \(X\) to \(X'\).

**Definition 3.** Let \((X, \oplus)\) be a r.m.s. Then \(\oplus\) is free if for all \(x \in X\), \(\lambda \in \mathbb{R}\),

\[
x \oplus \lambda = x \implies \lambda = 0.
\]

An r.m.s. with a free action will also be said to be free.

The following result is classical; we provide the proof for completeness.

**Proposition 1.** Let \((X, \oplus)\) be a r.m.s.. The following statements are equivalent:

(i) \(\oplus\) is free.

(ii) There exists a set \(A\) such that \(X\) is isomorphic to the canonical r.m.s. on \(A\).

**Proof.** All but very short proofs are gathered in appendix A.

It is easy to deduce from this proposition that the two first examples of r.m.s. we gave are isomorphic to a canonical r.m.s. However, this is not necessarily true for the r.m.s. discussed in example 3; this depends on the underlying r.m.s. The following example shows that not all r.m.s. are free, and therefore not all r.m.s. are representable by a canonical r.m.s.

**Example 5.** Let \(X = \mathbb{R}_+^*,\) and define \(x \oplus \lambda = x^{2^\lambda}\). Then \(x \oplus 0 := x^1 = x\) and \((x \oplus \lambda) \oplus \mu = (x^{2^\lambda})^{2^\mu} = x^{2^{\lambda+\mu}} = x \oplus (\lambda + \mu)\). However \(1 \oplus \lambda = 1\) for all \(\lambda \in \mathbb{R}\).

The next proposition provides a simple test for the fact that an r.m.s. is free. If \(X\) is an r.m.s., say that \(f : X \rightarrow \mathbb{R}\) is \(\oplus\)-homogeneous if it is a morphism from \(X\) to \(\mathbb{R}\) endowed with its natural r.m.s. structure. Call \(X^\oplus\) the set of \(\oplus\)-homogeneous functions. Then:

**Proposition 2.** The r.m.s. \(X\) is free if and only if \(X^\oplus \neq \emptyset\).
2.2 The Benefit Function and the Loss Function

2.2.1 The Benefit Function

Definition and examples  Let $X$ be an r.m.s. and let $\succ$ be any binary relation defined on $X$. Then, the benefit function (Luenberger, 1992) or buying price functional associated to $\succ$ is the function:

$$b_{\succ} : X^2 \rightarrow \mathbb{R},$$

$$\quad (x, r) \mapsto \sup B_{\succ}(x, r),$$

where

$$B_{\succ}(x, r) := \{ \lambda \in \mathbb{R} \mid x \ominus \lambda \succ r \}.$$

This function can be interpreted as measuring the willingness-to-pay (WTP) for $x$ of a decision maker whose preferences on $X$ are represented by $\succ$, when what he or she ends up with when refusing to pay is the object $r$ and when he or she accepts to pay $\lambda$ for $x$, he or she ends up with $x \ominus \lambda$. $r$ may therefore be interpreted as the default state of the agent, for instance his or her initial endowment.

Example 6 (Classical willingness to pay). In classical consumer theory, we consider for $X$ a set of bundles of $L$ goods, in non-necessarily nonnegative quantities. Often, one of these goods, say good 1, is considered to be the numeraire. Then it is easy to define an r.m.s. by setting

$$x \oplus \lambda = (x_1 + \lambda, x_2, \ldots, x_n).$$

Then, for a preference relation $\succ X$, the benefit function $b_{\succ}(x, r)$ is the willingness to pay for bundle $x$ when the endowment is bundle $r$.

Example 7 (Compensating variation). If there are $L$ goods, we can consider $A = \mathbb{R}_+^L$, the set of price vectors, and for $X$ the canonical r.m.s. on $A$. Assume the consumer has utility function $u$ and indirect utility function $v$. Consider $x = (p^1, w)$, $r = (p^0, w)$ and $\succ$ defined by

$$(p, w) \succ (p', w') \iff v(p, w) \geq v(p', w').$$

Then $b_{\succ}(x, r)$ is the Compensating Variation for moving from $p^0$ to $p^1$ while keeping the wealth unchanged.

Example 8 (Luenberger’s benefit function). Let $X$ be a subset of $\mathbb{R}_+^L$ bounded from below and $g \in X$. Let $u : X \rightarrow \mathbb{R}$ be a utility function for preference ordering $\succ$ and $\alpha \in \mathbb{R}$. Luenberger’s original definition of the benefit function (Luenberger, 1992) is:

$$b(x, \alpha; g) := \sup \{ \lambda \in \mathbb{R} \mid u(x - \lambda g) \geq \alpha \}.$$ 

Given $r \in X$, it is possible to define what Luenberger calls the compensating benefit function, namely $cb(x, r; g) = b(x, u(r); g)$. If $\succ$ is the preference ordering generated by $u$, Luenberger’s compensating benefit function is none other than $b_{\succ}(x, r)$ when the r.m.s. is defined by $x \oplus \lambda := x + \lambda g$. 

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Example 9 (Deaton’s distance function). Let $X$ be some subset of $\mathbb{R}_+^L$. Deaton’s distance function $D$ is defined, for some binary relation $\succ$ on $X$, by:

$$D(x, r) = \sup \{ \mu > 0 \mid \frac{1}{\mu} x \succ r \}$$

It can be associated with a benefit function. For $\lambda > 0$, define

$$x \oplus \lambda := \exp(\lambda)x.$$  

Then, we have

$$D(x, r) = \exp(b(x, r)).$$

Example 10 (Value at Risk, capital requirements and risk measures). Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X$ be the set $L^\infty(\Omega, \mathcal{F}, P)$ of bounded real random variables modeling financial assets. $X$ is a r.m.s. defining, for $Z \in X$ and $\lambda \in \mathbb{R}$, $Z \oplus \lambda = Z + \lambda$.

A standard tool for risk measurement and financial decision-making is Value-at-Risk. We will show that it can be related to the benefit function as we have defined it. Let $\alpha \in [0, 1]$ and $Z \in X$. Then, the $\alpha$-quantile of $Z$, $q_\alpha(Z)$ is defined by

$$q_\alpha(Z) = \inf \{ \lambda \mid P(Z \leq \lambda) > \alpha \} = \sup \{ \lambda \mid P(Z \leq \lambda) \leq \alpha \}.$$  

Then Value at Risk at level $\alpha$, $\text{VaR}_\alpha$ is defined by

$$\text{VaR}_\alpha(Z) = -q_\alpha(Z).$$

Now consider $R \in X$ and define the $R$-quantile of $Z$, $q_R(Z)$, to be

$$q_R(Z) = \inf \{ \lambda \mid P(Z \leq \lambda) > P(R \leq 0) \} = \sup \{ \lambda \mid P(Z \leq \lambda) \leq P(R \leq 0) \}.$$  

The idea of this notion is that it measures the maximal amount of money one is willing to pay to hedge the risk involved in $R$ by buying $Z$ instead. It is the buying price for a swap contract between $Z$ and $R$. In other words, defining the binary relation $\succ_P$ on $X$ by

$$Z \succ_P Z' \iff P(Z \leq 0) \leq P(Z' \leq 0),$$

we have that

$$q_R(Z) = b_{\succ_P}(Z, R).$$

$Z \succ_P Z'$ can be interpreted as $Z$ being less risky than $Z'$.

Now, we can analogously define Value at Risk relative to the benchmark asset $R$ as

$$\text{VaR}(Z, R) = -q_R(Z) = -b_{\succ_P}(Z, R).$$

It is easy to see that, if $P$ is non-atomic, for all $\alpha$ there exists $R^\alpha \in X$ such that $P(R^\alpha \leq 0) = \alpha$, so that the standard Value at Risk is a special case of the notion we defined here.

The notion of Value at Risk is a special case of the notion of capital requirement defined as follows. Let $\mathcal{A}$ be proper subset of $X$ such that $Z \geq Z' \in \mathcal{A}$ implies
$Z' \in \mathcal{A}$. $\mathcal{A}$ is called an acceptability set. Then the capital requirement associated to $\mathcal{A}$, $\rho_{\mathcal{A}}$, is defined as

$$\rho_{\mathcal{A}}(Z) = \inf\{\alpha \in \mathbb{R} \mid Z + \alpha \in \mathcal{A}\}$$

Now assume that each benchmark asset $R$ defines an acceptability set $\mathcal{A}_R$ and define the capital requirement associated to $R$ as

$$\rho(Z, R) := \rho_{\mathcal{A}_R}(Z).$$

Then, defining a relation $\succeq$ on $X$ by

$$Z \succeq Z' \iff Z \in \mathcal{A}_{Z'},$$

it is easy to see that $\rho(Z, R) = -b_{\succeq}(Z, R)$. Coherent risk measures introduced by Artzner, Delbaen, Eber, and Heath (1999) and convex risk measures introduced by Föllmer and Schied (2002) are special cases with respectively convex cones and convex sets as acceptability sets. They can be (and they indeed are) defined axiomatically, and one of their defining axioms relates them intimately with benefit functions. This will be shown in the next section. Indeed, the notion of capital requirement can be defined in just the same way in a general r.m.s. provided we redefine the notion of acceptability accordingly: a proper subset $\mathcal{A}$ of some r.m.s. $X$ is an acceptability set if for all $\lambda \geq 0$ and $x \in \mathcal{A}$, $x \oplus \lambda \in \mathcal{A}$.

Properties Benefit functions have the following fundamental property:

Property 1 (Translation Property or Vertical Equivariance). For all $\lambda \in \mathbb{R}$,

$$b_{\succeq}(x \oplus \lambda, r) = b_{\succeq}(x, r) + \lambda.$$

They can indeed be characterized by it, as shown by the following proposition:

Proposition 3. Let $b : X^2 \to \mathbb{R}$. The following are equivalent:

(i) $b$ satisfies the Translation property.

(ii) there exists a binary relation $\succeq$ on $X$ such that $b(x, r) = b_{\succeq}(x, r)$

(iii) there exists an acceptability set $\mathcal{A}_r$ for all $r \in X$ such that $b(x, r) = -\rho(x, r)$.

The interpretation is that the willingness to pay for an object is only linearly affected by the amount money this object “contains”.

An immediate consequence of this property is the following monotonicity property:

Property 2. For all $(x, r) \in X^2$, for all $\lambda, \mu \in \mathbb{R}$,

$$\lambda > \mu \implies b_{\succeq}(x \oplus \lambda, r) > b_{\succeq}(x \oplus \mu, r).$$
The benefit function also enjoys another (anti-)monotonicity property: if $R$ is a binary relation, let $R^-$ be the right trace of $R$ (Bouyssou and Pirlot, 2005), defined by:

$$x R^- y \iff \forall z \in X, z R x \implies z R y.$$  

Then:

**Property 3.** For all $r, r' \in X$, for all $x \in X$,

$$r \succ r' \implies b_{r'}(x, r') \geq b_{r}(x, r).$$

**Proof.** If $r \succ r'$, then, for all $\lambda \in \mathbb{R}$, $x \oplus \lambda \succ r$ $\implies$ $x \oplus \lambda \succ r'$. Therefore, $B(x, r) \subseteq B(x, r')$, so that $b_{r}(x, r) \leq b_{r'}(x, r')$. □

The interpretation is that, whenever $r \succ r'$, $r$ is somehow better than $r'$. Now, the willingness-to-pay for $x$ when endowed with $r$ may be interpreted as a monetary evaluation of the utility spread between $x$ and $r$. The better $r$, therefore, the smaller this spread.

Finally we address the natural question of the relation between a benefit function and the binary relation that generates it. It is straightforward that for all $x, y \in X$,

$$x \succ y \implies b_{\succ}(x, y) \geq 0.$$  

Does the converse hold however? It turns out that this is not always the case, as we will now show. Specifically, call exact a benefit function that has this property; call upper monotonic a binary relation $\succ$ on $X$ such that for all $x, y \in X$, $\lambda, \mu \in \mathbb{R}$,

$$x \oplus \mu \succ y \text{ and } \lambda \geq \mu \implies x \oplus \lambda \succ y$$  

and call it upper semicontinuous if for all $x, y \in X$, the set $\{ \lambda \mid x \oplus \lambda \succ y \}$ is closed. We then have the following characterization, that we give as a lemma since it will be of further use.

**Lemma 1.** A benefit function $b_{\succ}$ is exact if and only if $\succ$ is upper monotonic and upper semicontinuous.

### 2.2.2 The Loss Function

**Definition and examples** In just the same way as the benefit function measures the WTP of the decision maker, it is possible to construct a function that measures its willingness to accept (WTA). We shall call loss function or selling price functional associated to $\succ$ the function $s_{\succ}$ defined by

$$s_{\succ} : X^2 \rightarrow \mathbb{R}$$

$$(x, r) \mapsto \inf \{ \lambda \in \mathbb{R} \mid r \oplus \lambda \succ x \}$$

**Example 11** (Classical willingness-to-accept). In the same context as example 6, it is easy to see than the corresponding expression for willingness to accept is but the loss function.

**Example 12** (Equivalent variation). In the same context as example 7, it is easy to see that $s_{\succ}$ is no other that the equivalent variation for moving from $(p^0, w)$ to $(p^1, w)$.

Other examples can be built from the corresponding example for the benefit function.
Properties

This function enjoys properties fairly similar to those of the benefit function, that are derived mostly from the duality property $s_r(x, r) = -b_r(r, x)$. We omit their simple proof.

**Property 4** (Reverse Vertical Equivariance). For all $(x, r) \in X^2$, for all $\lambda \in \mathbb{R}$,

$s_r(x, r \oplus \lambda) = s_r(x, r) - \lambda$.

**Property 5.** For all $(x, r) \in X^2$, for all $\lambda, \mu \in \mathbb{R}$,

$\lambda > \mu \implies s_r(x, r \oplus \lambda) < s_r(x, r \oplus \mu)$.

**Property 6.** For all $x, y \in X$, for all $r \in X$,

$x \succeq^r y \implies s_r(x, r) \geq s_r(y, r)$.

3 Three Structural Representation Theorems

3.1 The Decision-Theoretic Framework

Let $X$ be the non-empty set of objects on which preferences are defined. Throughout the paper, we shall assume that $X$ is a r.m.s. with operation $\oplus$.

**Remark 1.** Essentially all results in this paper can be easily restated and proved in the more general context where we assume that for all $r \in X$, there exists a mapping $\oplus_r : X \times \mathbb{R} \to X$ such that $(X, \oplus_r)$ is a r.m.s. An example of such a structure is a subset $X$ of a vector space $V$ such that, for all $r \in X$, for all $x \in X$, for all $\lambda \in \mathbb{R}$, $x + \lambda r \in X$ so that one can define a r.m.s. by letting $x \oplus_r \lambda = x + \lambda r$.

In order not to riddle our formulas with indices and not to obscure the arguments, we will work in the simpler framework where the group action is the same for all $r \in X$.

We now introduce preferences: let $\{\succeq_r\}_{r \in X}$ be a family of binary relations defined over $X$. These binary relations model a set of reference-dependent preferences, RDP for short, that can be interpreted as modeling the observed choice behavior of the individual in a context where his or her reference point is an element $r$ of $X$.

The main objective of the paper is to study representations of RDP in terms of the benefit and loss function relative to some endogenously defined preference relation $\succeq$. To that effect, we introduce some axioms.

3.2 Basic Axioms

We first introduce three basic axioms that are simply adaptations to our setting to very standard rationality assumptions. The first axiom is the standard ordering assumption for observable preferences.

**Axiom 1 (Weak Order).** $\succeq_r$ is a weak order for all $r \in X$.

The second axiom is a form of continuity axiom. In essence it states that any object has a monetary equivalent in the sense that given his or her endowment the agent is indifferent between owning this object or owning a certain amount of money along with the same initial endowment:
Axiom 2 (Solvability). For all \((x, r) \in X^2\), there exists \(\lambda \in \mathbb{R}\) such that
\[ x \sim_r r \oplus \lambda. \]

The third axiom simply reflects the fact that more money is better than less.

Axiom 3 (Monotonicity). For all \(r \in X\), for all \(\lambda, \mu \in \mathbb{R}\),
\[ \lambda > \mu \implies r \oplus \lambda \succ r \oplus \mu. \]

Notice that this axiom asserts that more money is preferred only if the initial endowment is not changed.

Let us say that a family \((u_r)_{r \in X}\) of real valued functions on \(X\) is normalized if:
\[ u_r(r \oplus \lambda) = \lambda, \quad \forall r \in X, \quad \forall \lambda \in \mathbb{R} \]
and that it is normalizable if \((u_r - u_r(r))_{r \in X}\) is normalized. Then these three axioms are jointly necessary and sufficient for the existence of a normalized family of utility functions for \((\succ_r)_{r \in X}\).

Proposition 4. \((\succ_r)_{r \in X}\) satisfies Weak Order, Solvability and Monotonicity if and only if there exists a normalizable family \(\{u_r\}_{r \in X}\) such that for all \(r \in X\),
\[ x \succ_r y \iff u_r(x) \geq u_r(y). \]

Moreover, \(\{v_r\}_{r \in X}\) is another normalizable family of utility functions iff there exists \(\gamma : X \to \mathbb{R}\) such that \(v_r(x) = u_r(x) + \gamma(r)\) for all \(x \in X\).

3.3 The First Representation Theorem: The Permanent Income Hypothesis

3.3.1 Statement

We proceed now to introduce axioms that restrict behavior in a more substantial way. The first and strongest of the axioms we shall consider asserts that given a reference point, preference is not affected by adding the same monetary amount to each of the compared alternatives.

Axiom 4 (Income Independence). For all \(r \in X\), for all \(x, y \in X\), for all \(\lambda \in \mathbb{R}\),
\[ x \succ_r y \iff x \oplus \lambda \succ_r y \oplus \lambda. \]

Traditionally, the permanent income hypothesis as defined by Friedman (Friedman, 1957) states that the consumption behavior of the individual is not affected by unexpected gains (windfalls) or losses. Imposing the previous axiom can be seen as stating essentially the same idea if one interprets \(r\) as the individual’s permanent income. This interpretation would be in line with Koszegi and Rabin (2007)’s conception of the reference point as expected future consumption.
Remark 2. Under axiom 4 and under the requirement that \( \lambda \) in axiom 2 is unique, it is possible to weaken axiom 3, requiring only that, for all \( r \in X \), for all \( \lambda \in \mathbb{R} \),

\[
r \oplus \lambda \succeq_r r \quad \implies \quad \lambda \geq 0.
\]

The first structural theorem shows that the Permanent Income Hypothesis in our setting is essentially equivalent to saying that utility can be measured by the willingness to pay.

Theorem 1. \( \{\succeq_r\}_{r \in X} \) satisfies Weak Order, Solvability, Monotonicity and Income Independence iff there exists a reflexive, upper monotonic and upper semi-continuous binary relation \( \succcurlyeq \) on \( X \) such that \( b_\succ(x, r) \) is finite for all \( (x, r) \in X^2 \) and \( \forall x, y, r \in X \),

\[
x \succcurlyeq_r y \quad \iff \quad b_\succ(x, r) \geq b_\succ(y, r).
\]

Moreover, \( \succcurlyeq' \) is a reflexive, upper monotonic and upper semi-continuous binary relation such that \( b_\succ' \) represents the preferences iff there exists a function \( \gamma : X \to \mathbb{R}^+ \) such that, for all \( x, y \in X \),

\[
x \succcurlyeq' y \quad \iff \quad x \succcurlyeq y,
\]

where the relation \( \succcurlyeq' \) is defined by

\[
x \succcurlyeq' y \quad \iff \quad x \oplus \gamma(y) \succeq_y y,
\]

and

\[
\gamma \geq \gamma' \quad \implies \quad \succcurlyeq' \subseteq \succcurlyeq' \quad \text{and} \quad \succcurlyeq \subseteq \succcurlyeq'.
\]

In particular, if we denote by \( \gamma^0 \) the null function, \( \succcurlyeq^0 \) is the minimal binary relation satisfying the conditions of the theorem and is the unique reflexive, upper monotonic and upper semicontinuous binary relation such that

\[
b_\succ(x, r) \geq 0 \quad \iff \quad x \succeq_r r.
\]

Before proceeding to interpret the theorem we shall make some technical remarks.

One may wonder whether the conditions imposed on the preferences in this theorem can be met on any r.m.s. The answer is negative. In fact, they restrict substantially the structure, as shown by the following corollary, which is an immediate consequence of proposition 2 and the fact that \( b_\succ \) is a morphism from \( X \) to \( \mathbb{R} \).

Corollary 1. If the conditions of the theorem are met, then \( X \) is a free r.m.s.

By proposition 1, this corollary entails that there exists some set \( A \) such that \( X \) is isomorphic to \( A \times \mathbb{R} \). In other words, these axioms entail that the interpretation of the real numbers as money is grounded, in the sense that one can isolate a special good, the numeraire, in amount \( w \), from the others, which are gathered in a composite object \( a \in A \) and money acquired by selling or lost by buying modifies directly the amount of the numeraire owned by the decision maker.

Another corollary shows that, under the conditions of the theorem the monotonicity axiom, which applies in principle only to an element \( r \) which is the reference point, can be generalized to any object \( x \).
Corollary 2 (Global Monotonicity). If the conditions of the theorem are met, then for all \( x \in X, \lambda, \mu \in \mathbb{R} \),
\[
\lambda > \mu \implies x \oplus \lambda \succ_r x \oplus \mu.
\]

**Proof.** If \( \lambda > \mu \), \( b(x \oplus \lambda, r) - b(x \oplus \mu, r) = \lambda - \mu > 0 \), so that \( x \oplus \lambda \succ_r x \oplus \mu \).

### 3.3.2 Interpretation and Examples

Theorem 1 provides necessary and sufficient conditions for reference dependent preferences to be representable by the willingness to pay functional. This willingness to pay is defined with respect to some exact and reflexive binary relation \( \succ \). We would like to think of this relation as a representation of the decision maker’s reference independent unobservable preferences that induce observable preferences when a reference point is selected. This interpretation is in line with the psychological literature that distinguishes between attitudes, that are context-independent, and preferences, that are context-dependent and framing-dependent (Kahneman et al., 1999). Relation \( \succ \) can thus be seen as a representation of attitudes as opposed to preferences. This interpretation is further justified by the fact that \( b_\succ \) is exact, so that a positive willingness to pay indeed reflects a positive attitude toward a given object.

We now come back to the examples we gave for the benefit function to illustrate the potential interest of theorem 1.

It is a usual practice in Industrial Organization to (implicitly most of the time) assume that the willingness to pay of consumers for one unit of good faithfully represents their utility for that good. It is well-known that in classical consumer theory this is possible if and only if preferences are quasilinear. In other words, this is possible only if one good is assumed to be numeraire, i.e. money, and utility is linear (in fact additive would be the correct term) with respect to money (see example 6). Theorem 1 extends to an abstract setting and generalizes to the case of reference-dependent preferences this result, and this leads to potential generalizations of extant models of Industrial Organization to settings where objects of consumption cannot easily be described in terms of commodity bundles.

Another special case of theorem 1 is the classical result that when preferences are quasilinear the compensating variation is a sound measure of the welfare variation experienced by the consumer (see example 7 and Mas-Colell, Whinston, and Green (1995, p.83)).

Theorem 1 also provides foundations for the use of Luenberger’s benefit function and Deaton’s distance function as representation of preferences. More importantly, it provides foundations for the representation of preferences on financial assets (random variables) by a risk measure (not necessarily convex though; this would require extra axioms that are straightforward). In particular this provides an axiomatization of Value at Risk relative to a given benchmark, that we state as a corollary. Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(X\) be the set of all real random variables on \((\Omega, \mathcal{F}, P)\) with continuous cdf. It obviously is an r.m.s. with the natural addition of reals. Let \((\succ_R)_{R \in X}\) be a family of preference relations on \(X\). Say that this family satisfies **restricted stochastic dominance** if, for all \(Z, Z' \in X\),
\[
Z' \succ_Z Z + \text{Var}(Z, Z) \iff P(Z' \leq 0) \leq P(Z \leq 0).
\]
Corollary 3. \((\succeq_R)_{R \in X}\) satisfies all the axioms of theorem 1 and restricted stochastic dominance if and only if for all \(Z, Z', R \in X\),
\[
Z \succeq_R Z' \iff \text{VaR}(Z, R) \leq \text{VaR}(Z', R).
\]

### 3.3.3 Loss aversion

Sensitivity to losses is a well-documented psychological fact. Its usual manifestation is loss aversion, the fact that, as Kahneman and Tversky repeatedly put it, “losses loom larger than gain”, i.e. people are more displeased with a loss of a given size than they are pleased with a gain of the same size (e.g. Tversky and Kahneman (1991)). Put this way, loss aversion is clearly a cardinal notion, since it involves the comparison of utility levels. In the present context, it can be formalized as follows:

**Definition 4 (Loss Aversion (LA)).** The family \(\{u_r\}_{r \in X}\) representing observable preferences \(\{\succeq_r\}_{r \in X}\) exhibits loss aversion if:
\[
\forall x, y \in X, \ x \succeq y \implies u_x(x) - u_x(y) \geq u_y(x) - u_y(y).
\]

Suppose that an agent considers that \(x\) is somehow better than \(y\) when \(y\) is the default option. Therefore, moving from \(x\) to \(y\) when \(x\) is the default option corresponds to a loss that is objectively of the same absolute value, but the subjective absolute value of which may be (cardinally) measured by the utility difference \(u_x(x) - u_x(y)\), whereas the subjective absolute value of the gain from moving from \(y\) to \(x\) may be (cardinally) measured by the utility difference \(u_y(x) - u_y(y)\). Loss aversion entails that the utility value of the gain is smaller than the utility value of the corresponding loss, therefore \(u_x(x) - u_x(y) \geq u_y(x) - u_y(y)\).

One interesting problem is to provide an ordinal foundation for the cardinal notion of loss aversion. This can be done in various ways (see e.g. Schmidt and Zank (2005)). A necessary condition for loss aversion is the following axiom:

**Axiom 5 (Status quo Bias).** For all \((x, r) \in X^2\),
\[
x \succeq_r r \implies x \succeq_x r.
\]

This axiom conveniently expresses the idea that being the status quo gives an alternative an extra power against other alternatives besides its intrinsic merit: it says that if an alternative \(x\) beats the status quo \(r\), then a fortiori \(x\) must beat \(r\) when \(x\) is the status quo, since it is thus even more attractive than before. This is a kind of consistency axiom that rules out a very direct form of preference reversal: preferring \(x\) to \(y\) when \(y\) is the endowment and \(y\) to \(x\) when \(x\) is the endowment. Since this kind of preference reversal makes the decision-maker vulnerable to money pumps, ruling them out has been deemed necessary in the literature for a modeling of rational reference-dependent preferences, and therefore axioms akin to axiom 5 have been introduced by many authors, including Munro and Sugden (2002), Sagi (2006) and Masatlioglu and Ok (2005, 2006).

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6Masatlioglu and Ok (2006) criticize the descriptive power of Tversky and Kahneman (1991) based on a definition of loss aversion that involves a strict inequality. However, introducing a weak inequality does not significantly alter the intuition while it renders it immune to their criticism.
This necessary condition is not, however, a sufficient condition for loss aversion, as can be shown by the following trivial example:

**Example 13.** Let $E = \{x, y\}$ be a set and define the (reflexive) binary relations $\succsim_x$ and $\succsim_y$ by $x \succsim_y y$ and $x \succsim_x y$. Moreover, let $u_x(x) = u_y(y) = 0$, $u_x(y) = -1$ and $u_y(x) = 2$. Then axiom 5 is satisfied but $-u_x(y) < u_y(x)$. Therefore there is no loss aversion.

It is possible, however, to take advantage of the monetary structure of $X$ to obtain a characterization of loss aversion. Consider first the following modification of the definition of loss aversion:

**Definition 5 (Monetary Loss Aversion (MLA)).** The family $\{u_r\}_{r \in X}$ representing observable preferences $\{\succsim_r\}_{r \in X}$ exhibits monetary loss aversion if:

$$\forall x, y \in X, \forall \lambda \in \mathbb{R}, \quad x \succsim_r y \oplus \lambda \implies u_x(x) - u_x(y \oplus \lambda) \geq u_y(x) - u_y(y \oplus \lambda).$$

This definition amounts to allow for unexpected windfalls in the definition of loss aversion, with the interpretation that these shocks on income are not integrated in the endowment.

Consider now the following axiom:

**Axiom 6 (Strong Monetary Status quo bias (SMSQB)).** For all $(x, r) \in X^2$, for all $\lambda \in \mathbb{R}$,

$$x \succsim_r r \oplus \lambda \implies x \ominus \lambda \succsim_x r.$$ 

When $\lambda = 0$ this is just standard status quo bias. The introduction of money only smooths its effect. Indeed, when $\lambda > 0$, then the fact that $x$ beats $r$ even with a premium of $\lambda$ means that $x$ is significantly better than $r$. Therefore, it implies that when $x$ is the status quo it can beat $r$ even if it is worsened by the amount $\lambda$. In turn, if $\lambda < 0$, then $x$ beats $r$ only if $r$ is slightly worsened by the amount lambda. There is not, therefore, a strong preference for $x$, so that when it is the status quo $x$ will need to be improved by $\lambda$ in order to still beat $r$.

The next proposition will show that this axiom indeed characterizes both loss aversion and monetary loss aversion. In fact, it characterizes a much stronger property, defined as follows:

**Definition 6 (Unconditional Monetary Loss Aversion (UMLA)).** The family $\{u_r\}_{r \in X}$ representing observable preferences $\{\succsim_r\}_{r \in X}$ exhibits unconditional monetary loss aversion if:

$$\forall x, y \in X, \forall \lambda \in \mathbb{R}, \quad u_x(x) - u_x(y \oplus \lambda) \geq u_y(x) - u_y(y \oplus \lambda).$$

Then, we have the following proposition:

**Proposition 5.** Assume $\{\succsim_r\}_{r \in X}$ satisfies Weak Order, Solvability and Monotonicity. Then, the following are equivalent:

(i) all normalizable families of utility functions satisfy loss aversion.
(ii) all normalizable families of utility functions satisfy monetary loss aversion.

(iii) all normalizable families of utility functions satisfy unconditional monetary loss aversion.

(iv) \((\succeq_r)_{r \in X}\) satisfies Strong Monetary Status quo bias.

Proposition 5 and the fact that \(s_\succ(x, r) = -b_\succ(r, x)\) yield the following corollary to theorem 1:

Corollary 4. Assume \((\succeq_r)_{r \in X}\) satisfies Weak Order, Solvability, Monotonicity and Income Independence. Then it satisfies SMQSB if and only if

\[ b_\succ \leq s_\succ \]

It is noticeable that this corollary yields only an inequality between WTP and WTA since in classical consumer theory, quasilinearity of the utility function yields equality. It is even more noticeable that an additional axiom is needed to obtain the natural result that WTP is smaller than WTA even in the quasilinear case. The culprit for this rather surprising fact is of course the introduction of reference-dependence. When there is no reference dependence, indeed, i.e. when \(\succeq_r = \succeq_{r'}\) for all \(r, r' \in X\), Income independence implies SMSQB, so that the above inequality holds automatically.

A very similar result holds for the reverse inequality that is needed to establish equality. Consider indeed the following axiom:

Axiom 7 (Negative Monetary Status Quo Bias). For all \((x, r) \in X^2\), for all \(\lambda, \mu \in \mathbb{R}\),

\[ x \oplus \lambda \succeq x \quad r \oplus \mu = \Rightarrow \quad x \succeq r \oplus (\mu - \lambda). \]

This axiom clearly expresses an idea that opposes Status Quo Bias. Consider first the case where \(\lambda = \mu = 0\). Then if \(x\) beats \(r\) when \(x\) is the status quo, then a fortiori does it beat it when \(r\) is the status quo. Thus being the status quo is a disadvantage. Notice that this behavior has not been observed experimentally but it seems to make sense just as much as the opposite since it reflects an attitude that has been described extensively in novels and plays: the fact that one always longs more for what one does not own than for what one owns, and symmetrically the fact that an object automatically loses any attractiveness whatsoever as soon as it is owned. That this attitude is not observed in markets while it is repeatedly observed in many nonmarket conditions is a rather surprising fact that would indeed require some investigation.

The axiom also takes care of the case where \(\lambda\) and \(\mu\) are nonzero but its interpretation is similar to that of SMSQB so we will not develop it further.

Anyway, if one defines monetary gain love by reversing the inequalities in the definition of monetary loss aversion, then it is easy to prove the following dual of proposition 5:

Proposition 6. Assume \((\succ_r)_{r \in X}\) satisfies Weak Order, Solvability and Monotonicity. Then, the following are equivalent:
(i) all normalizable families of utility functions satisfy Unconditional Monetary Gain Love.

(ii) \((\succeq_r)_{r \in X}\) satisfies Negative Monetary Status Quo bias.

Notice that the introduction of \(\mu\) in the definition of NMSQB is needed to compensate for the very weak axioms used in the previous proposition.

Given this proposition, it is clear that NMSQB will be necessary and sufficient in the presence of Income independence for the willingness to pay to exceed the willingness to accept. Therefore, the conjunction of SMSQB and NMSQB is necessary and sufficient to yield equality. But this characterization does not take full advantage of Income Independence. To that effect, we introduce a notion of absence of status quo bias that under under Income Independence is equivalent to the conjunction of SMSQB and NMSQB:

**Axiom 8 (Weak Monetary Status Quo Indifference (WMSQI)).** For all \((x, r) \in X^2\), for all \(\lambda \in \mathbb{R}\),

\[x \oplus \lambda \succeq_x r \iff x \oplus \lambda \succeq_r r.\]

We therefore have (the proof is omitted in view of the previous discussion):

**Proposition 7.** Assume \((\succeq_r)_{r \in X}\) satisfies Weak Order, Solvability, Monotonicity and Income Independence. Then it satisfies Weak Monetary Status Quo Indifference if and only if

\[b_{\succeq} = s_{\succeq}\]

Notice that Weak Monetary Status Quo Indifference is automatically true under Income Independence if there is no reference-dependence. This explains why willingness to pay equals willingness to accept under quasilinearity in classical demand theory, but also why the introduction of reference-dependence, that is well-established empirical fact, may explain a failure of this equality even when there are non income effects.

### 3.4 The Second Representation Theorem: Dropping Income Independence

#### 3.4.1 Motivation

In the previous subsection, we have given conditions for the willingness to pay to be a sound measure of agent’s preferences. We have shown, in particular, that a necessary condition for this is that money be treated in a linear way. This assumption, however, is quite strong. In choice under risk it is well known since Bernoulli’s resolution of the St Petersburg Paradox that this assumption is not very satisfactory as it has unrealistic predictions. Similarly, Keynes (?) has elaborated the idea that preference for liquidity is intimately linked to uncertainty, so that the utility of money is not necessarily linear; the utility of money depends on the opportunity cost of its detention, and this opportunity cost may not vary linearly with the amount of money one owns. However, one could go further into the exploration of the causes of this non-linearity by considering the ideas of the causes of preference for liquidity.
as defined by Keynes: the transaction motive, the precautionary motive and the speculation motive. The relative weight of these three motives are likely to be very different for a rich and a poor person. The preference for liquidity of the poor person is likely to be driven by the transaction and precautionary motives; the preference or liquidity of the rich person, on the contrary, is likely to be more influenced by the speculation motive. This suggests that preference for liquidity is likely to be reference-dependent, and should therefore be a privileged field of application for a theory of reference-dependent preferences.

In our framework, linearity of money is captured by the Income Independence axiom. It is easy to give examples for which this axiom might fail. They would all imply a certain form of complementarity or interaction between money and the good considered. Assume for instance that $x$ is a large flat on Park Avenue and that $y$ is a charming and very ancient castle in the south of France. It is well known that ancient castles are costly to maintain. Therefore they are useless unless one has enough money to do it. Assume both can be bought at the same price. It is conceivable that one prefers the flat in New-York to the castle in France ($x \succ r$, $y$) due to the fact that one would not be able to maintain the castle, but would prefer the castle to the flat if some extra cash fell from the sky at the moment of buying either of these accommodations (i.e., for some $\lambda$ large enough $y \oplus \lambda \succ x \oplus \lambda$).

In this subsection, we intend to study thoroughly the interconnection between the non-linearity of the utility for money and preference for liquidity. We shall therefore explore the consequences of dropping the Income Independence axiom. This will be the first step of our study. In a second step, we shall provide a very general, model-free comparative definition of preference for liquidity and link it with the results of the previous step.

### 3.4.2 Statement of the Theorem

We consider the following weakening of Income independence:

**Axiom 9** (Buying Consistency). *For all $(x,r) \in X^2$, for all $\lambda \in \mathbb{R}$,*

\[
x \oplus \lambda \succ_r r \implies x \succ_r r \oplus \lambda.
\]

The idea of this axiom is that, if I am willing to pay $\lambda$ euros in order to buy $x$ when my current endowment is $r$, then this means that the subjective value of $x$ relative to $r$ is at least lambda, i.e. I prefer $x$ to $r$ even when I receive in addition a windfall amount of $\lambda$ euros.

In order to focus on behavior that is compatible with empirical data, we shall also impose a form of monetary status quo bias, though weaker than SMSQB:

**Axiom 10** (Monetary Status Quo Bias (MSQB)). *For all $(x,r) \in X^2$, for all $\lambda \in \mathbb{R}$,*

\[
x \oplus \lambda \succ_r r \implies x \oplus \lambda \succ_x r.
\]

We do not elaborate on its interpretation since it is very similar to the interpretation of SMSQB.

Let us turn now to the statement of the theorem.
Theorem 2. \((\succsim_r)_{r \in X}\) satisfies Weak Order, Solvability, Monotonicity, Buying Consistency and MSQB if and only if there exists a unique normalized family of utility functions \(u_r : X \rightarrow \mathbb{R}\) and a unique reflexive, upper monotonic and upper semicontinuous binary relation \(\succ\) such that

(i) \(b_\succ(x,r) \leq u_r(x) \leq s_\succ(x,r)\).

(ii) \(x \succ r \iff u_r(x) \geq 0 \iff b_\succ(x,r) \geq 0\).

In particular \(b_\succ(x,r)\) and \(s_\succ(x,r)\) are finite for all \((x,r) \in X^2\).

Remark 3. Since this theorem proves the existence of a morphism of r.m.s. when the axioms hold, these can only be true if the action \(\oplus\) is free. Therefore, once again the monetary interpretation is all the more warranted.

3.4.3 Discussion of the Theorem

Comparison with Theorem 1 In what sense are Buying Consistency and Monetary Status Quo Bias weakenings of the axioms introduced in the previous subsection? The missing link that will allow us to go from the weak form to the strong form of both axioms is the following counterpart to Income Independence:

Axiom 11 (Wealth Independence). For all \(x,y \in X\), for all \(r \in X\), for all \(\lambda \in \mathbb{R}\),

\[ x \succ_r y \iff x \succ_{r,\lambda} y. \]

A very natural interpretation of this axiom is that the wealth level of individuals does not affect their preferences. This is quite contrary to the natural intuition that, for instance, richer people are less risk averse than poorer people, since for identical lotteries what is at stake is lower. We do not use this axiom as a normative requirement but only as a tool to obtain some equivalence results. The following proposition indeed holds:

Proposition 8. Assume \(\{\succsim_r\}_{r \in X}\) satisfies Weak Order, Solvability and Monotonicity. Then:

(i) Income Independence implies Buying Consistency and they are equivalent if Wealth Independence holds. Moreover, if WMSQI holds, Income Independence implies Wealth Independence.

(ii) SMSQB implies MSQB and they are equivalent if Wealth Independence and Buying Consistency hold.

An important consequence of the previous proposition is that when there is no reference dependence, Income Independence and Buying Consistency are equivalent. Therefore, the subsequent results and discussion only make sense when there is reference-dependence. This illustrate the fact that introducing reference dependence allows to analyze phenomena that cannot be studied under the assumption of reference independence, as for instance preference for liquidity, as we shall see.
**Interpretation** Theorem 2 provides axiomatic foundations for the WTA/WTP discrepancy observed in the lab. Assuming SMSQB (and indeed MSQB suffices), theorem 1 also provided such foundations. However, it did it assuming linearity of the utility of money, a consequence of which being that willingness to pay was a good measure of utility.

As we noted when we discussed the motivation for theorem 2, linearity of the utility of money is not necessarily a sensible assumption. Theorem 2 does without it, and, as a consequence utility cannot be identified with willingness to pay anymore. This can be summed up by the idea that “Money matters”. In other words, money has some intrinsic utility that affects the definition of the selling and buying prices. Consider the buying price. If the utility of the good is strictly greater than the buying price, this means that evaluating the buying price requires taking into account the utility gain from owning the good and the utility loss from loosing money. A symmetric argument can be made for the selling price. If this utility is linear in the amount of money, the results coincide. But if money has some intrinsic (non-linear) utility, then the results will probably not coincide. Therefore the utility of money is likely to explain the endowment effect.

As we noted in the motivation section, the notion of nonlinear utility of money can be related to the notion of preference for liquidity. This notion can be studied formally in our model, and we will proceed to do this in the next subsection. We shall give a precise formal definition for this concept and relate it to a parameter of the utility function. This parameter is actually linked to the fact that theorem 2 constitutes a generalization of theorem 1 also with respect to a formalization of the notion of attitudes as opposed to preferences. But this shall be made clearer in the subsequent developments.

### 3.4.4 A formal definition and characterization of preference for liquidity

The notion of preference for liquidity can be given a precise definition in the context of our model adopting a comparative approach (akin to now standard definitions of risk aversion (Rotschild and Stiglitz, 1970) and ambiguity aversion (Epstein, 1999; Ghirardato and Marinacci, 2002)) based on the following intuition: a person 1 with endowment \( r \) has a stronger preference for liquidity than a person 2 with endowment \( r' \) if, given any object \( x \) for which they both have the same willingness to pay and willingness to accept and given any amount of money \( w \), whenever person 1 prefers having \( x \) to having \( w \), then so does person 2.

Formally,

**Definition 7.** Let \( \succsim^1_r \) and \( \succsim^2_r \) be the observable preference profiles of two individuals and \( r, r' \in X \). Then 1 has a stronger preference for liquidity at \( r \) than 2 at \( r' \) if

\[
x \succsim^1_r (\succsim^1_r) r \oplus w \implies x \succsim^2_r (\succsim^2_r) r' \oplus w
\]

for all \( w \in \mathbb{R} \) and for all \( x \in X \) such that, for all \( \lambda \in \mathbb{R} \),

\[
x \oplus \lambda \succsim^1_r x \iff x \oplus \lambda \succsim^2_{r'} x \quad \text{and} \quad r \oplus \lambda \succsim^1_x x \iff r' \oplus \lambda \succsim^2_x x.
\]
This notion can be characterized in the model of theorem 2. However, this require a more thorough study of this model. To that effect, let us first introduce a definition.

Definition 8. $k \in X$ is crisp at $r$ if it satisfies the following two conditions:

(i) For all $\lambda \in \mathbb{R}$, $k \ominus \lambda \succeq r \iff k \succeq r \oplus \lambda$

(ii) For all $\lambda \in \mathbb{R}$, $r \oplus \lambda \succeq k \iff r \oslash \lambda \succeq k$

We denote by $K_r$ the set of elements of $X$ that are crisp at $r$.

Crisp elements can be characterized in a way that justifies their name.

Proposition 9. Assume $(\succeq_r)_{r \in X}$ satisfies Weak Order, Solvability and Monotonicity. Then $k$ is crisp at $r$ only if $b_\succ(k, r) = s_\succ(k, r)$ where $\succ$ is as in theorem 2.

The converse holds if $(\succeq_r)_{r \in X}$ satisfies all the conditions of theorem 2.

We are now in position of stating the following corollary to theorem 2, the proof of which is trivial and therefore omitted:

Corollary 5. $(\succeq_r)_{r \in X}$ satisfies Weak Order, Solvability, Monotonicity, MSQB and Buying Consistency iff there exists a function $a_r : X \to [0, 1]$ such that, for all $x, y, r \in X$,

$$x \succeq_r y \iff u_r(x) \geq u_r(y),$$

where

$$u_r(x) := a_r(x)b_\succ(x, r) + (1 - a_r(x))s_\succ(x, r),$$

$\succ$ is defined by

$$x \succ y \iff x \succ y.$$

Moreover, $a_r(x)$ is unique iff $x \not\in K_r$.

This corollary introduces a new object in the picture along with willingness to pay and willingness to accept. It would then be desirable to have an interpretation for it. It turns out that it has a natural interpretation as a coefficient of preference for liquidity, as shown by the following proposition.

Proposition 10. Assume $(\succeq^1_r)_{r \in X}$ and $(\succeq^2_r)_{r \in X}$ satisfy the axioms of theorem 2. Then, the following are equivalent:

(i) 1 has a stronger preference for liquidity at $r$ than 2 at $r'$.

(ii) For all $x \in X \setminus (K^1_r \cup K^2_r)$ such that $b_{\succ^1}(x, r) = b_{\succ^2}(x, r')$ and $s_{\succ^1}(x, r) = s_{\succ^2}(x, r')$, $a^1_r(x) \geq a^2_r(x)$.

It is quite natural to think of preference for liquidity as a property that depends in principle on money only and not on the objects to be bought with it. One would therefore like to have a characterization of preference for liquidity independent of objects in $X$. This leads to the following question: to what extent does the function $a_r$ depend on $x$? To answer this question, we must first introduce a definition.
Definition 9. Let \( x, y, r \in X \). \( x \) and \( y \) are affinely related at \( r \), denoted \( x \preceq_r y \), if there exists \( m > 0 \) and \( p \in \mathbb{R} \) such that for all \( \lambda \in \mathbb{R} \):

(i) \( x + \lambda \succ_r r \iff y \ominus (m\lambda + p) \succ_r r \);

(ii) \( r \oplus \lambda \succ_x x \iff r \ominus (m\lambda + p) \succ_y y \);

(iii) \( x \succ_r r \ominus \lambda \iff y \succ_r r \oplus (m\lambda + p) \);

We now give a characterization of this relation that shows that it is an equivalence relation and justifies its name, which is a direct consequence of lemma 3 proved in the appendix:

Proposition 11. Assume \((\succ_r)_{r \in X}\) satisfies the conditions of theorem 2. Then the following are equivalent:

(i) \( x \succ_r y \);

(ii) \( b_\succ(y, r) = m b_\succ(x, r) + p, s_\succ(y, r) = m s_\succ(x, r) + p, \) and \( u_r(y) = m u_r(x) + p \)

where \( \succ \) is as in theorem 2 and \((u_r)_{r \in X}\) is a normalized family of utility functions.

Proposition 12. Under the conditions of theorem 2, for all \( r \in X \), for all \( x, y \in X \setminus K_r \), \( x \succ_r y \implies a_r(x) = a_r(y) \).

This result suggests that it is possible to characterize axiomatically the case where \( a_r \) is constant, and therefore to have a single parameter measuring preference for liquidity. For this we will introduce a dominance axiom. Before stating it, for all \( x, y \in X \) and \( m > 0, p \in \mathbb{R} \), we write

\[ m.x + p \succ_r y \]

if whenever \( \tau \in \mathbb{R} \) is such that \( x \sim_r r \oplus \tau \) then \( x' \succ_r y \) for all \( x' \in X \) such that \( x' \sim_r r \oplus (m\tau + p) \).

Axiom 12 (Dominance). For all \( x, y, r \in X \), if there exists \( m > 0, p \in \mathbb{R} \) such that for all \( \lambda \in \mathbb{R} \):

(i) \( y \ominus (m\lambda + p) \succ_r r \implies x \ominus \lambda \succ_r r \);

(ii) \( r \oplus \lambda \succ_x x \implies r \ominus (m\lambda + p) \succ_y y \);

then \( m.x + p \succ_r y \).

If \( m = 1 \) and \( p = 0 \), and assuming Monotonicity and Weak order, this axiom merely says that if the buying price and the selling price of \( x \) are greater than the buying price and the selling price of \( y \), then \( x \) is preferred to \( y \) given \( r \). This axiom is stronger as it implies that if the buying price and the selling price of \( y \) are smaller than some increasing affine transformation of the buying price and the selling price of \( x \), then any object whose monetary equivalent is the same increasing affine transformation of the monetary equivalent of \( x \) is preferred to \( y \).

The following result holds:
Proposition 13. The following are equivalent:

(i) \( (≽_r)_r∈X \) satisfies Weak Order, Solvability, Monotonicity, MSQB, Buying Consistency and Dominance.

(ii) For all \( r \in X \), there exists \( α_r ∈ [0,1] \), unique if \( K_r \neq X \), such that, for all \( x, y ∈ X \),

\[
x ≽_r y \iff α_r b_>(x,r) + (1 − α_r) s_>(x,r) ≥ α_r b_>(y,r) + (1 − α_r) s_>(y,r)
\]

where \( ≽ \) is defined by

\[
x ≽ y \iff x ≽_r y.
\]

3.4.5 Application: the Preference Reversal Phenomenon

The preference reversal phenomenon is a very well-known and robust anomaly (Lichtenstein and Slovic, 1971; Grether and Plott, 1979): consider the following lotteries

\[
P = (0, \frac{1}{100}; 4\$, \frac{99}{100}), \ S = (0, \frac{83}{100}; 16\$, \frac{17}{100}).
\]

When asked to express a preference, people usually respond that they prefer \( P \) to \( S \); however, when asked to give a selling price for both lotteries, they usually value the \( S \) bet higher.

Let us see how our theory may lead to a rationale for such a choice. Tversky, Slovic, and Kahneman (1990) emphasize that preference reversals are mainly due to the fact that the selling price of a lottery is not necessarily a certainty equivalent for it, but merely a cash equivalent. In other words, preference reversals may be explained by the endowment effect as portrayed by theorem 2. It should therefore be possible to show that preferences satisfying the axioms of theorem 2 may exhibit preference reversals, and to relate these reversals to properties of these preferences. Specifically, we wish to show how preference for liquidity as defined in the previous section can account for preference reversal. To that effect consider two decision makers \( (≽_1^r)_r∈X \) and \( (≽_2^r)_r∈X \) that satisfy the axioms of proposition 13 and denote their preference for liquidity coefficients by \( (α_i^r)_{r∈X}, i = 1, 2 \). Suppose that for some reference point \( r ∈ X \) these two decision makers differ only by their preference for liquidity, i.e. they have the same WTP and the same WTA relatively to \( r \) for all \( x ∈ X \). For any preference profile \( (≽_r)_r∈X \) and for \( r ∈ X \), define the preference reversal set at \( r \), denoted \( PR((≽_r)_r∈X, r) \), by

\[
PR((≽_r)_r∈X, r) = \{(x,y) ∈ X | s_>(y,r) > s_>(x,r) \text{ and } x ≽_r y\},
\]

where \( ≽ \) is as in theorem 2. Then, the following holds:

Proposition 14. If 1 has a greater preference for liquidity than 2 at \( r \), i.e. \( α_1^r ≥ α_2^r \), then \( PR((≽_1^r)_r∈X, r) \) \( ⊆ \) \( PR((≽_2^r)_r∈X, r) \).

This proposition therefore says that, under suitable comparability conditions, more preference for liquidity entails more preference reversals. A standard psychological explanation of preference reversal is that the procedure used to elicit

\footnote{The argument easily extends to the case where this holds for two different reference points \( r \) and \( r' \).}
preferences leads to a greater salience of one attribute (probabilities or outcomes), and specifically that asking for the selling price emphasizes the monetary dimension of the lottery (due to what psychologists call scale compatibility: the fact that the selling price and the outcomes are expressed in the same unit of measure). It is to be expected, therefore, that the more the agent likes money, the more sensitive will he or she be to the monetary dimension of the lottery, and the more effective will be the salience effect generated by scale compatibility, leading to more preference reversals.

Another interest of this proposition is that it provides a testable implication of the theory of reference-dependent preferences outlined in this section. Moreover, it also paves the way to an indirect test of the scale compatibility hypothesis, provided that one accepts the idea that it is the rationale that links liquidity preference and preference reversals, because then if this relationship is not to be found in data then some doubt may be cast on scale compatibility as a plausible explanation of preference reversals.

3.4.6 Application: Reference-Dependence under Risk

Adding some structure on the set $X$ may lead to more precise results and in particular to applications to specific decision making contexts like for instance decision under risk. In this section, we shall assume that $X$ is a convex subset of a locally convex Hausdorff space. The following axiom provides the articulation between the r.m.s. structure of $X$ and its additional vector and topological structure:

**Axiom 13** (Monetary Global Convexity (MGConv)). The set

$$
\mathcal{G}((\succsim_r)_{r \in X}) := \{(x, r, \lambda) \in X^2 \times \mathbb{R} \mid x \ominus \lambda \succsim_r r\}
$$

is convex.

**Axiom 14** (Monetary Continuity (MCont)). The set

$$
\mathcal{H}((\succsim_r)_{r \in X}, \lambda) := \{(x, r) \in X^2 \mid x \ominus \lambda \succsim_r r\}
$$

is closed for each $\lambda$.

For a convex set $C$ in a topological vector space, let $\mathcal{A}(C)$ be the set of affine continuous functionals on $C$. The following corollary of theorem 2 holds:

**Corollary 6.** The following are equivalent:

(i) $(\succsim_r)_{r \in X}$ satisfies Weak Order, Solvability, Monotonicity, MSQB, Buying Consistency, Dominance, MGConv and MCont.

(ii) For $\succsim$ defined by

$$
x \succsim y \iff x \succsim_y y,
$$

there exists a set $\mathcal{B} \subseteq \mathcal{A}(X^2)$ such that

a) $b_{\succsim}(x, r) = \inf_{\beta \in \mathcal{B}} \beta(x, r)$,
b) \( s_\beta(x, r) = \sup_{\beta \in \mathcal{B}} \beta(x, r), \)

where for all \( \beta \in \mathcal{B}, \beta(x, y) = -\beta(y, x) \) for all \( (x, y) \in X^2 \) and, for all \( r \in X \), there exists \( \alpha_r \in [0, 1] \), unique if \( K_r \neq X \), such that, for all \( x, y \in X \),

\[
x \succ_{r} y \iff \alpha_r s_\beta(x, r) + (1 - \alpha_r)s_\beta(y, r) \\
\geq \alpha_r s_\beta(x, r) + (1 - \alpha_r)s_\beta(y, r).
\]

The set \( \mathcal{B} \) can be interpreted as usual in this case as a set of utility functions representing criteria with respect to which alternatives are compared. However, in this particular setting their value must not be considered as ordinal but as cardinal, since they bear a particular relationship with the willingness to pay and the willingness to accept that are cardinal concepts\(^8\). This relation is made clear by the following two consequences of corollary 6:

\[
\forall (x, r, \lambda) \in X^2 \times \mathbb{R}, x \otimes \lambda \succ_{r} r \iff \beta(x, r) \geq \lambda, \quad \forall \beta \in \mathcal{B}
\]

and

\[
\forall (x, r, \mu) \in X^2 \times \mathbb{R}, r \oplus \mu \succ_{x} x \iff \bar{\beta}(x, r) \leq \mu, \quad \forall \beta \in \mathcal{B}.
\]

Interpreting \( \beta(x, r) \) and \( \bar{\beta}(x, r) \) as, respectively, the relative advantage of owning \( x \) instead of \( r \) in dimension \( \beta \) and the relative disadvantage of having \( r \) instead of \( x \) in dimension \( \beta \), equations 2 and 3 say that, when the status quo is \( r \), a buying price for \( x \) is acceptable if whatever the criterion or dimension considered, it is smaller than the utility surplus enjoyed by having \( x \) instead of \( r \); similarly, a selling price for \( x \) is acceptable if, whatever the dimension considered, the loss of surplus incurred by owning \( r \) instead of \( x \) is smaller than this price. Thus, an acceptable buying price must be low enough for the surplus in all dimensions to outweigh it, while an acceptable selling price must be high enough to compensate for the surplus loss in all dimensions.

It is noticeable that gains are privileged when one is considering to buy \( x \) while losses are emphasized when considering to sell it, although one could consider both aspects at once. This shows that the attention is not focused on the same criteria in both activities. This is reminiscent of the results presented in Tversky, Simonson, and Shafir (2000), where, depending on the question asked (for instance “To which parent would you award sole custody of the child?” vs “Which parent would you deny sole custody of the child?”), people tend to focus on reasons pro or reasons con: the decision problem generates its own salience effect.

The main application of this corollary is the modeling of reference-dependent preferences under risk. For concreteness, we shall consider a free r.m.s. \( Y \) and let \( X = \Delta_0(Y) \), the set of simple probability measures on \( Y \), endowed with the norm \( \| \cdot \| \) defined by

\[
\| p \| := \sup \sum_{i=1}^{n} |p(E_i)|,
\]

where the supremum is taken over all finite sequences \( \{E_1 \ldots E_n\} \) of pairwise disjoints subsets of \( Y \). The vector space of all measures such that this quantity is finite

---

\(^8\)The normalization in corollary 8 below is also a consequence of this cardinality.
is a Banach space, and it is therefore a locally convex Hausdorff space. Thus we can apply the previous result.

**Corollary 7.** The following are equivalent:

(i) \((\succcurlyeq_r)_{r \in \Delta_0(Y)}\) satisfies Weak Order, Solvability, Monotonicity, MSQB, Buying Consistency, Dominance, MGConv and MCont.

(ii) There exists a family \((v_\beta)_{\beta \in \mathcal{B}}\) of bounded functions from \(Y^2\) to \(\mathbb{R}\) such that

\[
\begin{align*}
\text{a)} & \quad b_\succ(p, r) = \inf_{\beta \in \mathcal{B}} \int_{Y^2} v_\beta dp \otimes r, \\
\text{b)} & \quad s_\succ(p, r) = \sup_{\beta \in \mathcal{B}} \int_{Y^2} w_\beta dp \otimes r,
\end{align*}
\]

where for all \(\beta \in \mathcal{B}\), \(w_\beta(y, y') = -v_\beta(y', y)\) for all \((y, y') \in Y^2\) and, for all \(r \in X\), there exists \(\alpha_r \in [0, 1]\), unique if \(K_r \neq X\), such that, for all \(p, q \in \Delta_0(Y)\),

\[
p \succ_r q \iff \alpha_r b_\succ(p, r) + (1 - \alpha_r) s_\succ(p, r) \geq \alpha_r b_\succ(q, r) + (1 - \alpha_r) s_\succ(q, r). \tag{4}
\]

Some recent papers on reference-dependent preferences (Sagi, 2001; Masatlioglu and Ok, 2005; Giraud, 2006) relate them to a set of functions that are interpreted as criteria with respect to which alternatives are compared to one another and to the status quo. In particular Sagi (2001) and Giraud (2006) axiomatize related functional forms for reference-dependence under risk. The formulation of equation 4 can be seen as a generalization of both models: it is a generalization of Sagi’s model for which \(\alpha_r = 1\) for all \(r\) and \(v_\beta(y, y') = \psi_\beta(y) - \psi_\beta(y')\) for some function \(\psi_\beta\), and a generalization\(^9\) of the model of Giraud (2006) for which \(v_\beta = w_\beta\) for all \(\beta\). The present axiomatization of this functional form for reference dependence under risk is more illuminating than the one in Giraud (2006) because it allows for a clearer comparison of its axioms to those of Sagi (2001) (bearing in mind that strictly speaking no rigorous comparison is allowed because Sagi (2001) does not use a r.m.s. structure and the axioms are not expressed in a “monetary” form).

The main axiom in Sagi (2001) is the “no regret” axiom, thoroughly studied in Sagi (2006), and that is a stronger form of SQB:

**Axiom 15** (No Regret (NR) (Sagi, 2006)). For all \(x, y \in X\),

\[x \succ y \implies x \succ_z y, \forall z \in X.\]

What are the implications of this axiom in our setting? In order to fully exploit its implication, we need to specify more the link between the r.m.s. and the affine structure of the set \(\Delta_0(Y)\). We therefore introduce the following axiom:

**Axiom 16** (Calibration). There exist two lotteries \(e_0\) and \(e_1\) such that:

\(^9\)Strictly speaking the formulation of equation 4 generalizes the model of Giraud (2006) only for the case of constant preference for liquidity. But in fact we could have stated the same result for non-constant preference for liquidity by dropping dominance. We did not do it in order to simplify the statement of the corollary.
(i) $e_1 \succ e_0$:

(ii) for all $p, q \in \Delta_0(Y)$, for all $\lambda \in \mathbb{R}$,

\[
p \otimes \lambda \succ_q q \iff \begin{cases} 
\frac{1}{1+\lambda} p + \frac{\lambda}{1+\lambda} e_0 \succ \frac{1}{1+\lambda} q + \frac{\lambda}{1+\lambda} e_1, & \text{if } \lambda \geq 0; \\
\frac{1}{1-\lambda} p - \frac{\lambda}{1-\lambda} e_1 \succ \frac{1}{1-\lambda} q - \frac{\lambda}{1-\lambda} e_0, & \text{if } \lambda \leq 0.
\end{cases}
\]

The following corollary holds:

**Corollary 8.** The following are equivalent:

(i) $(\succ_r)_{r \in \Delta_0(Y)}$ satisfies Weak Order, Solvability, Monotonicity, Buying Consistency, Dominance, MGConv, MCont, No Regret and Calibration;

(ii) There exists a convex weak*-closed set $\Psi$ of bounded functions from $Y$ to $\mathbb{R}$ such that

a) $b_\succ(p, r) = \inf_{\psi \in \Psi} (\int \psi dp - \int \psi dr)$,

b) $s_\succ(p, r) = \sup_{\psi \in \Psi} (\int \psi dp - \int \psi dr)$,

and, for all $r \in X$, there exists $\alpha_r \in [0, 1]$, unique if $K_r \neq X$, such that, for all $p, q \in \Delta_0(Y)$,

\[
p \succ_r q \iff \alpha_r b_\succ(p, r) + (1 - \alpha_r) s_\succ(p, r) \geq \alpha_r b_\succ(q, r) + (1 - \alpha_r) s_\succ(q, r).
\]

Moreover, $\Psi \subseteq Y^\oplus$ and each $\psi \in \Psi$ is defined up to addition of a constant.

Three remarks about this corollary are in order. First, Sagi (2006) emphasizes the fact that many well known reference dependent models of decision under risk, including Prospect Theory, do not satisfy the “No Regret” axiom and presents an example of a model that does. This corollary provides another example of a model that does satisfy the axiom. Second, in this corollary the link between the willingness to pay and the willingness to accept is more precise than in the previous results, since in this case they are both generated by the same set of univariate utility function, so that the willingness to pay is measured by the smallest expected utility difference between the endowment and the candidate alternative and the willingness to accept by largest expected utility difference. This is consistent with the idea that when stating a buying price in a bargaining process, one would tend to emphasize the liabilities of the candidate alternative, whereas when stating a selling price one will insist on the assets of what one is selling. Third, the fact that a calibration axiom is needed to derive this result is significant of the fact that willingness to pay and willingness to accept are cardinal and monetary notions and thus cannot be represented by sets of utility functions of arbitrary scale.
3.4.7 Application: The Strength of the Status Quo Bias and Its Welfare Interpretation

The status quo bias is the fact that the decision maker has a tendency to stick to the status quo. However, there is no reason for this bias to be of equal strength for any status quo. To model this fact, we shall introduce a comparative definition of the strength of this status quo. To be precise, we shall in fact introduce several definitions, all based on the same intuition: the status quo is stronger for \( r \) than for \( r' \) if, whenever the status quo bias is not strong enough for the decision maker to stick to \( r \), then neither can it be for the decision maker to stick to \( r' \). Formally, we have the following definitions:

**Definition 10.**

- The status quo bias is stronger for \( r \) than for \( r' \) in the first sense (denoted \( r \succ_{SQB1} r' \)) if:
  \[
  \forall x \in X, x \succ r \Rightarrow x \succ_{r'} r'.
  \]

- The status quo bias is stronger for \( r \) than for \( r' \) in the second sense (denoted \( r \succ_{SQB2} r' \)) if:
  \[
  \forall x \in X, x \succ r \Rightarrow x \succ_{r'} r'.
  \]

- The status quo bias is strictly stronger for \( r \) than for \( r' \) (denoted \( r \succ_{SSQB} r' \)) if:
  \[
  \forall x \in X, x \succeq r \Rightarrow x \succ_{r'} r'.
  \]

Let us first give some very obvious properties of these relations (we omit their simple proofs):

**Proposition 15.**

- Assume that for all \( r \in X \), \( \succ_r \) is a weak order. Then, \( \succeq_{SQB1} \) and \( \succeq_{SQB2} \) are reflexive and transitive relations and \( \succ_{SSQB} \) is irreflexive and transitive.

- For all \( r, r' \in X \), \( r \succ_{SSQB} r' \Rightarrow (r \succ_{SQB1} r' \text{ and } r \succ_{SQB2} r') \).

We present some characterizations of these relations in terms of the willingness to pay, in the context of the axioms introduced in the previous section:

**Proposition 16.** Assume that \((\succeq_r)_{r \in X}\) satisfies the axioms of theorem 2. Then, \( r \succeq_{SQB1} r' \iff b_{x}(x,r) \leq b_{x}(x,r'), \forall x \in X. \)

If, in addition, the set \( \{ \lambda \in \mathbb{R} \mid x \ominus \lambda \succ r \} \) is open for all \( x, r \in X \), then
\( r \succeq_{SQB2} r' \iff r \succ_{SSQB} r' \iff b_{x}(x,r) < b_{x}(x,r'), \forall x \in X. \)
3.5 The Third Representation Theorem: Imprecise Valuation

3.5.1 Motivation

It is usual in economics to assume that people know the exact value they attribute to a good (independently of the fact that they are willing to report this value in a sincere way). However, this assumption is but an idealization: in general, people assess the value of a good with a certain amount of imprecision. This imprecision in the valuation of objects is usually attributed to an imperfect ability to discriminate between very similar objects, by analogy to the imperfect discrimination power of human perception of sounds, colors or scents. The consequences of this limitation in human perception were first studied by Luce in a famous article (Luce, 1956) where he introduced the concept of semiorder. This concept was generalized by Fishburn who introduced the concept of interval order (Fishburn, 1970, p.18). Interval orders have the property that, under suitable separability conditions, they can be represented by two functions $u$ and $v$ in the sense that

$$x \succ y \iff u(x) > v(y).$$

Each object $x$ is therefore associated with an interval $[u(x), v(x)]$ that can be seen as an imprecise utility function. However, the interpretation of the functions $u$ and $v$ is not as transparent as the interpretation of standard utility functions. In this section, we which to provide an interpretation for these functions as willingness to pay and willingness to accept, and to the interval $[u(x), v(x)]$ as the set of equivalent monetary valuations for object $x$.

Indeed, imprecise utility is related in two natural way to imprecise monetary valuation. First, it is very likely that an imprecise utility valuation would translate into an imprecise monetary one, as it would be surprising that someone who cannot discriminate clearly between very similar objects would however be able to give them a precise monetary value. Second, money itself is a dimension on which discrimination is not always possible or easy. For instance, one would probably be as happy of buying a given accommodation if its price were 300 000 euros as if it were 299 000 euros, since for an amount of this size a difference of 1000 euros is relatively negligible.

To make this relation precise, we shall therefore introduce axioms that imply that one possible interval utility representation is indeed the interval of values defined by the willingness to pay and the willingness to accept.

3.5.2 Statement

For this third representation theorem, we will take as a primitive the asymmetric part of $\succ_r$, $\succ_r$.

**Axiom 17** (Strict Partial Order). For all $r \in X$, $\succ_r$ is irreflexive and transitive.

**Axiom 18** (Strong Separability). For all $x, y, r \in X$, if $x \succ_r y$, then there exists $\lambda > \mu$ such that

$$x \ominus \lambda \succ_r r \quad \text{and} \quad r \oplus \mu \succ_y y.$$
Axiom 19 (Strict Buying Consistency). For all \((x, r) \in X^2\), for all \(\lambda \in \mathbb{R}\),

\[ x \oplus \lambda \succ_r r \implies x \succ_r r \oplus \lambda. \]

Axiom 20 (Strict Monetary Status quo Bias). For all \((x, r) \in X^2\), for all \(\lambda \in \mathbb{R}\),

\[ x \oplus \lambda \succ_r r \implies x \oplus \lambda \succ_r r. \]

Strict Partial Order has the usual interpretation. Strict Buying Consistency and Monetary Status Quo Bias have the same interpretation as their weak forms. Finally, Strong Separability says that if I strictly prefer \(x\) to \(y\) given endowment \(r\), then there is always a price for which I would be willing to pay for \(x\) and would be willing to sell \(y\) for a lower price. This means that strict preference really means something in terms of price discrimination.

Theorem 3. \(\{\succ_r\}_{r \in X}\) satisfies Strict Partial Order, Strong Separability, Monotonicity, Strict Buying Consistency and Strict MSQB if and only if there exists a unique irreflexive, upper monotonic and upper semicontinuous binary relation \(\succ\) such that

(i) \(b_\succ(x, r) \leq s_\succ(x, r)\), for all \(x, r \in X\);
(ii) \(x \succ r y \iff b_\succ(x, r) > s_\succ(y, r)\);
(iii) \(b_\succ(r, r) = 0\) and \(s_\succ(r, r) = 0\) for all \(r \in X\);
(iv) \(x \succ r r \iff b_\succ(x, r) > 0\).

3.5.3 Interpretaion

The representation found in theorem 3 for binary relation \(\succ\) shows that the axioms imply that it be an interval order. As we said above, interval orders are generalizations of weak orders that are suitable to model imprecise (utility) valuation of objects. Theorem 3 allows to relate imprecise utility valuation to imprecise monetary valuation. Indeed, define first the indifference relation \(\sim\), by

\[ x \sim_r y \iff \neg(x \succ_r y) \text{ and } \neg(y \succ_r x). \]

Then we have the following corollary of theorem 3 (the simple proof is omitted).

Corollary 9. Under the conditions of theorem 3, for all \(\lambda \in \mathbb{R}\),

\[ b_\sim(x, r) \leq \lambda \leq s_\sim(x, r) \iff x \sim_r r \oplus \lambda. \]

Because the indifference relation of an interval order is not necessarily transitive, we cannot conclude from this that \(b_\sim(x, r) = s_\sim(x, r)\), thus maintaining the possibility of the WTA/WTP discrepancy.

The idea that that the decision maker may have only a vague and imprecise idea of the monetary equivalent of a given object is perfectly in agreement with psychological intuition and has important implications for various domains of economic theory, ranging from industrial organization, where monetary equivalents for goods are routinely used, to auction theory, where it is commonly assumed that people
know their valuation for the good sold. This also has implications for experimental economics where auction theory is used to elicit monetary equivalents and for contingent valuation studies in environmental economics.

For the sake of completeness, we show how greater willingness to pay and greater willingness to accept can be characterized in terms of preference only. Define two new binary relations $\succ^b_r$ and $\succ^s_r$ by:

\[
x \succ^b_r y \iff \exists z \in X, x \succ r z, z \sim r y,
\]

and

\[
x \succ^s_r y \iff \exists z \in X, x \sim r z, z \succ r y.
\]

These binary relations can be seen as approximations of the true preference relation: $x$ is approximately preferred to $y$ in the sense of $\succ^b_r$ if it is preferred to an object similar to $y$; likewise, $x$ is approximately preferred to $y$ in the sense of $\succ^s_r$ if there exists an object similar to $x$ that is preferred to $y$. Then:

**Corollary 10.** Under the conditions of theorem 3,

\[
x \succ^b_r y \iff b_r(x, r) > b_r(y, r)
\]

and

\[
x \succ^s_r y \iff s_r(x, r) > s_r(y, r).
\]

### 3.5.4 Application: Preference for Liquidity Again

Now, let us come back to preference for liquidity. It is possible to characterize it in the context of theorem 3, as shown by the next proposition (in the definition of preference for liquidity $x \succ r y$ means $x \succ r y$ or $x \sim r y$):

**Proposition 17.** Assume $\{\succ^1_r\}_{r \in X}$ and $\{\succ^2_r\}_{r \in X}$ satisfy the axioms of theorem 3. Then, the following are equivalent:

(i) 1 has a stronger preference for liquidity at $r$ than 2 at $r'$.

(ii) For all $x \in X$, $b_r(x, r) \leq b_{r'}(x, r')$ and $s_r(x, r) \leq s_{r'}(x, r')$.

In words, this proposition says that if agent 1 has greater preference for liquidity than 2, his or her buying price for a given item will systematically be lower than agent 2’s buying price, and so will his or her selling price. This is intuitive since, when I sell a good, I lose the item I possess but I am able to enjoy liquidity. Therefore, the greater my preference for liquidity, the smaller will the selling price need to be in order for the pleasure enjoyed from liquidity to compensate the pain incurred from losing the object. A similar reasoning can be done for the buying price: if a buy a good, I lose the advantages of liquidity. The higher these advantages for me, the less will my willingness to abandon them be, and therefore the less my willingness to pay.
3.5.5 Application: The Strength of the Status Quo Bias

The strength of the status quo bias can also be characterized in the context of theorem 3. This characterization is of the same nature than in the context of theorem 2 and is proved similarly.

**Proposition 18.** Assume that $(\succ_r)_{r \in X}$ satisfies the axioms of theorem 3. Then,

$$r \succeq_{SQB2} r' \iff b_r(x, r) \leq b_r(x, r'), \quad \forall x \in X.$$

4 Concluding Remarks

We have developed here a framework that allows to precisely state the relationship between leading concepts of the theory and empirical research on reference-dependent preferences, namely the status quo bias, loss aversion and the endowment effect. Loss aversion and the endowment effect can be seen, of at least are often expressed as, cardinal concepts. We have shown that, essentially, the status quo bias (under some form or another) is their ordinal counterpart, just as convexity of preferences is the ordinal counterpart of decreasing marginal utility. As a by-product, we provided a comparative definition of the notion of preference for liquidity, based on ideas from the theory of risk aversion and ambiguity aversion (preference for liquidity can indeed be seen as a form of illiquidity aversion) and characterized it in terms of a single parameter that relates utility, on the one hand, and WTP and WTA, on the other hand. Studying this parameter leads to predictions on the preference reversal phenomenon.

Many research paths can be followed at this point.

The first and foremost is to test the above mentioned predictions linking preference for liquidity and preference reversals.

The second is to apply our results to fields of the economic literature where notions like the willingness to pay play an important role, such as auction theory, electoral competition theory and models of product differentiation, where it might be of interest to study the impact of introducing reference-dependence and status quo effects.

The third research path is related to the formal framework we introduced. This framework can be generalized to allow for different notions of money, i.e. for different assets circulating between the objects of choice: one can think of time, information or language transformations. Replacing $\mathbb{R}$ by another group or semi-group and showing how this can lead to fruitful analysis of these other contexts is thus the object of our future research.

A Proofs

A.1 Results on r.m.s.

*Proof of proposition 1. (ii) $\implies$ (i): Trivial.*
(i) $\implies$ (ii): Consider the relation $\approx$ defined on $X$ by

$$x \approx y \iff \exists \lambda \in \mathbb{R}, \ y = x \oplus \lambda.$$ 

Clearly this is an equivalence relation. The equivalence classes are called orbits. Let $A$ be the set of orbits, i.e. $A := X/\approx$. For each $a \in A$, by the axiom of choice, it is possible to fix $x_a \in a$. We will consider that it is fixed for the rest of the proof.

Now take $x \in X$. There exists a unique $a \in A$ such that $x \in a$. There exists therefore by definition a $w \in \mathbb{R}$ such that $x = x_a \oplus w$. Let therefore $\varphi(x) = (a, w)$. Does this properly define a mapping? If $w'$ is such that $x = x_a \oplus w'$, then, because $X$ is a free r.m.s., $w = w'$. Therefore, $\varphi$ is well-defined. It is clearly one-to-one. To see that it is onto, take $(a, w) \in A \times \mathbb{R}$. Then $\varphi(x_a \oplus w) = (a, w)$. Therefore $\varphi$ is bijective. Let us show that it is a morphism. Take $x \in X$ and $\lambda \in \mathbb{R}$. Then, by definition, if $x \in a$, then $x \oplus \lambda \in a$ as well. Moreover, if $x = x_a \oplus w$, then $x \oplus \lambda = x_a \oplus (w + \lambda)$. Therefore,

$$\varphi(x \oplus \lambda) = (a, w + \lambda) = (a, w) \oplus \lambda = \varphi(x) \oplus \lambda.$$ 

This completes the proof.

**Proof of proposition 2.** Assume $X$ is free. Then, $X$ is isomorphic to $A \times \mathbb{R}$ for some set $A$. Therefore, w.l.o.g. we can write $x = (a, w)$ for a unique pair $(a, w)$.

Define $f : X \to \mathbb{R}$ letting $f(x) = w$. Then for all $\lambda \in \mathbb{R}, f(x \oplus \lambda) = f(a, w + \lambda) = w + \lambda = f(x) + \lambda$. Therefore, $f \in X^\oplus$.

Conversely, if $X^\oplus \neq \emptyset$, take $f \in X^\oplus$, $x \in X$ and $\lambda \in \mathbb{R}$ such that $x \oplus \lambda = x$. Then, $f(x \oplus \lambda) = f(x) + \lambda = f(x)$, so that $\lambda = 0$.

**A.2 Other results**

**Proof of Proposition 3.** (i) $\implies$ (ii) Define $\succ$ by $x \succ y$ if and only if $b(x, y) \geq 0$.

Then by the Translation property $B_\succ(x, r) = \{\lambda \mid b(x, r) \geq \lambda\}$, and $b(x, r) \in B_\succ(x, r)$, therefore $b_\succ(x, r) = b(x, r)$.

(i) $\implies$ (iii) Defined $\mathcal{A}_r = \{x \in X \mid x \succ r\}$, with $\succ$ as above. Notice that the Translation Property implies that $\mathcal{A}_r$ is an acceptability set since $\lambda \geq 0$ and $x \in \mathcal{A}_r$ imply that $b(x \oplus \lambda, r) = b(x, r) + \lambda \geq 0$ and moreover $b(x \oplus (b(x, r) + 1), r) = -1 < 0$ so that $\mathcal{A}_r$ must be a proper subset of $X$. Now by the previous argument $b(x, r) = b_\succ(x, r)$, and $b_\succ(x, r) = -\rho(x, r)$, so that $b(x, r) = -\rho(x, r)$. 

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(ii) ⇒ (i) If $b = b_x$, then

$$b(x \oplus \lambda, r) = b_x(x \oplus \lambda, r)$$

$$= \sup \{ \lambda' \in \mathbb{R} \mid (x \oplus \lambda) \ominus \lambda' \succ r \}$$

$$= \sup \{ \lambda' \in \mathbb{R} \mid x \ominus (\lambda - \lambda') \succ r \}$$

$$= \sup \{ \lambda' \in \mathbb{R} \mid x \ominus (\lambda' - \lambda) \succ r \}$$

$$= \sup \{ \mu \in \mathbb{R} \mid x \ominus \mu \succ r \} = \lambda + \sup \{ \mu \in \mathbb{R} \mid x \ominus \mu \succ r \} + \lambda$$

$$= b_x(x, r) + \lambda.$$ 

(iii) ⇒ (i) Proved similarly. □

**Proof of Lemma 1.** If Assume $\succ$ is upper monotonic and upper semicontinuous and take $x, y \in X$ such that $b_x(x, y) \geq 0$. By upper semi-continuity, $x \ominus b_x(x, y) \succ y$. Therefore, since $-b(x, y) \leq 0$, if follows by upper monotonicity that $x \succ y$.

**Only if** Let $(\lambda_n)$ be a sequence converging to $\lambda \in \mathbb{R}$ and such that $x \ominus \lambda_n \succ y$. Then, by definition $b_x(x, y) \geq -\lambda_n$, and therefore $b_x(x, y) \geq -\lambda$ i.e. $b_x(x \ominus \lambda, y) \geq$ and therefore, since $b_x$ is exact, $x \ominus \lambda \succ y$. Therefore $\succ$ is upper semicontinuous. To show upper monotonicity, take $x, y \in X, \lambda, \mu \in \mathbb{R}$, such that $x \ominus \mu \succ y$ and $\lambda \geq \mu$. Then $b_x(x, y) \geq -\mu \geq -\lambda$, therefore since $b_x$ is exact, $x \ominus \lambda \succ y$. □

**Proof of Proposition 4.** We begin by the existence part of the theorem.

By Solvability, there exists $\lambda \in \mathbb{R}$ such that: $x \sim r \ominus \lambda$. Monotonicity implies that it is unique.

Moreover, Monotonicity and Weak order imply that, for all $\lambda, \mu \in \mathbb{R}$,

$$\lambda \geq \mu \iff r \ominus \lambda \succ r \ominus \mu.$$ 

Indeed, the first implication follows immediately from Monotonicity if $\lambda > \mu$ and from reflexivity if $\lambda = \mu$. Conversely, if $r \ominus \lambda \succ r \ominus \mu$, it is impossible to have $\mu > \lambda$ because, by Monotonicity, this would imply $r \ominus \mu \succ r \ominus \lambda$, which, by weak order, is contrary to the assumption.

It suffices therefore to let $u_r(x) := \lambda$. This clearly defines a normalized family of utility functions. This completes the proof of sufficiency. Necessity is straightforward.

Now for the uniqueness part. Sufficiency is straightforward. We prove necessity. Assume w.l.o.g. that $u_r(r) = 0$ for all $r \in X$ so that $(u_r)$ is normalized. Then,

$$u_r(r \ominus u_r(x)) = u_r(x),$$

and therefore since $u_r$ is a utility function, $x \sim_r r \ominus u_r(x)$. Therefore,

$$v_r(x) = v_r(r \ominus u_r(x)) = v_r(r) + u_r(x).$$

Define $\gamma(r) = v_r(r)$ □
Proof of Theorem 1. We begin by the existence part.

Sufficiency of the axioms: Define $\succ$ by
\[ x \succ y \iff x \succeq y. \]
Reflexivity of $\succ$ follows from reflexivity of $\succeq$ for all $r \in X$; upper monotonicity and upper semi-continuity will follow from lemma 1 once proved that $b_\succ(.,r)$ is a utility function for $\succeq_r$, since this will imply that $b_\succ$ is exact.
For notational simplicity, we let $b(x,r) := b_\succ(x,r)$ for all $(x,r) \in X^2$. Throughout the proof, we fix $r \in X$.
Consider the utility function defined in the proof of proposition 4. It suffices to show that $u_r(x) = b(x,r)$.
By Income Independence: $x \oplus u_r(x) \sim_r r$. Therefore:
\[ b(x,r) \geq u_r(x). \]
To show the reverse inequality, take $\mu \in \mathbb{R}$. Then, as $x \sim_r r \oplus u_r(x)$, we have, by Income Independence again,
\[ x \oplus \mu \sim_r r \oplus (u_r(x) - \mu). \]
Therefore, by Monotonicity and Weak Order,
\[ x \oplus \mu \succeq_r r \implies u_r(x) - \mu \geq 0, \]
hence $b(x,r) \leq u_r(x)$. This completes the proof of sufficiency.

Necessity of the axioms: Assume there exists a binary relation $\succ$ such that $b_\succ(.,r)$ is finite for all $r$ and represents $\succeq_r$ for all $r$. Then Weak Order is trivially satisfied. The other axioms follow from the translation property of $b_\succ$.
Indeed, Monotonicity follows immediately, as
\[ \lambda > \mu \implies b_\succ(r,r) + \lambda > b_\succ(r,r) + \mu \implies b_\succ(r \oplus \lambda, r) > b_\succ(r \oplus \mu, r) \implies r \oplus \lambda \succ_r r \oplus \mu. \]
Income independence trivially follows from the translation property.
Finally, for Solvability, $b_\succ(r \oplus (b_\succ(x,r) - b_\succ(r,r)), r) = b_\succ(r,r) + b_\succ(x,r) - b_\succ(r,r) = b_\succ(x,r)$, hence
\[ x \sim_r r \oplus (b_\succ(x,r) - b_\succ(r,r)). \]
Now for uniqueness. We first prove sufficiency. Let $\gamma : X \to \mathbb{R}$ be a function and consider $b_\gamma := b_\succ \gamma$ and denote $b_\gamma := b$, where $\succ$ is defined as above. Then $b_\gamma(x,r) = b(x,r) + \gamma(r)$ and therefore is a utility function. Moreover it is easily seen that $b_\gamma$ is exact, and therefore $\succ_\gamma$ is upper monotonic and upper semi-continuous. Finally, $\succ_\gamma$ is reflexive since $\gamma(r) \geq 0$ and therefore by Monotonicity $r \oplus \gamma(r) \succ_r r$. Now for necessity, let $b_\succ' := b'$. By the proposition 4, there exists a function $\gamma :
$X \rightarrow \mathbb{R}$ such that $b'(x, r) = b(x, r) + \gamma(r)$; $\gamma(r)$ is nonnegative since, because $\succeq'$ is reflexive, $r \succeq' r \Rightarrow b'(r, r) = \gamma(r) \geq 0$. Now, since $\succeq'$ is upper monotonic and upper semi-continuous, it follows from lemma 1 that $b'$ is exact, and therefore:

$$
\begin{align*}
x \succeq' y & \iff b'(x, y) \geq 0 \\
& \iff b(x, y) + \gamma(y) \geq 0 \\
& \iff b(x \oplus \gamma(y), y) \geq 0 \\
& \iff x \oplus \gamma(y) \succeq y.
\end{align*}
$$

The ordering property on relations $\succeq'$ follows immediately from the Translation Property and the representation result. Moreover, if

$$
b_{\succeq}(x, r) \geq 0 \iff x \succeq_r r,
$$

then since $\succeq$ is reflexive, upper monotonic and upper semicontinuous, $b_{\succeq}$ is exact and therefore

$$
x \succeq r \iff x \succeq_r r.
$$

**Proof of Corollary 3.** Sufficiency is a straightforward corollary of theorem 1. The only thing to prove is necessity of restricted stochastic dominance. If preferences can be represented by Value at Risk, then $\succeq_P$ is a binary relation such that $b_{\succeq_P}$ represents preferences. It is obviously reflexive and upper monotonic because cdf is nonnegative. It remains to show that it is upper semi-continuous. Take any normalized family $(\lambda_n)$ converging to $\lambda$ such that $Z + \lambda_n \succeq_P Z'$. Therefore, $P(Z + \lambda_n \leq 0) \leq P(Z' \leq 0)$. In other words, $F_Z(-\lambda_n) \leq F_{Z'}(0)$, where $F_Z$ is the cdf of $Z$. By continuity therefore, $F_Z(-\lambda) \leq F_{Z'}(0)$, i.e. $Z + \lambda \succeq_P Z'$.

It is now possible to apply the uniqueness part of theorem 1 to find a function $\gamma : X \rightarrow \mathbb{R}^+$ such that $\succeq_P = \succeq^\gamma$. In fact, inspection of the proof of proposition 4 and theorem 1 reveals that $\gamma(Z) = b_{\succeq_P}(Z, Z) = -\text{VaR}(Z, Z)$, and this completes the proof.

**Proof of Proposition 5.** (i) $\Rightarrow$ (iv). Let $(x, r) \in X^2$. Take first $\lambda \geq 0$ such that $x \succeq_r r \oplus \lambda$ and take a normalized family of utility functions $(u_r)$. Then, $u_x(x) \geq \lambda \geq 0 = u_r(r)$. By loss aversion therefore $-u_x(r) \geq u_r(x) \geq \lambda$, so that $u_x(r) \leq -\lambda$, i.e. $u_x(r) \leq u_x(x \oplus \lambda)$: $x \oplus \lambda \succeq_x r$. Now take $\lambda < 0$ such that $r \succ_x x \ominus \lambda$. Then, $u_x(r) > -\lambda > 0 = u_x(x)$. Therefore, by loss aversion, $-u_r(x) > -\lambda$, so that $u_r(x) < \lambda = u_r(r \oplus \lambda)$: $r \oplus \lambda \succ_r x$. Therefore, for $\lambda < 0$ the contrapositive holds.

(iv) $\Rightarrow$ (iii). Take any normalized family $(u_r)$, $x, y \in X$ and $\lambda \in \mathbb{R}$. Then $y \oplus \lambda \sim_x x \oplus u_x(y \oplus \lambda)$. Then:

$y \oplus \lambda \sim_x x \oplus u_x(y \oplus \lambda) \quad \Rightarrow \quad y \oplus (\lambda - u_x(y \oplus \lambda)) \preceq_y x \quad \text{by SMSQB}$

$y \oplus (\lambda - u_x(y \oplus \lambda)) \preceq y \oplus u_y(x) \quad \text{by Solvability}$

$\lambda - u_x(y \oplus \lambda) \geq u_y(x) \quad \text{by Monotonicity}$

$u_x(x) - u_x(y \oplus \lambda) \geq u_y(x) - u_y(x \oplus \lambda)$.

(iii) $\Rightarrow$ (ii). Straightforward.
(ii) $\Rightarrow$ (i). Straightforward.

**Proof of Theorem 2.**

**Sufficiency of the axioms:** By proposition 4, there exists a unique normalized family of utility functions $(u_r)_{r \in X}$. Define $\succ$ as usual by $x \succ y \iff x \succ y$ and let $s := s_\succ$ and $b := b_\succ$. We want to show now that $s(x, r) \geq u_r(x) \geq b(x, r)$ for all $x, r \in X$.

Take $\mu$ such that $r \oplus \mu \succeq x$. Then, by MSQB, $r \oplus \mu \succ r x$. But $x \sim r \oplus u_r(x)$. Therefore, by Weak Order, $r \oplus \mu \succ r \oplus u_r(x)$ and, by Monotonicity, $\mu \geq u_r(x)$ and, finally, $s(x, r) \geq u_r(x)$.

Take $\lambda$ such that $x \ominus \lambda \succeq_r r$. Then, by Buying Consistency, $x \succ r \oplus \lambda$. But $x \sim_r r \oplus u_r(x)$. Therefore, by Weak Order, $r \oplus u_r(x) \succeq_r r \oplus \lambda$ and, by Monotonicity, $u_r(x) \geq \lambda$ and, finally, $u_r(x) \geq b(x, r)$.

As for the second claim, assume first $u_r(x) \geq 0$. Then $x \succ_r r$, implying, by definition of $b(x, r)$, $b(x, r) \geq 0$. But, since $b(x, r) \leq u_r(x)$, we have $u_r(x) \geq 0$. This shows in particular by lemma 1 that $\succ$ is reflexive, upper monotonic and upper semicontinuous.

Uniqueness of $\succ$ follows from the fact that if $\succ'$ is another reflexive, upper monotonic and upper semicontinuous binary relation satisfying the properties of the theorem, then, by lemma 1 and by property (ii),

$$x \succ' y \iff b_\succ'(x, y) \geq 0 \iff x \succ y.$$

**Necessity of the axioms:** Assume the conclusion of the theorem holds. Given proposition 4, the only thing to prove is that MSQB and Buying Consistency hold.

Let us turn to MSQB. Assume $x \ominus \lambda \succeq_r r$. Then, $s(r, x) \leq \lambda$. But $s(r, x) \geq u_x(r)$, so this implies $u_x(r) \leq \lambda = u_x(x \ominus \lambda)$, that is $x \ominus \lambda \succeq_x r$.

For Buying Consistency, assume $x \ominus \lambda \succeq_r r$. Then $b(x, r) \geq \lambda$. But, as $u_r(x) \geq b(x, r)$, it follows that $u_r(x) \geq \lambda = u_r(r \ominus \lambda)$: $x \succ_r r \ominus \lambda$.

**Proof of Proposition WI.** In all this proof we consider a normalized family $(u_r)$ of utility functions.

(i) If. The result follows from the following lemma:

**Lemma 2.** If $\{\succ_r\}_{r \in X}$ satisfies Weak Order, Solvability and Monotonicity, Buying Consistency and Wealth Independence, then

$$x \ominus \lambda \succ_r r \iff x \succ_r r \ominus \lambda.$$

**Proof.** One direction is Buying consistency and the other one is proved as follows:

$$x \succ_r r \ominus \lambda \implies x \succ_{r \ominus \lambda} r \ominus \lambda \quad \text{by Wealth Independence}$$

$$\implies x \ominus \lambda \succ_{r \ominus \lambda} r \quad \text{by Buying Consistency}$$

$$\implies x \ominus \lambda \succ_r r \quad \text{by Wealth Independence again}$$

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Now, for all \( x, r \in X \), \( x \sim_r r \oplus u_r(x) \), therefore, by the previous lemma, \( x \oplus u_r(x) \succ_r r \), and therefore \( b_\succ(x, r) \geq u_r(x) \), where \( \succ \) is defined as in theorem 1. Conversely, if \( \lambda \) is such that \( x \ominus \lambda \succ_r r \), then by buying consistency \( x \succ_r r \ominus \lambda \), and therefore \( u_r(x) \geq \lambda \), and as these holds for any such \( \lambda \), \( u_r(x) \geq b_\succ(x, r) \). Therefore \( b_\succ \) is a utility function, so we conclude by theorem 1.

**Only if.** Income Independence obviously implies Buying Consistency. Moreover, if WMSQI holds, then by proposition 7 and theorem 1, \( s_\succ (\cdot, r) \) is a utility function for \( \succ_r \), so the result follows from the Translation Property for \( s_\succ \).

**(ii) SMSQB \Rightarrow MSQB.** Take \( x, r \in X \) and \( \lambda \in \mathbb{R} \) such that \( x \ominus \lambda \succ_r r \). Then, \( u_r(x \ominus \lambda) \geq 0 \), and therefore, since \( b \) is crisp at \( r \ominus \lambda \), we omit the reference to \( u \).

\( \Rightarrow s \): Conversely, assume that all the conditions of theorem 2 hold and that \( WMSQI \) holds. Then this implies that \( b(k, r) = u_r(k) \). Let us first prove condition (i). Since Buying Consistency holds, we only have to prove that \( k \succ_r r \ominus \lambda \) implies \( k \ominus \lambda \succ_r r \). This follows from the Translation Property.

\( \Rightarrow \): Conversely, assume that all the conditions of theorem 2 hold and that \( b(k, r) = s(k, r) \). Then this implies that \( b(k, r) = s(k, r) \). Let us first prove condition (i). Since Buying Consistency holds, we only have to prove that \( k \succ_r r \ominus \lambda \) implies \( k \ominus \lambda \succ_r r \). This follows from the Translation Property.

**Proof of proposition 9.** Since \( \{\succ_r\}_{r \in X} \) satisfies Weak Order, Solvability and Monotonicity, fix a normalized family of utility functions \( (u_r)_{r \in X} \).

Let \( k \) be crisp at \( r \). We will show that \( b_\succ(k, r) = u_r(k) = s_\succ(k, r) \). For simplicity we omit the reference to \( \succ \). Take \( \lambda \in \mathbb{R} \) such that \( k \ominus \lambda \succ_r r \). Then \( k \succ_r r \ominus \lambda \), therefore \( r \oplus u_r(k) \succ_r r \ominus \lambda \), so that \( u_r(k) \geq \lambda \), yielding \( u_r(k) \geq b(k, r) \). Conversely, \( k \sim_r r \oplus u_r(k) \) implies \( k \ominus u_r(k) \succ_r r \), so that \( u_r(k) \geq b(k, r) \). Therefore \( b(k, r) = u_r(k) \).

To show that \( s(k, r) = u_r(k) \), note that \( k \sim_r r \ominus u_r(k) \) implies that \( r \ominus u_r(k) \succ_r r \) and therefore, since \( k \) is crisp at \( r \), \( r \ominus u_r(k) \succ_k k \), we have \( r \ominus u_r(k) \succ_k r \), so that \( s(k, r) \) is defined as such.

**Proof of Proposition 10.** (i) \( \Rightarrow \) (ii) Take \( \{\succ^1_r\}_{r \in X} \) and \( \{\succ^2_r\}_{r \in X} \) satisfying the axioms of theorem 2 such that 1 has a stronger preference for liquidity at \( r \).
than 2 at \( r' \) and \( x \in X \setminus (K^1_r \cup K^2_r) \) satisfying the premises of the definition of preference for liquidity. Then, by lemma 3 below, \( b_{\geq 1}(x, r) = b_{\geq 2}(x, r') \) and \( s_{\geq 1}(x, r) = s_{\geq 2}(x, r') \):

**Lemma 3.** Under the assumptions of theorem 2, for all \( x, r \in X \),

\[
B(x, r) = (-\infty, b_{\geq}(x, r)],
\]

\[
S(x, r) = [s_{\geq}(x, r), +\infty),
\]

\[
\{\lambda \in \mathbb{R} \mid x \succ_r \lambda \} = (-\infty, u_r(x)]
\]

with \((u_r)_r \in X\) a normalized family of utility functions.

**Proof.** Let \( \lambda \in B(x, r) \). Then \( \lambda \leq b_{\geq}(x, r) \). Conversely, take \( \lambda \leq b_{\geq}(x, r) \). Then \( b_{\geq}(x \oslash \lambda, r) \geq 0 \), implying \( x \oslash \lambda \succ_r r \) according to theorem 2. Therefore \( \lambda \in B(x, r) \).

Consider now \( \mu \in S(x, r) \). Then by definition \( s_{\geq}(x, r) \leq \mu \). Conversely, take \( \mu \geq s_{\geq}(x, r) \). Then \( -\mu \leq -s_{\geq}(x, r) = b_{\geq}(r, x) \), therefore \( b_{\geq}(r \oplus \mu, x) \geq 0 \), i.e. \( r \oplus \mu \gtrless_r x \): \( \mu \in S(x, r) \).

Finally, \( x \succ_r r \oplus \lambda \) iff \( \lambda \leq u_r(x) \) by definition of \( u_r(x) \).

Now, we have that \( x \sim^1_r r \oplus u^1_r(x) \). Therefore, \( x \succ^2_r r' \oplus u^1_r(x) \), so that \( u^2_r(x) \geq u^1_r(x) \), i.e. \( a^1_r(x) \geq a^2_r(x) \) since \( b_{\geq 1}(x, r) = b_{\geq 2}(x, r') \) and \( s_{\geq 1}(x, r) = s_{\geq 2}(x, r') \).

(ii) \( \implies \) (i) Take \( x \in X \) such that for all \( \lambda \in \mathbb{R} \),

\[
x \oplus \lambda \succ^1_r r \iff x \oplus \lambda \succ^2_r r'
\]

and

\[
r \oplus \lambda \succ^1_r x \iff r' \oplus \lambda \succ^2_r x
\]

Then \( b_{\geq 1}(x, r) = b_{\geq 2}(x, r') \) and \( s_{\geq 1}(x, r) = s_{\geq 2}(x, r') \). Therefore, if \( x \notin (K^1_r \cup K^2_r) \), \( a^1_r(x) \geq a^2_r(x) \), and, since \( b_{\geq 1}(x, r) < s_{\geq 1}(x, r) \),

\[
x \succ^1_r r \oplus w \iff a^1_r(x)(b_{\geq 1}(x, r) - s_{\geq 1}(x, r)) + s_{\geq 1}(x, r) \geq w
\]

\[
\implies a^2_r(x)(b_{\geq 1}(x, r) - s_{\geq 1}(x, r)) + s_{\geq 1}(x, r) \geq w
\]

\[
\implies a^2_r(x)(b_{\geq 2}(x, r') - s_{\geq 2}(x, r')) + s_{\geq 2}(x, r') \geq w
\]

\[
\implies x \succ^2_r r' \oplus w.
\]

If \( x \in K^1_r \cup K^2_r \), then \( b_{\geq 1}(x, r) = b_{\geq 2}(x, r') = s_{\geq 2}(x, r') = s_{\geq 1}(x, r) \). Therefore,

\[
x \succ^1_r r \oplus w \iff b_{\geq 1}(x, r) \geq w \iff b_{\geq 2}(x, r') \geq w \iff x \succ^2_r r' \oplus w.
\]

The case of strict preference can be proved similarly. 

\[\boxed{}\]
**Proof of Proposition 12.** Let \( r \in X \), and \( x, y \in X \setminus K_r \) such that \( x \succeq_r y \). Then there exists \( m > 0 \), \( p \in \mathbb{R} \) such that

\[
\begin{align*}
  b_\succ(y, r) &= m b_\succ(x, r) + p, \\
  s_\succ(y, r) &= m s_\succ(x, r) + p, \\
  u_r(y) &= m u_r(x) + p
\end{align*}
\]

Therefore, by definition,

\[
a_r(y) = \frac{u_r(y) - s_\succ(y, r)}{b_\succ(y, r) - s_\succ(y, r)} = \frac{mu_r(x) + p - ms_\succ(x, r) - p}{mb_\succ(x, r) + p - ms_\succ(x, r) - p} = \frac{u_r(x) - s_\succ(x, r)}{b_\succ(x, r) - s_\succ(x, r)} = a_r(x). \]

**Proof of Proposition 13.** Given corollary 5, the only thing to prove is that \( a_r(x) = a_r(y) \) for all \( x, y, r \) with \( x, y \notin K_r \), if and only if Dominance holds.

**Sufficiency of Dominance** By proposition 12, it suffices to show that Dominance implies that all \( x, y \notin K_r \) are affinely related at \( r \). Take \( x, y \notin K_r \). Since \( b(x, r) < s(x, r) \) and \( b(y, r) < s(y, r) \), there exists \( m > 0 \) and \( p \) such that

\[
b_\succ(y, r) = m b_\succ(x, r) + p \quad \text{and} \quad s_\succ(y, r) = m s_\succ(x, r) + p.
\]

Therefore, for all \( \lambda \in \mathbb{R} \):

- (i) \( y \oplus (m\lambda + p) \succ_r r \iff x \oplus \lambda \succ_r r \);
- (ii) \( r \oplus \lambda \succ_x x \iff r \oplus (m\lambda + p) \succ_y y \).

Then, two applications of Dominance show (using Monotonicity and transitivity) that, if \( x \sim_r r \oplus \tau \) and \( y \sim_r r \oplus \tau' \), then \( \tau' = m\tau + p \). This shows that the utility function \( u_r \) in the proof of theorem 2 satisfies \( u_r(y) = mu_r(x) + p \). But this implies that \( x \) and \( y \) are affinely related at \( r \), and therefore that \( a_r(x) = a_r(y) =: a_r \).

**Necessity of Dominance** Take \( r \in X \), \( x, y \in X \) that satisfy the premises of Dominance for some \( m > 0 \) and \( p \in \mathbb{R} \). Then, it is easy to show that

\[
b_\succ(y, r) \leq m b_\succ(x, r) + p \quad \text{and} \quad s_\succ(y, r) \leq m s_\succ(x, r) + p.
\]

But this implies that

\[
\begin{align*}
u_r(y) &= a_r b_\succ(y, r) + (1 - a_r) s_\succ(y, r) \\
&\leq m(a_r b_\succ(x, r) + (1 - a_r) s_\succ(x, r)) + p \\
&= mu_r(x) + p.
\end{align*}
\]

Therefore, since \( x \sim_r r \oplus u_r(x) \) and \( y \sim_r r \oplus u_r(y) \), if \( x' \sim_r r \oplus (mu_r(x) + p) \), then by monotonocity \( x' \sim_r r \oplus (mu_r(x) + p) \succ_r r \oplus u_r(y) \sim_r y \) and we conclude by transitivity.

**Proof of Proposition 14.** The proof relies on the following lemma:
Lemma 4. If $(\succeq_r)_{r \in X}$ satisfies all the conditions of proposition 13, then for all $x, y, r \in X$

$$(x, y) \in PR((\succeq_r)_{r \in X}, r) \iff \begin{cases} s_{\succeq}(y, r) - s_{\succeq}(x, r) > 0 \\ b_{\succeq}(x, r) - b_{\succeq}(y, r) > 0 \\ \alpha_r > \bar{\alpha}_r(x, y) := \frac{s_{\succeq}(y, r) - s_{\succeq}(x, r)}{s_{\succeq}(y, r) - s_{\succeq}(x, r) + b_{\succeq}(x, r) - b_{\succeq}(y, r)}. \end{cases}$$

Proof. $\Rightarrow$: If $(x, y) \in PR((\succeq_r)_{r \in X}, r)$, then by definition $s_{\succeq}(y, r) - s_{\succeq}(x, r) > 0$. Moreover, if $b_{\succeq}(y, r) - b_{\succeq}(x, r) \geq 0$, then by proposition 13, $y \succ_r x$, a contradiction. Now, a simple computation shows that $x \succ_r y \Rightarrow \alpha_r > \bar{\alpha}_r(x, y)$.

$\Leftarrow$: $b_{\succeq}(y, r) - b_{\succeq}(x, r) < 0$ and $s_{\succeq}(y, r) - s_{\succeq}(x, r) > 0$ implies that $s_{\succeq}(y, r) - s_{\succeq}(x, r) + b_{\succeq}(x, r) - b_{\succeq}(y, r) > 0$, and therefore a simple computation shows that $\alpha_r > \bar{\alpha}_r(x, y) \Rightarrow x \succ_r y$.

Now assume that $(x, y) \in PR((\succeq_r^2)_{r \in X}, r)$. Then by the previous lemma,

$$\begin{cases} s_{\succeq}(y, r) - s_{\succeq}^2(x, r) > 0 \\ b_{\succeq}^2(x, r) - b_{\succeq}^2(y, r) > 0 \\ \alpha_r^2 > \bar{\alpha}_r^2(x, y). \end{cases}$$

But since 1 and 2 differ only by their preference for liquidity, this implies

$$\begin{cases} s_{\succeq}^1(y, r) - s_{\succeq}^1(x, r) > 0 \\ b_{\succeq}^1(x, r) - b_{\succeq}^1(y, r) > 0 \\ \alpha_r^1 > \bar{\alpha}_r^1(x, y). \end{cases}$$

and since $\alpha_r^1 \geq \alpha_r^2$, we have that $\alpha_r^1 > \bar{\alpha}_r^1(x, y)$, and therefore

$$(x, y) \in PR((\succeq_r^1)_{r \in X}, r).$$

Proposition of Corollary 6. Given proposition 13, it suffices to prove necessity and sufficiency of MGCont and MConv for points $a), b)$ and $c)$. Since, by theorem 2, $x \ominus \lambda \succeq_r r \Leftrightarrow b_{\succeq}(x, r) \geq \lambda$, MGConv and MCont hold if and only if $b_{\succeq}$ is a concave and upper semi continuous function, which holds, in a locally convex Hausdorff space, if and only if $b_{\succeq}(x, r) = \inf_{\beta \in \mathcal{B}} \beta(x, r)$, where $\mathcal{B}$ is defined as follows:

$$\mathcal{B} := \{ \beta \in \mathcal{M}(X^2) \mid \beta(x, r) \geq b_{\succeq}(x, r), \text{ for all } (x, r) \in X^2 \}.$$ 

Since, moreover, $s_{\succeq}(x, r) = -b_{\succeq}(r, x)$, we have that $s_{\succeq}(x, r) = -\inf_{\beta \in \mathcal{B}} \beta(r, x) = \sup_{\beta \in \mathcal{B}} -\beta(r, x)$.

Proof of Corollary 7. $\Delta_0(Y)$ is a r.m.s. as shown in example 3. It is free: since $Y$ is free, there exists a morphism $\varphi : Y \rightarrow \mathbb{R}$. Let $\Phi : X \rightarrow \mathbb{R}$ be defined by $\Phi(p) = \int \varphi dp$. Then $\Phi$ is also a morphism, and $X$ is therefore free; it is convex. Therefore, it satisfies all the requirements of the previous corollary. For all $\beta \in \mathcal{B}$ and $y, y' \in Y$, let $v_{\beta}(y(y')) = \beta(\delta_y, \delta_{y'})$, where $\delta_y$ is the Dirac measure at $y$, and $w_{\beta}(y(y')) = -v_{\beta}(y', y)$. Then $v_{\beta}$ and $w_{\beta}$ are bounded as images of a bounded subset by a norm-continuous affine functional.
Proof of Corollary 8. Notice first that No Regret trivially implies MSQB, so that it is possible to apply corollary 6. We only need to show that No Regret and Calibration are necessary and sufficient for the specific form for $b_\succ$ and $s_\succ$.

For sufficiency, notice that axiom NR implies that the binary relation $\succ$ is transitive, and it is reflexive by axiom Weak Order, therefore it is a preorder. Moreover, $\succ$ satisfies independence by axiom MGConv and norm continuity as defined by Evren (2005) by axiom MCont (in fact this axiom implies that $\succ$ has closed graph in the norm topology. Therefore it is possible to apply theorem 3.4 in Evren (2005) to find a set $\Phi$ of bounded utility functions such that

$$b_\succ(p, q) \geq 0 \iff p \succ q \iff \int \varphi dp \geq \int \varphi dq, \forall \varphi \in \Phi. \quad (5)$$

We will now show that

$$b_\succ(p, q) = \inf_{\psi \in \Psi} \int \psi dp - \int \psi dq,$$

for some set $\Psi$ of bounded functions. In the sequel, to simplify notations, we let $\varphi(p) = \int \varphi dp$ for $p \in \Delta_0(Y)$.

We first show that $\varphi(e_1) > \varphi(e_0)$ for all $\varphi \in \Phi$, where $e_0$ and $e_1$ come from the Calibration axiom. Notice first that, since $e_1 \succ_{e_0} e_0$, it is possible to suppose that $\Phi$ does not contain any constant function. Now, since $e_1 \succ_{e_0} e_0$, we have $\varphi(e_1) \geq \varphi(e_0)$ for all $\varphi \in \Phi$. Assume that there exists $\varphi_0$ such that $\varphi_0(e_1) = \varphi_0(e_0)$. Since $\varphi_0$ is non-constant, there exists $s, t \in \Delta_0(Y)$ such that $\varphi_0(s) < \varphi_0(t)$. Then, for any $\alpha, \beta \in (0, 1)$, we have

$$\varphi_0((1 - \alpha)e_1 + \alpha(\beta s + (1 - \beta)t)) - \varphi_0((1 - \alpha)e_0 + \alpha t)$$

$$= (1 - \alpha)(\varphi_0(e_1) - \varphi_0(e_0)) + \alpha \beta(\varphi_0(s) - \varphi_0(t)) < 0$$

Therefore, by equation 5 and by the Weak Order axiom, we have that

$$(1 - \alpha)e_0 + \alpha t \succ_{(1-\alpha)e_0 + \alpha t} (1 - \alpha)e_1 + \alpha(\beta s + (1 - \beta)t). \quad (6)$$

Now, consider the

$$\mathcal{H}''(\succ_r)_{r \in X, 0} := \{(x, r) \in X^2 \mid x \succ_x r\}.$$

Clearly, we have that

$$\mathcal{H}''(\succ_r)_{r \in X, 0} := \{(x, r) \in X^2 \mid s_\succ(r, x) \leq 0\}.$$

But, since by axiom MCont, the function $b_\succ$ is upper semi-continuous, it follows by duality that $s_\succ$ is lower semi-continuous, and therefore the set $\mathcal{H}''(\succ_r)_{r \in X, 0}$ is closed. This implies that, by equation 6, when $\beta$ goes to 0,

$$(1 - \alpha)e_0 + \alpha t \succ_{(1-\alpha)e_0 + \alpha t} (1 - \alpha)e_1 + \alpha t. \quad (7)$$
Now, again since $\mathcal{P}((\succ_r)_{r \in X}, 0)$ is closed by lower semi-continuity of $s_\succ$, equation 7 implies as $\alpha$ goes to 0, that $e_0 \succ_{e_0} e_1$. But this is contrary to the assumption that $e_1 \succ_{e_0} e_0$. Therefore we can conclude that $\varphi(e_1) > \varphi(e_0)$ for all $\varphi \in \Phi$.

Now, it is possible to normalize each function $\varphi \in \Phi$ into a function $\psi$ such that $\psi(e_1) = 1$ and $\psi(e_0) = 0$ replacing each $\varphi$ by $\frac{1}{\varphi(e_1)-\varphi(e_0)} \varphi - \frac{\varphi(e_0)}{\varphi(e_1)-\varphi(e_0)} := \psi$. So we let $\Psi$ be the set of functions defined in this way.

Now, it is straightforward to show that, by part (ii) of the Calibration axiom, we have, for all $p, q \in \Delta_0(Y)$, for all $\lambda \in \mathbb{R},$

$$p \oplus \lambda \succ_q q \iff \psi(p) - \psi(q) \geq \lambda, \forall \psi \in \Psi.$$ 

Therefore, since $b_(\varphi)(p \ominus b_{\varphi}(p, q), q) = 0$, we have $\psi(p) - \psi(q) \geq b_{\varphi}(p, q)$, for all $\psi \in \Psi$. Conversely, for all $\eta > 0$, $b_{\varphi}(p \ominus (b_{\varphi}(p, q) - \eta), q) = \eta < 0$, and therefore there exists $\psi \in \Psi$ such that $\psi(p) - \psi(q) < b_{\varphi}(p, q) - \eta$, thus $\inf_{\psi \in \Psi} \int \psi dp - \int \psi dq \leq b_{\varphi}(p, q)$. This concludes the proof of sufficiency, since $s_{\varphi}(p, q) = -b_{\varphi}(q, p)$.

Let us now show that $\Psi \subseteq Y^\oplus$. Indeed, we have that $\Psi \subseteq \Delta_0(Y)^\oplus$. Indeed, consider $p \in \Delta_0(Y)$ and $\psi \in \Psi$. Then, for all $\lambda \in \mathbb{R},$

$$\lambda = b_{\varphi}(p \ominus \lambda, p) \leq \psi(p \ominus \lambda) - \psi(p) \leq s_{\varphi}(p \oplus \lambda, p) = s_{\varphi}(p \oplus \lambda, p \oplus (\lambda - \lambda)) = \lambda.$$ 

Therefore, $\psi(p \ominus \lambda) = \psi(p) + \lambda$.

For necessity, the only thing to show is necessity of No regret and Calibration. For NR, take $p, q$ such that $p \succ_q q$. Then, $\psi(p) \geq \psi(q)$, for all $\psi \in \Psi$. Let $r \in \Delta_0(Y)$. Then this implies $\psi(p) - \psi(r) \geq \psi(q) - \psi(r)$, for all $\psi \in \Psi$. Fix $\psi_0 \in \Psi$. Then $\psi_0(p) - \psi_0(r) \geq \inf_{\psi \in \Psi} \psi(q) - \psi(r)$, and, since this is true for all $\psi_0$, $\inf_{\psi \in \Psi} \psi(p) - \psi(r) \geq \inf_{\psi \in \Psi} \psi(q) - \psi(r)$, and therefore $b_{\varphi}(p, r) \geq b_{\varphi}(q, r)$. We show similarly that $s_{\varphi}(p, r) \geq s_{\varphi}(q, r)$, and we conclude by dominance that $p \succ_r q$.

For Calibration, take any $e_0 \in \Delta_0(Y)$. Clearly, since the only thing that matters are the values $\psi(p) - \psi(q)$, it is possible to replace each $\psi \in \Psi$ by $\psi - \psi(e_0)$. Therefore, we can assume that $\psi(e_0) = 0$ for all $\psi \in \Psi$. Now, since $\psi \in \Delta_0(Y)^\oplus$, $\psi(e_0 \oplus 1) = 1$. Therefore, since $e_0 \oplus 1 \succ_{e_0} e_0$, we can choose $e_1 := e_0 \oplus 1$. Part (ii) of Calibration is then easily verified once we note that

$$p \ominus \lambda \succ_q q \iff \psi(p) - \psi(q) \geq \lambda, \forall \psi \in \Psi.$$ 

**Proof of Proposition 16.** Let us prove the first part of the proposition.

Assume first that $r \succ_{\text{SQB}1} r'$ and take $x \in X$ and $\lambda \in \mathbb{R}$ such that $x \ominus \lambda \succ_r r'$. Then, by definition of $\succ_{\text{SQB}1}$, we have $x \ominus \lambda \succ_r r'$. Therefore $b_{\varphi}(x, r) \leq b_{\varphi}(x, r')$.

Conversely, assume $b_{\varphi}(x, r) \leq b_{\varphi}(x, r')$ and suppose $x \not\succ r$. Then $b_{\varphi}(x, r) \geq 0$, and therefore $b_{\varphi}(x, r') \geq 0$, implying $u_{r'}(x) \geq 0$ and thus $x \not\succ r'$.

Let us now prove the second part of the proposition.

We first prove the second equivalence.

Assume first that $r \succ_{\text{SSQB}} r'$. For all $x \in X$, $x \ominus b_{\varphi}(x, r) \succ_r r$, and therefore, since $r \succ_{\text{SSQB}} r'$, $x \ominus b_{\varphi}(x, r) \succ_{r'} r'$. Since $\{\lambda \in \mathbb{R} \mid x \ominus \lambda \succ_r r'\}$ is open, there exists $\varepsilon > 0$ such that $x \ominus (b_{\varphi}(x, r) + \varepsilon) \succ_{r'} r'$ and therefore $b_{\varphi}(x, r') \geq b_{\varphi}(x, r) + \varepsilon > b_{\varphi}(x, r)$.

Conversely, assume that for all $x \in X$, $b_{\varphi}(x, r) < b_{\varphi}(x, r')$. Then, $b_{\varphi}(x, r) < u_{r'}(x)$. Now take $x$ such that $x \succ_{r} r$. Then $b(x, r) \geq 0$, therefore $u_{r'}(x) > 0$, so that $x \not\succ_{r'} r'$. 

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We now prove the first equivalence. We already know that \( r \succ_{SSQB} r' \Rightarrow r \succ_{SQB} r' \). We need therefore only show the converse. Notice first that \( \lambda < b_{\succ}(x,r) \Rightarrow x \oplus \lambda \succ r \). Indeed, \( r \succ x \oplus \lambda \Rightarrow 0 \geq u_r(x \oplus \lambda) \Rightarrow 0 \geq b_{\succ}(x \oplus \lambda, r) \). Now assume \( r \succ_{SQB} r' \). Then, for all \( x \in X \), for all \( \lambda < b_{\succ}(x,r) \), \( x \oplus \lambda \succ r \), and therefore, \( x \oplus \lambda \succ r' \). Since \( \{ \lambda \in \mathbb{R} \mid x \oplus \lambda \succ r \} \) is open, there exists \( \varepsilon > 0 \) such that \( x \oplus (\lambda + \varepsilon) \succ r' \) and therefore \( b_{\succ}(x,r) \geq \lambda + \varepsilon > b_{\succ}(x,r) + \varepsilon > b_{\succ}(x,r) \). Therefore, \( r \succ_{SSQB} r' \).

**Proof of Theorem 3. Sufficiency of the axioms** Define \( \succ \) by \( x \succ y \) if and only if \( x \succ y \). Assume \( x \succ y \). Then, by Strong Separability, there exists \( \lambda > \mu \) such that

\[
x \oplus \lambda \succ r \quad \text{and} \quad r \oplus \mu \succ y.
\]

This implies, on the one hand, that \( b_{\succ}(x,r) \geq \lambda \) and, on the other hand, that \( \mu \geq s_{\succ}(y,r) \). But, since \( \lambda > \mu \), this implies that \( b_{\succ}(x,r) > s_{\succ}(x,r) \).

Now, assume that \( b_{\succ}(x,r) > s_{\succ}(x,r) \). Then, since \( b_{\succ}(x,r) \) is the supremum of the set \( B_{\succ}(x,r) := \{ \lambda \in \mathbb{R} \mid x \oplus \lambda \succ r \} \), we cannot have \( s_{\succ}(y,r) \geq \lambda \) for all \( \lambda \in B_{\succ}(x,r) \), since this would contradict the minimality of \( b_{\succ}(x,r) \). Therefore there exists \( \lambda \in B_{\succ}(x,r) \) such that \( b_{\succ}(x,r) \geq \lambda > s_{\succ}(y,r) \). By a similar reasoning, there exists \( \mu \in \mathbb{R} \) such that \( r \oplus \mu \succ y \) and \( \lambda > \mu \geq s_{\succ}(y,r) \). By Strict Buying Consistency, we have that \( x \succ r \oplus \mu \) and by Strict MSQB, we have that \( r \oplus \mu \succ y \). Therefore, since \( \lambda > \mu \), Monotonicity and Strict Partial Order imply that \( x \succ y \).

Irreflexivity of \( \succ \) for all \( r \) implies first that \( \succ \) is irreflexive and second that not \( b_{\succ}(x,r) > s_{\succ}(x,r) \), and therefore \( b_{\succ}(x,r) \leq s_{\succ}(x,r) \).

Finally, for all \( r \in X \), Monotonicity and Irreflexivity imply that

\[
B_{\succ}(r,r) = (-\infty,0),
\]

so that its supremum, \( b_{\succ}(r,r) \) is 0. For the same reason,

\[
\{ \mu \mid r \oplus \mu \succ r \} = (0,\infty),
\]

so that its infimum \( s_{\succ}(r,r) \) equals 0.

\((iv)\) follows from \((ii)\) and \((iii)\). It implies upper monotonicity and upper semicontinuity by lemma 1, and uniqueness again by lemma 1.

**Necessity of the axioms**

**Strict Partial Order** Irreflexivity follows from \((i)\). For transitivity, take \( x,y,z,r \in X \) such that \( x \succ y \) and \( y \succ z \). Then, \( b_{\succ}(x,r) > s_{\succ}(y,r) \) and \( b_{\succ}(y,r) > s_{\succ}(z,r) \). Since, by \((i)\), \( b_{\succ}(y,r) \leq s_{\succ}(y,r) \), we have that \( b_{\succ}(x,r) > s_{\succ}(z,r) \).

**Strong Separability** If \( x \succ y \), then \( b_{\succ}(x,r) > s_{\succ}(y,r) \), so we can repeat the argument given in the sufficiency part.
Monotonicity Let $\lambda > \mu$. Then, for all $r \in X$, $b_\prec(r \oplus \lambda, r) - s_\prec(r \oplus \mu, r) = b_\prec(r \oplus \lambda, r) - s_\prec(r \oplus \mu, (r \oplus \mu) \ominus \mu) = \lambda - \mu > 0$ because $b_\prec(r, r) = s_\prec(r \oplus \mu, r \ominus \mu) = 0$. Therefore $r \oplus \lambda \succ_r r \oplus \mu$.

Strict Buying Consistency Let $x, r \in X$ and $\lambda \in \mathbb{R}$ be such that $x \ominus \lambda \succ_r r$. Then, $b(x \ominus \lambda, r) > s_\prec(r, r) = 0$. Therefore, $b(x, r) > \lambda = s_\prec(r \oplus \lambda, r)$.

Strict MSQB Let $x, r \in X$ and $\lambda \in \mathbb{R}$ be such that $x \ominus \lambda \succ_r r$.

Then $b(x, r) + \lambda > 0$, therefore $-b(x, r) - \lambda < 0$, i.e. $s_\prec(r, x) < \lambda = b_\prec(x \oplus \lambda, x)$.

\[\square\]

\textbf{Proof of Corollary 10.} We prove the first result, the second is proved similarly.

Assume first $x \succ^b_r y$. Then there exists $z \in X$ such that $x \succ_r z$, and therefore $b_\prec(x, r) > s_\prec(z, r)$, and such that $z \sim_r y$, so that $-(y \sim_r z)$, i.e. $b_\prec(y, r) \leq s_\prec(z, r)$. Combining both yields $b_\prec(x, r) > b_\prec(y, r)$.

Assume now $b_\prec(x, r) > b_\prec(y, r)$. Then, there exists $\lambda \in \mathbb{R}$ such that $x \ominus \lambda \succ_r r$ and $b_\prec(x, r) \geq \lambda > b_\prec(y, r)$. By Strict Buying Consistency, this implies $x \succ_r r \ominus \lambda$, and therefore, by Monotonicity and transitivity, $x \succ_r r \ominus b_\prec(y, r)$. But, by the preceding corollary, we have $y \sim_r r \ominus b_\prec(y, r)$. Therefore, $x \succ^b_r y$.

\[\square\]

\textbf{Proof of Proposition 17.} (i) $\Rightarrow$ (ii)

\[x \sim^2_r r' \oplus b_\prec(z(x, r')) \Rightarrow \neg(x \succ_2^r r' \oplus b_\prec(z(x, r'))) \Rightarrow \neg(x \succ^1_r r \oplus b_\prec(z(x, r'))) \Rightarrow b_\prec(z(x, r)) \leq b_\prec(z(x, r')).\]

Similarly, \[x \sim^1_r r \oplus s_\prec(z(x, r)) \Rightarrow x \succ^2_r r' \oplus s_\prec(z(x, r)) \Rightarrow \neg(x \succ^2_r r' \oplus s_\prec(z(x, r))) \Rightarrow s_\prec(z(x, r)) \leq s_\prec(z(x, r')).\]

(ii) $\Rightarrow$ (i) Straightforward.

\[\square\]

\section{Relation to Briec and Gardères (2004)}

\subsection{Definition of The Generalized Benefit Function of Briec and Gardères}

The previous definitions of the benefit function constitute a generalization of the definition of Luenberger (1992) as explained above. Briec and Gardères (2004) presents another generalization. We need to adapt their definition to our framework to allow for a comparison. Overlooking some details, the more general way to adapt their definition is the following. Let $X$ be a non-empty set and let $\Delta : X^2 \rightarrow \mathbb{R}$ be a skew-symmetric function (i.e. $\Delta(y, x) = -\Delta(x, y, r)$ for all $x, y \in X$). We may interpret $\Delta$ as a generalization of the notion of variation in $\mathbb{R}$, where, if $x, y \in \mathbb{R}$, $\Delta(x, y) = y - x$. We therefore refer to $\Delta$ as a \textit{variation function}. We shall refer to a pair $(X, \Delta)$ as a variational space, v.s. for short.
Let $\succ$ be a binary relation on $X$. Then, given a variation function $\Delta$, we define the Briec-Gardère benefit function associated to $\succ$, $BG_\succ$, by:

$$BG_\succ(x, r) = \sup \{ \Delta(y, x) \mid y \succ r \}.$$ 

In the sequel, we will consider $\succ$ as fixed and will drop the reference to it.

### B.2 Relationship Between both Generalizations

The question of the relationship between both generalizations is quite subtle, and only partial answers will be given to it. We will approach the problem from two points of view: the structural point of view and the functional point of view.

#### B.2.1 The Structural Point of View

The structural point of view consists in looking on conditions on $\oplus$ and/or on $\Delta$ that ensure that $b = BG$. Let us first give some definitions.

**Definition 11.** Let $(X, \oplus)$ be a r.m.s. and $\Delta$ a variation. 

$\Delta$ is said to agree with $\oplus$ if for all $x \in X$ and $\lambda \in \mathbb{R}$:

$$\Delta(x, x \oplus \lambda) = \lambda. \tag{8}$$

$\oplus$ is said to agree with $\Delta$ if for all $x, y \in X$,

$$y = x \oplus \Delta(x, y) \tag{9}$$

$\oplus$ and $\delta$ agree if both (8) and (9) hold.

$\Delta$ satisfies the so-called “Chasles relation$^{10}$” if:

$$\forall x, y, z \in X, \Delta(x, y) + \Delta(y, z) = \Delta(x, z).$$

$\oplus$ is transitive if for all $x, y \in X$, there exists $\lambda \in \mathbb{R}$ such that $y = x \oplus \lambda$.

The following proposition shows how all these structural properties are related to the question we are investigating.

**Proposition 19.** Let $(X, \oplus)$ be a r.m.s. such that $\oplus$ is transitive and $\Delta$ a variation that satisfies the Chasles relation. Then, the following are equivalent:

(i) $\Delta$ agrees with $\oplus$.

(ii) $b = BG$.

**Proof.** We first prove a lemma that we shall freely use in the sequel without mentioning it.

---

$^{10}$See Ritzenthaler (2004) for this name. We disagree with the opinion defended in this paper that Chasles does not deserve to give his name to this equality because he did not know vectors: the Chasles relation has many other applications, for instance in integration, so it is not directly solely associated with vectors.
Lemma 5. Let \((X, \Delta)\) be a v.s. Then, the following are equivalent:

(i) \(\Delta\) satisfies Chasles.

(ii) For all \(x, y, r \in X\), \(\Delta(x, y) = BG(y, r) - BG(x, r)\).

Proof.

(i) \(\Rightarrow\) (ii):

\[
BG(y, r) = \sup\{\Delta(z, y) \mid z \succeq r\}
= \sup\{\Delta(z, x) + \Delta(x, y) \mid z \succeq r\}
= \sup\{\Delta(z, x) \mid z \succeq r\} + \Delta(x, y)
= BG(x, r) + \Delta(x, y).
\]

(ii) \(\Rightarrow\) (i): Straightforward. \(\square\)

We now proceed with the proof of the proposition.

(i) \(\Rightarrow\) (ii): Fix \((x, r) \in X^2\). Let \(\lambda\) be such that \(x \ominus \lambda \succeq r\). Then, by (8), \(\lambda = \Delta(x \ominus \lambda, x)\), so that \(\lambda \leq BG(x, r)\), and therefore \(b(x, r) \leq BG(x, r)\). Let us prove the reverse inequality. Let \(\lambda = \Delta(y, x)\) for some \(y \in X\) such that \(y \succeq r\). By transitivity of \(\oplus\), there exists \(\mu\) such that \(x = y \oplus \mu\). By (8), \(\lambda = \Delta(y, x) = \mu\). Therefore, \(x \ominus \lambda = y \succeq r\) and \(\lambda \leq b(x, r)\), implying that \(BG(x, r) \leq b(x, r)\).

(ii) \(\Rightarrow\) (i): By Chasles, we have that for all \(x \in X\) and \(\lambda \in \mathbb{R}\), \(\Delta(x, x \oplus \lambda) = BG(x \oplus \lambda, r) - BG(x, r) = b(x \oplus \lambda, r) - b(x, r) = \lambda\). \(\square\)

This proposition shows that, in a precisely described subset of all r.m.s. and all v.s., agreement of \(\Delta\) with \(\oplus\) is a necessary and sufficient conditions for both generalizations to coincide. It does it by considering simultaneously both structures. We shall now further explore the relationship between both generalizations considering only one structure at a time.

Proposition 20. If \((X, \oplus)\) be a r.m.s., then the following are equivalent

(i) There exists a variation \(\Delta\), satisfying Chasles and such that \(\Delta(x, \cdot)\) is injective for each \(x\), such that \(b = BG\).

(ii) \(\oplus\) is transitive.

Proof. (i) \(\Rightarrow\) (ii): Let \(x, y \in X\). Since \(\Delta\) satisfies Chasles and \(b = BG\), for all \(r \in X\), \(\Delta(x, y) = b(y, r) - b(x, r)\). Therefore, \(\Delta(x, x \oplus \Delta(x, y)) = \Delta(x, y)\) and, therefore \(y = x \oplus \Delta(x, y)\) since \(\Delta(x, \cdot)\) is injective. This shows that \(\oplus\) is transitive.
(ii) \(\implies\) (i): Since \(\oplus\) is transitive, for all \(x, y \in X\), there exists \(\lambda\) such that \(y = x \oplus \lambda\). But then, for all \(r \in X\), \(\lambda = b(y, r) - b(x, r)\). This implies that the difference \(b(y, r) - b(x, r)\) is independent from \(r\). We may there let \(\Delta(x, y) := b(y, r) - b(x, r)\). \(\Delta\) trivially satisfies Chasles. Let us show that \(\Delta(x, \cdot)\) is injective. Let \(y, y'\) be such that \(\Delta(x, y) = \Delta(x, y')\). Then \(b(y, r) = b(y', r)\). But, since \(\oplus\) is transitive, there exists \(\mu\) such that \(y' = y \oplus \mu\). Then \(\mu = b(y', r) - b(y, r) = 0\), hence \(y = y'\). Finally, by construction \(\Delta(x, x \oplus \lambda) = \lambda\), and this implies, by proposition 19, that \(b = BG\).

\[\Box\]

**Proposition 21.** If \((X, \Delta)\) be a v.s., then, the following are equivalent:

(i) There exists an action \(\oplus\) such that \(\oplus\) and \(\Delta\) agree and \(b = BG\).

(ii) \(\Delta\) satisfies Chasles and \(\Delta(x, \cdot)\) is bijective for each \(x\).

**Proof.**

(i) \(\Rightarrow\) (ii) : Fix \(x \in X\).

- \(\Delta(x, \cdot)\) is one-to-one: since for all \(y \in X\), \(y = x \oplus \Delta(x, y)\), for all \(y, y' \in X\), \(\Delta(x, y) = \Delta(x, y')\), then \(y = x \oplus \Delta(x, y) = x \oplus \Delta(x, y') = y'\).
- \(\Delta(x, \cdot)\) is onto: let \(\lambda \in \mathbb{R}\). Then \(\Delta(x, x \oplus \lambda) = \lambda\).

Now take \(x, y, z \in X\). Since \(y = x \oplus \Delta(x, y)\), \(b(y, r) - b(x, r) = \Delta(x, y)\). But since \(b = BG\), this implies \(BG(y, r) - BG(x, r) = \Delta(x, y)\): \(\Delta\) satisfies Chasles.

(ii) \(\Rightarrow\) (i) : Let \(x \in X\), \(\lambda \in \mathbb{R}\). Then, by assumption there exists a unique \(y \in X\) such that \(\Delta(x, y) = \lambda\). We define \(x \oplus \lambda := y\). Then \(x \oplus 0 = x\) because \(\Delta(x, x) = 0\). Moreover, let \(\lambda, \mu \in \mathbb{R}\). Let \(y := x \oplus \lambda\) and \(z := y \oplus \mu\). Then \(x \oplus (\lambda + \mu) = x \oplus (\Delta(x, y) + \Delta(y, z)) = x \oplus \Delta(x, z) = z = y \oplus \mu = (x \oplus \lambda) \oplus \mu\). Therefore, the second axiom of an r.m.s. is satisfied. \(\oplus\) and \(\Delta\) agree by construction. That \(b = BG\) follows from the fact that \((X, \oplus)\) is transitive and (8) holds (see proposition 19).

\[\Box\]

**B.2.2 The Functional Point of View**

The functional point of view consists in investigating whether some functional properties of, for instance, \(b\), implies that it coincides with \(BG\) for some \(\Delta\), or vice-versa. We have first the following proposition. Say that \(b\) is of constant variation if for all \(x, y, r \in X\), the difference \(b(y, r) - b(x, r)\) does not depend on \(r\).

**Proposition 22.** \(b\) is of constant variation if and only if there exists \(\Delta\) satisfying Chasles and agreeing with \(\oplus\) such that \(b = BG\).

**Proof.** Assume \(b\) is of constant variation and set \(\Delta(x, y) := b(y, r) - b(x, r)\). \(\Delta\) agrees with \(\oplus\) by the translation property and it trivially satisfies the Chasles relation. Let us show that \(b = BG\). Fix \((x, r) \in X^2\). Take \(\lambda \in \mathbb{R}\) such that \(x \ominus \lambda \geq r\). Then \(\Delta(x \ominus \lambda, x) = \lambda\), so that \(\lambda \leq BG(x, r)\). This implies \(b(x, r) \leq BG(x, r)\). Conversely, let \(\lambda \in \mathbb{R}\) be such that there exists \(y \geq r\) such that \(\Delta(y, x) = \lambda\). Then, \(\lambda = b(x, r) - b(y, r)\). But \(y \geq r\), so that \(b(y, r) \geq 0\). Therefore, \(\lambda \leq b(x, r)\), which implies \(BG(x, r) \leq b(x, r)\).

The converse is trivial.

\[\Box\]
Proposition 23. BG satisfies \( BG(y, r) - BG(x, r) = \Delta(x, y) \) for all \( x, y, r \in X \) and for all \( x \in X, \lambda \in \mathbb{R} \), there exists a unique \( y \) such that \( BG(y, r) - BG(x, r) = \lambda \) if and only if there exists an action \( \oplus \) agreeing with \( \Delta \) such that \( b = BG \).

**Proof.** Let \( x \in X, \lambda \in \mathbb{R} \). Define \( x \oplus \lambda \) := \( y \) for \( y \) such that \( BG(y, r) - BG(x, r) = \lambda \). Clearly \( x \oplus 0 = x \). Let \( \mu \in \mathbb{R} \) and let \( z := y \oplus \mu = (x \oplus \lambda) \oplus \mu \). Then, we have that \( z \) is the unique element of \( X \) such that \( BG(z, r) - BG(y, r) = \mu \). But then \( BG(z, r) - (BG(x, r) + \lambda) = \mu \), and therefore \( BG(z, r) - BG(x, r) = \lambda + \mu \) and thus \( z = x \oplus (\lambda + \mu) \) by uniqueness. \( \oplus \) agrees with \( \Delta \) by construction and in fact \( \Delta(x, x \oplus \lambda) = BG(x \oplus \lambda, r) - BG(x, r) = \lambda \) by construction as well, so that \( \Delta \) agrees with \( \oplus \). Therefore, \( \oplus \) is transitive, \( \Delta \) satisfies Chasles by definition and agrees with \( \oplus \), so by proposition 19 \( b = BG \).

The converse is trivial. \( \square \)

B.3 The General Case

The interpretation we gave of Briec and Gardères’s definition is the most natural in our framework, and we have seen that with this interpretation the relationship between our definition and theirs is clear enough only if one is ready to assume Chasles as part of the definition of \( \Delta \). In that case, neither theory is a generalization of the other. However, it is possible to define a generalization of Briec and Gardères’s definition in the following way. Suppose there exists a family \( (\Delta_r)_{r \in X} \) of variation functions. Then we define \( BG \) by:

\[
BG(x, r) := \sup \{ \Delta_r(y, x) \mid y \succeq r \}.
\]

With that definition, \( b \) is a special case of \( BG \):

**Proposition 24.** Let \( (X, \oplus) \) be a r.m.s. Then, there exists a family \( (\Delta_r)_{r \in X} \) of variation functions that agree with \( \oplus \) and that satisfy the Chasles relation such that \( b = BG \).

**Proof.** Let

\[
\Delta_r(x, y) = b(y, r) - b(x, r).
\]

\( \Delta_r \) agrees with \( \oplus \) by the translation property and it trivially satisfies the Chasles relation. Let us show that \( b = BG \). Fix \( (x, r) \in X^2 \). Take \( \lambda \in \mathbb{R} \) such that \( x \oplus \lambda \succeq r \). Then \( \Delta_r(x \oplus \lambda, x) = \lambda \), so that \( \lambda \leq BG(x, r) \). This implies \( b(x, r) \leq BG(x, r) \). Conversely, let \( \lambda \in \mathbb{R} \) be such that there exists \( y \succeq r \) such that \( \Delta_r(y, x) = \lambda \). Then, \( \lambda = b(x, r) - b(y, r) \). But \( y \succeq r \), so that \( b(y, r) \geq 0 \). Therefore, \( \lambda \leq b(x, r) \), which implies \( BG(x, r) \leq b(x, r) \).

This shows that, if one adapts Briec and Gardères’s definitions of the generalized benefit function along the lines proposed here, then our generalization is a special case of theirs. However, it must be emphasized that the focus of their work is completely different from ours. Briec and Gardères work in the context of a Euclidean space, and intensively use convex analysis to prove their results (duality, consumer theory). On the other hand, we work in an abstract set \( X \) only endowed with the structure relevant to the study of benefit functions in a decision-theoretic framework.
References


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