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Smoothing Spline Method for Measuring Prospect Theory Components

Yao Thibaut Kpegli

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Prospect theory is today the main descriptive model for decision making under risk and uncertainty. Measurement methods of its components are key to many behavioral applications. This paper presents a smoothing spline method for measuring utility function, weighting function and loss aversion. The method is nonparametric and includes a penalty term to control the collinearity between the value and the weighting functions. It is applicable to both risk and uncertainty. We apply the method to individual data of Tversky and Kahneman (1992) and Gonzalez and Wu (1999). In line with original prospect theory, the probability weighting function is not sign-dependent. The value function is S-shaped with a loss aversion coefficient of 1.6.

Keywords:
prospect theory, risk attitudes elicitation, smoothing spline

Working Paper

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Abstract

Prospect theory is today the main descriptive model for decision making under risk and uncertainty. Measurement methods of its components are key to many behavioral applications. This paper presents a smoothing spline method for measuring utility function, weighting function and loss aversion. The method is nonparametric and includes a penalty term to control the collinearity between the value and the weighting functions. It is applicable to both risk and uncertainty. We apply the method to individual data of Tversky and Kahneman (1992) and Gonzalez and Wu (1999). In line with original prospect theory, the probability weighting function is not sign-dependent. The value function is S-shaped with a loss aversion coefficient of 1.6.

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1 Introduction

Empirical violations of expected utility theory (EUT; see Starmer, 2000, for a review) explain the development of alternative theories of decision makings. Cumulative Prospect Theory (CPT; see Tversky and Kahneman, 1992, henceforth TK92) emerged as the theory with more descriptive validity (e.g. Blavatskyy, 2021; Attema et al., 2013).

Under CPT, Risk attitudes result simultaneously from the value and weighting functions. As result, several combinations of the shapes of the value and weighting functions can lead to the same level of risk-attitudes. A main challenge when measuring CPT is how to deal with the collinearity between the value and weighting functions (e.g. Zeisberger et al., 2012; Abdellaoui et al., 2011a).

Measurement methods of CPT can be done under three approaches: parametric (with parametric specification of the utility and probability weighting functions)[1], semi-parametric (with parametric specification of the utility function and parameter-free probability weighting function) and non-parametric (no parametric specification for either function).

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[1]For the utility function, Power and Exponential are popularly used and sometime a mixture of them called Expo-Power (Saha, 1993; Holt and Laury, 2002; Abdellaoui et al., 2007a). For the distortion of probability, one-parameter (e.g. TK92) and two-parameter (Prelec, 1998; Goldstein and Einhorn, 1987; Chateauneuf et al., 2007; Abdellaoui et al., 2010) weighting functions (for a review, see table 4 of Epper and Fehr-Duda, 2020) are available.

Parametric (e.g. TK92) and semi-parametric (Abdellaoui et al., 2008) methods are more often used because their implementations are quick and easy. Yet, collinearity between the value and the weighting functions makes estimation results sensitive to parametric specifications (e.g. Abdellaoui, 2000; Abdellaoui et al., 2008). A pragmatic and limited way to get rid of this collinearity is to assume a linear value function (e.g. l'Haridon and Vieider, 2019) in line with the dual theory of Yaari (1987).

The so-called tradeoff method of Wakker and Deneffe (1996) is an alternative. This method is non-parametric and the collinearity between utility and weighting functions does not play a role in the elicitation procedure. This explains why most non-parametric methods (e.g. Abdellaoui, 2000; Abdellaoui et al., 2007b, 2016; Blavatskyy, 2021) are built upon the tradeoff method (see Kpegli et al., 2023, for detailed discussions on elicitation methods). The nonparametric method of Gonzalez and Wu (1999, henceforth GW99) is a notable exception which does not rely on the tradeoff method.[2] However, the collinearity between utility and weighting function takes a stronger form in this method. Irrespective to the data under consideration, the method can lead to unrealistic concave value function and probability weights close to 1 (see footnote 4).

This paper establishes an alternative nonparametric method to the tradeoff-type methods. The proposed method approximates the value function with smoothing spline. The smooth parameter allows to control for the collinearity between the value and weighting functions by penalizing deviation from the linear value function. In contrast to the tradeoff-type methods, the proposed method accounts for response errors and rely on simple questions. The method remains applicable under uncertainty where probabilities of events are unknown. The method is applied on the two datasets of TK92 and GW99. Results indicate that the probability weighting function is not sign-dependent. The value function is S-shaped with a loss aversion coefficient of 1.6.

The rest of the paper is organized as follows. Section 2 presents the spline value function. Section 3 develops the smoothing spline method for eliciting cumulative prospect theory components. Section 4 presents the key features of the method. We illustrate the method in Section 5 using the data of TK92 and GW99. Sections 6 and 7 provide discussion and conclusion.

2 Spline value function for CPT

2.1 Prospect theory for binary lottery

Consider a binary lottery $L = (x, y; p, 1-p)$ yielding outcome x with probability p and outcome y with probability $1-p$, both outcomes being real numbers.[3] For notational convenience, let $x > y \geq 0$ ($x < y \leq 0$) for non-mixed prospects involving only gains (losses). For mixed prospects (*i.e.*, involving both gains and losses), outcomes are denoted with an asterisk and $y^* < 0 < x^*$. $\succsim$ is a preference relation over prospects with $\succ$ ($\sim$) denoting strict preference (indifference). Preferences are represented by CPT with a probability weighting function w^i and a value function v as defined in equation (1) for non-mixed prospects and in equation (2)

[2]See the appendix of Fehr-Duda and Epper (2012) for a similar procedure.

[3]This notation is related to decision under risk. In the case of decisions under uncertainty, one would simply replace p and $1-p$ by E and E^c respectively. E denotes an event in a state space Ω and E^c denotes its complement in Ω. In that case, $L = (x, y; E, E^c)$ is a binary prospect that gives outcome x if E occurs, and y otherwise.

for the mixed ones:

$$CPT(L) = (v(x) - v(y))\, w^i(p) + v(y) \quad (1)$$

$$CPT(L) = w^+(p)v(x) + w^-(1-p)v(y) \quad (2)$$

where w^i and v are both continuous, strictly increasing and satisfying $v(0) = 0$, $w^i(0) = 0$ and $w^i(1) = 1$, and $i =$"+" ($i =$"−") stands for the gain (loss) domain. CPT makes no explicit link between weighting functions $w^+(.)$ and $w^-(.)$ which makes it more general than the original version of prospect theory (OPT, Kahneman and Tversky, 1979) in which $w^+(p) = w^-(p)$, or rank dependent utility theory (RDU, Quiggin, 1982; Gilboa, 1987; Schmeidler, 1989) that includes the duality condition $w^+(p) = 1 - w^-(1-p)$.

Following the seminal study by TK92 and the meta-analysis of Brown et al. (2021), the value function $v(.)$ is composed of the loss aversion index $\lambda > 0$, which reflects the exchange rate between gain and loss utility units, and the (basic) utility function $u(.)$ that reflects the intrinsic value of outcomes:

$$v(x) = \begin{cases} u_+(x) & \text{if } x \geq 0 \\ -\lambda u_-(-x) & \text{if } x < 0 \end{cases} \quad \text{and} \quad u(x) = \begin{cases} u_+(x) & \text{if } x \geq 0 \\ -u_-(-x) & \text{if } x < 0 \end{cases} \quad (3)$$

with $u_+ : \mathcal{R}_+ \longrightarrow \mathcal{R}_+$, $u_- : \mathcal{R}_+ \longrightarrow \mathcal{R}_+$ and $u_+(0) = u_-(0) = 0$. Let's assume that the basic utility function u is twice differentiable over $\mathbb{R} - \{0\}$ and differentiable at 0. Assuming that the utility function is differentiable at 0 means that the loss aversion index in the relation 3 corresponds to the ratio of the left and right derivative of the value function as defined by Köbberling and Wakker (2005). Loss aversion (loss seeking) corresponds to $\lambda > 1 (\lambda < 1)$, whereas $\lambda = 1$ captures loss neutrality.

2.2 Spline value function

The spline approximation of the function $u_i(.)$ corresponds to (e.g., Ahamada and Flachaire, 2010; Ruppert, 2002; Green and Silverman, 1993):

$$u_i(z) = a_0^i + \sum_{j=1}^{J_i} a_j^i z^j + \sum_{t=1}^{Q_i} b_t^i (z - q_t^i)_+^{J_i} \qquad \text{for} \qquad z \geq 0 \qquad \text{and} \qquad i = +, - \quad (4)$$

with $J_i \geq 1$ the order of the spline, Q_i the number of internal knots, $q_1^i < q_2^i < ... < q_T^i$ and

$$(z - q_t^i)_+^{J_i} = \begin{cases} (z - q_t^i)^{J_i} & \text{if} \quad z \geq q_t^i \\ 0 & \text{otherwise} \end{cases}$$

To accommodate with $u(0) = 0$ of Prospect Theory, the scaling $a_0^i = 0$ is used. Without loss of generality, the scaling $a_1^i = 1$ is used to allow identification of the loss aversion index à la Köbberling and Wakker (2005). This scaling means that the utility function is differentiable at 0, with $u'(0) = 1$.

A main advantage of (4) is that the shape of the utility function is very flexible. The following three quadratic splines (5) - (7) illustrate this flexibility. The spline (5) leads to a concave utility function (Figure 1, panel (a)). The spline (6) leads to a convex utility function (Figure 1, panel (b)). The spline (7) leads to a concave utility function over two intervals $[0, 0.25]$ and $[0.5, 0.75]$, and also leads to a convex utility over the two intervals $[0.25, 0.50]$ and $[0.75, 1]$ (Figure 1, panel (c)).

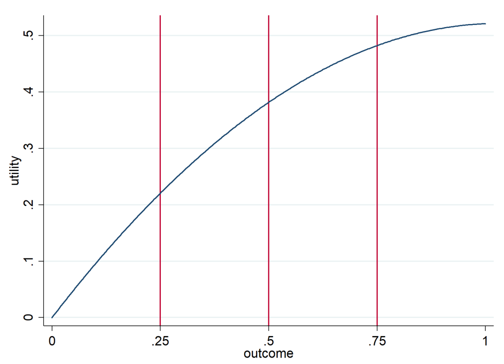
$$u_i(z) = z - 0.47z^2 - 0.01(z - 0.25)_+^2 - 0.01(z - 0.25)_+^2 - 0.01(z - 0.25)_+^2 \tag{5}$$

$$u_i(z) = z + z^2 + 0.1(z - 0.25)_+^2 + 0.1(z - 0.50)_+^2 + 0.1(z - 0.75)_+^2 \tag{6}$$

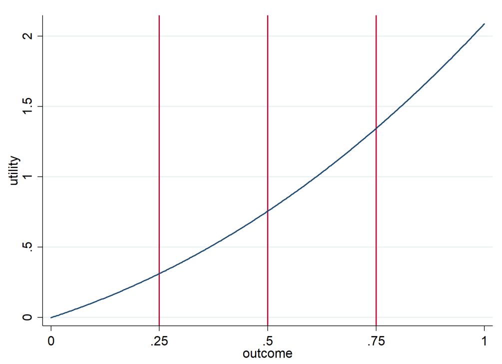
$$u_i(z) = z - 1.75z^2 + 3(z - 0.25)_+^2 - 2.5(z - 0.25)_+^2 + 2.5(z - 0.25)_+^2 \tag{7}$$

Figure 1: Spline: exemple of shapes

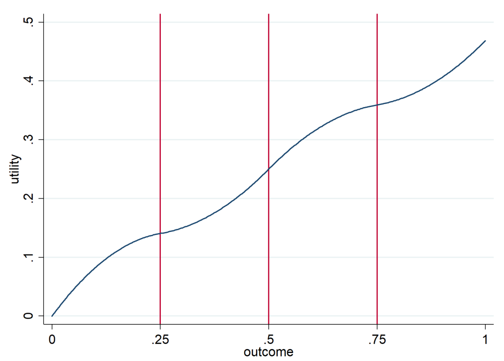
(a) Spline function (5)



(b) Spline function (6)



(c) Spline function (7)



3 Elicitation method

3.1 Step 1: utility and weighting functions in the gain domain

In the gain domain, the method can be implemented by following three substeps. First, a set of probabilities $\{p_k : k = 1, 2, ..., K\}$ are selected, with $p_k < p_{k+1}$. Second, at least two certainty equivalents for each probability p_k are elicited:

$$ce_{j,k} \sim (x_{j,k}, y_{j,k}; p_k, 1 - p_k) \quad , \quad j = 1, 2, ..., N_k^+ \quad \text{and} \quad N_k^+ \geq 2 \tag{8}$$

where N_k^+ stands for the number of certainty equivalents for positive outcomes $x_{j,k}$ and $y_{j,k}$ such that $x_{j,k} > y_{j,k} \geq 0$. Thus, in total $N^+ = \sum_{k=1}^{K} N_k^+ \geq 2 \times K$ certainty equivalents are elicited. Using (1) and (3), these certainty equivalents satisfy the following condition:

$$u_+(ce_{j,k}) = (u_+(x_{j,k}) - u_+(y_{j,k}))\, w^+(p_k) + u_+(y_{j,k}) \tag{9}$$

Let $\boldsymbol{ce}, \boldsymbol{x}$ and $\boldsymbol{y}$ be the column vectors containing all the realizations of $ce_{j,k}$, $x_{j,k}$ and $y_{j,k}$, respectively. Let also $\boldsymbol{I}^k$ be a dummy variable set to 1 if the probability equals p_k and 0 otherwise. For $k = 1, .., K$, denote $\delta_k^+ \equiv w^+(p_k) \in (0, 1)$. Arranging the terms in equation (9) and taking the log leads to the following equation:

$$log\Big(u_+(ce_{j,k}) - u_+(y_{j,k})\Big) = log\Big(u_+(x_{j,k})) - u_+(y_{j,k})\Big) + log\Big(w^+(p_k)\Big) \tag{10}$$

Adding an error term $\boldsymbol{e}$ to the log-transformation (10) leads to the following empirical equation:[4]

$$log\Big(u_+(\boldsymbol{ce}_l) - u_+(\boldsymbol{y}_l)\Big) = log\Big(u_+(\boldsymbol{x}_l) - u_+(\boldsymbol{y}_l)\Big) + log\Big(\sum_{k=1}^{K} \delta_k^+ \boldsymbol{I}_l^k\Big) + \boldsymbol{e}_l \tag{11}$$

where l is the l^{th} line in $\boldsymbol{ce}$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{I}^k$ and $\boldsymbol{e}$. Denote by θ^+ the vector of parameters $(\{a_j^+\}_{j=1}^{J_+}, \{b_t^+\}_{t=1}^{Q_+})$ associated with the smooth approximation of the utility function $u_+(.)$. Minimizing the following penalized sum of squared error allows to estimate the utility function and the probability weights as follows:[5]

$$\min_{\theta^+, \delta_1^+, ..., \delta_K^+} \sum_{l=1}^{N^+} \boldsymbol{e}_l^2 + \rho^+ \int_0^{\overline{x}} \Big[u_+''(z)\Big]^2 dz \tag{12}$$

where $\rho^+ \geq 0$ is the smooth parameter, $u_+''(.)$ the concavity (second derivative) of the utility function and $[0, \overline{x}]$ the range over which the utility function is elicited. Each level of the smooth parameter ρ^+ corresponds to a specific combination of utility curvature and probability weighting function. For example, when subject exhibits risk-aversion, the case $\rho^+ \longrightarrow +\infty$ corresponds to the linear utility function and the less elevated probability weighting function that will result from an estimation based on the dual theory of Yaari (1987). Then, the smooth parameter allows to chose the combination of shapes of the utility and weighting functions by penalizing deviation from the linear utility function.

Following the literature (e.g. Ahamada and Flachaire, 2010; Green and Silverman, 1993), the optimal values of the smooth parameter ρ^+ and the order of the spline Q_+ corresponds to the ones that minimize the leave-one cross-validation (CV). Formally, the optimal ρ^+ and Q_+ provide the smallest value for:

$$CV = \frac{1}{N^+} \sum_{l=1}^{N^+} \Big|\widehat{\boldsymbol{ce}}_{-l} - \boldsymbol{ce}_l\Big| \tag{13}$$

[4]An alternative approach would be to introduce the response error term at the utility level (GW99, eq. 7): $u(\boldsymbol{ce}_l) = (u(\boldsymbol{x}_l) - u(\boldsymbol{y}_l)) \sum_{k=1}^{K} \delta_k^+ \boldsymbol{I}_l^k + u(\boldsymbol{y}_l) + \boldsymbol{e}_l$. However, defining the response error at the utility level is problematic when using certainty equivalents data because it produces solutions that are characterized by unrealistic concavity of the utility and probability weighting functions. To illustrate this point, suppose that we are interested in eliciting utility only over strictly positive outcomes with a power utility function $u(z) = z^\alpha$. For an extremely concave utility function (i.e., $\alpha > 0$ and $\alpha \longrightarrow 0$) and an extremely concave weighting function (i.e., $\delta_k^+ = 1$ for $k = 1, 2, .., K$) along with the PT assumptions $w^+(0) = 0$ and $w^+(1) = 1$, we have $\boldsymbol{e}_l = 0$ for all $l = 1, 2, ..., N^+$. For the non-parametric method of GW99 which aims at estimating $u(z)$ for $z \in A(z) \equiv \{\$25, \$50, \$75, \$100, \$150, \$200, \$400, \$800\}$ and the probability weights $w(p)$ for $p \in B(p) \equiv \{0.01, 0.05, 0.10, 0.25, 0.40, 0.50, 0.60, 0.75, 0.90, 0.95, 0.99\}$, it follows that an extremely concave utility function (i.e., $u(z) = constant > 0$ for $z \in A(z)$ and $u(0) = 0$) and an extremely concave weighting function (i.e., $w(p) = 1$ for $p \in B(p)$ and $w(0) = 1 - w(p) = 0$) are solutions of the optimization problem (Kpegli et al., 2023).

[5]Note that we do not require monotonicity of the utility and weighting functions as is commonly done in the literature (see e.g. GW99, p. 147).

where $\widehat{\boldsymbol{ce}}_{-l}$ is the predicted value of $\boldsymbol{ce}_l$ based on data without the l^{th} observation.[6]

The following measure of utility curvature over $[0, \overline{x}]$ in the gain domain (see Kpegli et al., 2023; Abdellaoui et al., 2016) is used to classify the utility function as concave or convex:

$$\alpha = \frac{1}{\overline{x} u_+(\overline{x})} \int_0^{\overline{x}} u_+(t)dt \tag{14}$$

The utility function $u_+(.)$ is considered to be concave, linear and convex on the interval $[0, \overline{x}]$ when $\alpha > 0.5$, $\alpha = 0.5$ and $\alpha < 0.5$ respectively.

3.2 Step 2: utility and weighting functions in the loss domain

This step is similar to the first step. At least two certainty equivalents for each probability p_k are elicited:

$$ce_{j,k} \sim (x_{j,k}, y_{j,k}; p_k, 1-p_k) \quad , \quad j = 1, 2, ..., N_k^- \quad \text{and} \quad N_k^- \geq 2 \tag{15}$$

where N_k^- stands for the number of certainty equivalents for negative outcomes $x_{j,k}$ and $y_{j,k}$ such that $x_{j,k} < y_{j,k} \leq 0$. Thus, in total $N^- = \sum_{k=1}^{K} N_k^- \geq 2 \times K$ certainty equivalents are elicited.

Consider the following log-transformation of prospect theory functional form with an additive error term:

$$log\Big(u_-(-\boldsymbol{ce}_l) - u_-(-\boldsymbol{y}_l)\Big) = log\Big(u_-(-\boldsymbol{x}_l) - u_-(-\boldsymbol{y}_l)\Big) + log\Big(\sum_{k=1}^{K} \delta_k^- \boldsymbol{I}_l^k\Big) + \boldsymbol{e}_l \tag{16}$$

where l is the l^{th} line in $\boldsymbol{ce}$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{I}^k$ and $\boldsymbol{e}$. Denote by θ^- the vector of parameters $(\{a_j^-\}_{j=1}^{J_-}, \{b_t^-\}_{t=1}^{Q_-})$ associated with the smooth approximation of the utility function $u_-(.)$. Minimizing the following penalized sum of squared error provides the estimate of the utility function and the probability weights:[7]

$$\min_{\theta^-, \delta_1^-, ..., \delta_K^-} \sum_{l=1}^{N^-} \boldsymbol{e}_l^2 + \rho^- \int_0^{-\overline{x}} \Big[u_-''(z)\Big]^2 dz \tag{17}$$

where $\rho^- \geq 0$ is a (fixed) smooth parameter, $u_-''(.)$ the concavity (second derivative) of the utility function and $[\overline{x}, 0]$ the range on which the utility function is elicited. As in the gain domain, the smooth parameter controls the concavity of the utility function by penalizing deviation from linear utility function or the dual theory of Yaari (1987).

As in the gain domain, the optimal values of the smooth parameter ρ^- and the order of the spline Q_- correspond to the ones that minimize the leave-one cross-validation (CV):

$$CV = \frac{1}{N^-} \sum_{l=1}^{N^-} \Big|\widehat{\boldsymbol{ce}}_{-l} - \boldsymbol{ce}_l\Big| \tag{18}$$

[6]Denote by $\widehat{u}_+(.)$ and $\widehat{\delta}_k^+$ $(k = 1, 2, ..., K)$ the estimated utility function and decision weights. The predicted certainty equivalent $\widehat{\boldsymbol{ce}}_l$ is the solution to the equation $\widehat{u}_+(\widehat{\boldsymbol{ce}}_l) = (\widehat{u}_+(\boldsymbol{x}_l) - \widehat{u}_+(\boldsymbol{y}_l)) \sum_{k=1}^{K} \widehat{\delta}_k^+ \boldsymbol{I}_l^k + \widehat{u}_+(\boldsymbol{y}_l)$.

[7]As in gain domain, the method does not require monotonicity of the utility and weighting functions as is commonly done in the literature (see e.g. GW99, p. 147).

where $\widehat{\boldsymbol{ce}}_{-l}$ is the predicted value of $\boldsymbol{ce}_l$ based on data without the l^{th} observation.[8]

As in the gain domain, the following measure of utility curvature over the loss interval $[\underline{x}, 0]$ is used to classify the utility function as concave or convex (see Kpegli et al., 2023; Abdellaoui et al., 2016):

$$\beta = \frac{1}{\underline{x}u_-(\underline{x})} \int_{\underline{x}}^{0} u_-(t)dt \tag{19}$$

The utility function $u_-(.)$ is considered to be concave, linear and convex on the interval when $\beta > -0.5$, $\beta = -0.5$ and $\beta < -0.5$ respectively.

3.3 Step 3: measuring loss aversion

The third step allows to measure the loss aversion index λ as defined in (3). Following Abdellaoui et al. (2007b), the estimation of the loss aversion index can be done using a set of K indifference relationships that involve mixed binary prospects:

$$ce_k \sim (x_k, y_k, p_k, 1-p_k)\ , \quad k = 1, 2, ..., K \tag{20}$$

with $y_k < 0 < x_k$. Under CPT these indifferences imply that:

$$\left(1_{(ce_k \geq 0)} + \lambda 1_{(ce_k < 0)}\right) u(ce_k) = w^+(p_k)u(x_k) + \lambda w^-(1-p_k)u(y_k) \tag{21}$$

where $1_{(.)}$ refers to the indicator function. Denote by $\boldsymbol{D}^+$ a dummy variable that takes the value 1 if the certainty equivalent is positive (or zero) and 0 otherwise. Similarly, $\boldsymbol{D}^-$ is a dummy variable that takes value 1 if the certainty equivalent is negative and 0 otherwise. Assuming additive error at the basic utility scale ($\boldsymbol{e}_k$), the empirical counterpart of equation (21) then becomes:

$$\left(\boldsymbol{D}_k^+ + \lambda \boldsymbol{D}_k^-\right) \widehat{u}(\boldsymbol{ce}_k) = \widehat{\boldsymbol{\delta}}_k^+ \widehat{u}(\boldsymbol{x}_k) + \lambda \widehat{\underline{\boldsymbol{\delta}}}_k^- \widehat{u}(\boldsymbol{y}_k) + \boldsymbol{e}_k \tag{22}$$

with $\widehat{u}$, $\widehat{\boldsymbol{\delta}}_k^+$ and $\widehat{\underline{\boldsymbol{\delta}}}_k^-$ the estimates of the utility function $\widehat{u}$, the probability weights in the gain domain $w^+(p_k)$ and the probability weights in the loss domain $w^-(1-p_k)$ from steps 1 and 2.

Estimate of loss aversion index λ of Köbberling and Wakker (2005) is given by the minimization of the sum of squared error:

$$\min_{\lambda} \sum_{k=1}^{K} \boldsymbol{e}_k^2 \tag{23}$$

4 Key features of the method

This section highlights six features of the method.
Robust to collinearity. The method controls for the collinearity between utility and weighting functions (e.g. Zeisberger et al., 2012) with smooth parameters. Indeed, these smooth parameters allow to choose optimal combination of curvatures of utility and weighting functions in the gain and loss domains.
Applicability to unknown probabilities. The use of fixed effects for estimating decision weights is taken from Kpegli et al. (2023). The method is then directly applicable to cases of

[8] Denote by $\widehat{u}_-(.)$ and $\widehat{\delta}_k^-$ $(k = 1, 2, ..., K)$ the estimated utility function and decision weights. The predicted certainty equivalent $\widehat{\boldsymbol{ce}}_l$ is solution to the equation $\widehat{u}_-(-\widehat{\boldsymbol{ce}}_l) = (\widehat{u}_-(-\boldsymbol{x}_l) - \widehat{u}_-(-\boldsymbol{y}_l)) \sum_{k=1}^{K} \widehat{\delta}_k^- \boldsymbol{I}_l^k + \widehat{u}_-(-\boldsymbol{y}_l)$.

uncertainty where probabilities are unknown. The probability dummy variables can then be replaced by the event dummy variables in equations (11) and (16). It does not require setting any specific conditions on the event space, and hence can be applied to real-life uncertainty situations (Baillon et al., 2018).[9]

Error-robust. In contrast to the tradeoff method (Wakker and Deneffe, 1996), the smoothing spline method considers that subjects can make error in their responses. Estimations of prospect theory components result from a minimization of (penalized) sum of squared response errors.

***Easy* (not cognitively demanding).** The method is a certainty equivalent method. As such, it uses simple questions that involves the lowest possible number of outcomes (i.e., 3): comparisons of certain outcomes and binary lotteries. Hence, the method is less cognitively demanding than the tradeoff method that rely on comparison of two non-degenerate lotteries (e.g. Abdellaoui et al., 2008).

Tractable. The method can be implemented by using optimization programs available in standard statistical software. For example, the ML routine for Stata popularized by Harrison and Rutström (2008) and Moffatt (2015) for parametric risk elicitation can be used to implement the smoothing spline method.

Data-inefficient. The method builds upon the smoothing spline literature. Application of the method is more data consuming than parametric methods (e.g. Ahamada and Flachaire, 2010; Green and Silverman, 1993). This method should thus be used in two main cases. First, behavioral studies often elicit few observations per subjects. In such case, the method can be used to derive estimates at the aggregate level by pooling subjects as in TK92. The dataset include 25 subjects. For each subject, only 28 observations are available to derive estimates of utility and weighting function in each domain. The pooled data with 700 ($= 25 \times 28$) observations per domain can then be used to derive estimate at the aggregate level. Second, the method can be performed at the individual level if the number of observations per subjects is sufficiently large as in GW99. For each subject, the dataset include 165 observations to measure utility and weighting functions in the gain domain.

5 Application

This section applies the smoothing spline method on the data of TK92 and GW99.

5.1 Data

5.1.1 Data of TK92

Subjects: TK92 run a computerized experiment with 25 graduate students from Berkeley and Stanford with no particular training in decision theory. Each subject participated in three separate one-hour sessions organized over several days, and received \$25 for participation.

Procedure: the data are generated via the switching outcome procedure in which an indifference value is inferred through a list of equally spaced certain outcomes, ranging from the admissible maximum indifference value to the admissible minimum indifference value. Internal

[9]Previous methods require that the universal event is an interval of real numbers (e.g. temperature in a town) which is most suitable to deal with artificial uncertainty situations that can be created in the laboratory (Van De Kuilen and Wakker, 2011).

consistency of the responses to each prospect is monitored by a computer software to reduce response errors.

Data for the first and second steps: all outcomes are expressed in US dollars. For each subject, there are 28 values of certainty equivalents for binary lotteries that involve 7 pairs of positive monetary outcomes (0, 50), (0, 100), (0, 200), (0, 400), (50, 100), (50, 150) and (100, 200), and 9 probabilities of getting the higher outcome: 0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95 and 0.99. Also, there are, for each subject, 28 values of certainty equivalents for binary lotteries that involve 7 pairs of negative monetary outcomes (0, -50), (0, -100), (0, -200), (0, -400), (-50, -100), (-50, -150) and (-100, -200), and the same list of 9 probabilities as in the gain domain. These probabilities are now associated to losing the higher outcome.

Data for the third step: for mixed prospects, individual data are not available. We then use the median values reported in Table 6 of TK92. We use the four indifferences on mixed prospects that consist in eliciting the values of gains x to make subjects indifferent between the mixed prospects $(x, y; 0.5, 0.5)$ and 0. The values of y are -25, -50, -100 and -150.

5.1.2 Data of GW99

Subjects and procedure: GW99 run a computerized experiment with 10 graduate students in psychology. They also use the switching outcome procedure for eliciting certainty equivalents.

Data for the first step (only gain): Each subject has 165 certainty equivalents. The 165 values of certainty equivalents correspond to binary lotteries that involve 15 pairs of positive monetary outcomes (0, 25), (0, 50), (0, 75), (0, 100), (0, 150), (0, 200), (0, 400), (0, 800), (25, 50), (50, 75), (50, 100), (50, 150), (100, 150), (100, 200) and (150, 200) and 11 probabilities of obtaining the higher outcome: 0.01, 0.05, 0.1, 0.25, 0.4, 0.5, 0.6, 0.75, 0.9, 0.95 and 0.99.

5.2 Results

5.2.1 Utility function

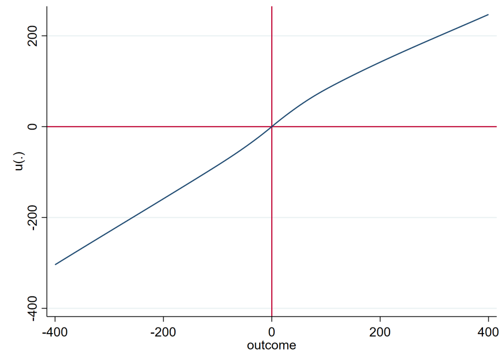
Figure 2 provides the estimated utility functions on both datasets.[10] The pool estimate on the data of TK92 leads to a concave (resp. convex) utility in the gain (resp. loss) domain. The utility curvature in gain domain is $\alpha = 0.554$ and is significantly different from linearity $(p - value = 0.0011)$[11], which corresponds to $\alpha = 0.5$. In the loss domain, the mean of utility curvature in the gain domain is $\beta = -0.517$ and is significantly different from linearity $(p - value = 0.0036)$ where $\beta = -0.5$. Furthermore, partial reflection is rejected $(H_0 : \alpha + \beta = 0, p - values = 0.0355)$.

For the data of GW, the mean of individual utility function in the gain domain is also concave.[12] The utility curvature is $\alpha = 0.623$ and is significantly different from linearity $(p - value < 0.0001)$.

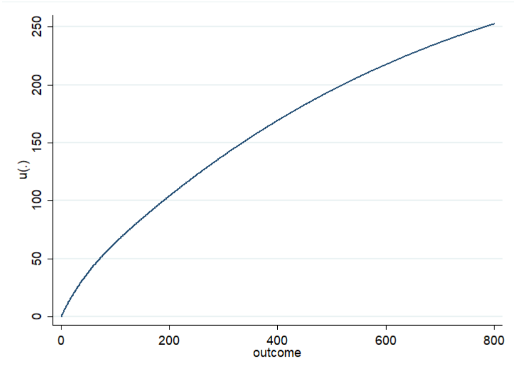
[10]Appendixes A and B provides detailed informations about estimates.
[11]All p-values are computed on the basis of Bootstrap with 1000 replications.
[12]see appendix for detailed results at the individual level

Figure 2: Utility function

(a) Data of TK92 (pool estimate)



(b) Data of GW99 (mean of individual estimates)



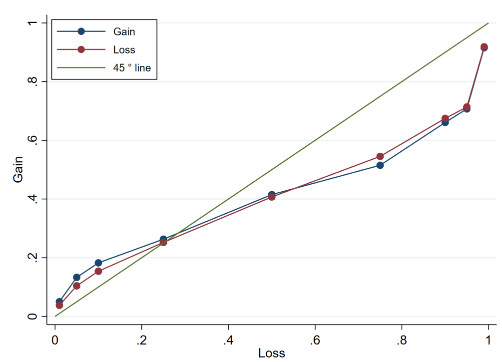
5.2.2 Weighting function

Figure 3 provides the estimated probability functions. For the data of TK92, probabilistic risk neutrality in the gain domain $w^+(p) = p$ is rejected for most probabilities (all $p - values < 0.0004$), except for 0.25 ($p - value = 0.3978$). Similar patterns emerge in the loss domain. Probabilistic risk neutrality $w^-(p) = p$ is rejected for most probabilities (all $p - values < 0.0014$), except for 0.25 ($p - value = 0.8598$). Hence, the weighting function in both domains is inverse S-shaped with overweighting for $p \in (0, 0.25]$ and underweighting for $p \in (0.25, 1)$. The data of GW99 in the gain domain also leads to a crossing point at $p = 0.25$ ($p - value = 0.6189$). For the data of GW99, probabilistic risk neutrality in the gain domain $w^+(p) = p$ is rejected for most probabilities (all $p - values < 0.0127$), except for 0.40 ($p - value = 0.3710$).

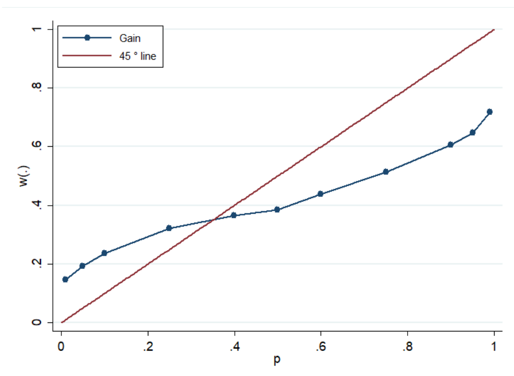
Over the 9 probabilities in the data of TK92, the hypothesis of identical probability weights across domains ($w^+(p_k) = w^-(p_k)$) of OPT cannot be rejected for all probabilities (all $p - values > 0.0514$). In contrast, the hypothesis of duality ($w^+(p_k) = 1 - w^-(1 - p_k)$) of RDU is rejected for all probabilities (all $p - values < 0.0327$).

Figure 3: Probability weighting function

(a) Data of TK92



(b) Data of GW99



5.2.3 Loss aversion

The estimated value of loss aversion is $\lambda = 1.636$ and is significantly different from loss neutrality ($p - value < 0.0001$). Hence, subjects exhibit loss aversion.

6 Discussion

The estimation result of concave utility function in the gain domain is a very common finding in the literature, irrespective to the approach: parametric (e.g. l'Haridon and Vieider, 2019, TK92), semi-parametric (e.g. Abdellaoui et al., 2008, 2011a; Kpegli et al., 2023) and non-parametric (e.g. Wakker and Deneffe, 1996; Abdellaoui et al., 2016; Blavatskyy, 2021, GW99).

In the loss domain, the finding of convex utility function corroborates the original parametric estimates of TK92. Though, the evidences about utility curvature in loss domain are mixed in the literature. Results based on non-parametric methods (e.g. Abdellaoui et al., 2011b; Attema et al., 2018; Abdellaoui et al., 2016; Blavatskyy, 2021; Hajimoladarvish, 2017) tend to provide evidences in favor of a convex utility function. In contrast, results based on semi-parametric methods (e.g. Abdellaoui et al., 2008; Attema et al., 2013, 2016; Kpegli et al., 2023) tend to provide evidences in favor of a concave utility function.

The methods also leads to the rejection of partial reflection ($\alpha + \beta = 0$). Empirical evidences on partial reflection in the literature is mixed. The rejection of partial reflection is consistent with some studies (Abdellaoui et al., 2013, 2016; Attema et al., 2013, 2016, ABL), but not with others (e.g. Abdellaoui, 2000; Andersen et al., 2006; Abdellaoui et al., 2007b; Booij and Van de Kuilen, 2009; Harrison and Rutström, 2009; Booij et al., 2010).

The estimation results provide support for identical probability weighting function across domains of OPT and reject the duality condition of RDU. Tests of identical probability weightings and duality are scarce in the literature. Abdellaoui (2000) and Kpegli et al. (2023) reject both duality and identical probability weighting functions across domains under risk. Abdellaoui et al. (2005) do not reject duality under uncertainty although they reject identical weighting functions across domains. The observation of identical probability weights and the rejection of duality provide support for OPT.

The method confirms loss aversion, with a loss aversion index of $\lambda = 1.636$. This estimate is less than the 2.25 reported by TK92. Our estimate is close to the estimated value of 1.6 that was elicited in both Booij et al. (2010) who use structural estimation techniques, and Abdellaoui et al. (2008) for pooled data. It is also close to the estimates of $\lambda = 1.8$ reported by Pennings and Smidts (2003). The evidence of loss aversion is a very common finding in the literature as only few studies find evidences for loss seeking (e.g. Abdellaoui et al., 2013; Nilsson et al., 2011).

7 Conclusion

In sum, this paper introduces a smoothing spline method to elicit the utility function, the weighting function and the loss aversion. The method allows to control for the collinearity between utility and weighting function and can be applied under both risk and uncertainty. Its application on experimental data provides reliable results.

Appendix

Appendix A- Data of TK92

Table 1 provides detailed results on the data of TK92. We use three internal knots that correspond to quartile of the certainty equivalent (e.g. Ahamada and Flachaire, 2010).

Table 1: Individual estimates: data of TK92

Domain	Gain ($i = +$)	Loss ($i = -$)
Utility function		
a_2^i	-0.00207	-0.00220
a_3^i	0.00000662	0.00000898
b_1^i	-0.0000120	-0.00000555
b_2^i	0.0000152	0.00000192
b_3^i	-0.00000901	-0.00000510
Probability weights		
$w^i(0.01)$	0.0501	0.0378
$w^i(0.05)$	0.133	0.104
$w^i(0.10)$	0.183	0.154
$w^i(0.25)$	0.263	0.252
$w^i(0.50)$	0.415	0.407
$w^i(0.75)$	0.515	0.545
$w^i(0.90)$	0.661	0.675
$w^i(0.95)$	0.707	0.714
$w^i(0.99)$	0.916	0.919
Order of the spline and smooth parameter		
Q_+	3	3
ρ^+	9500	9000
CV	12.42	11.49
N	700	700

Appendix B- Data of GW99

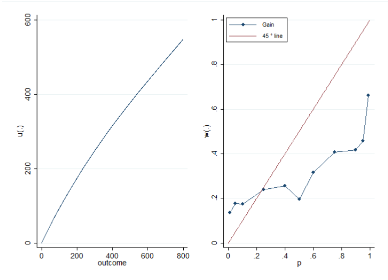
Table 2 provides detailed results for the 10 subjects in the data of GW99. We use three internal knots that correspond to quartile of the certainty equivalent (e.g. Ahamada and Flachaire, 2010).

Table 2: Individual estimates: data of GW99

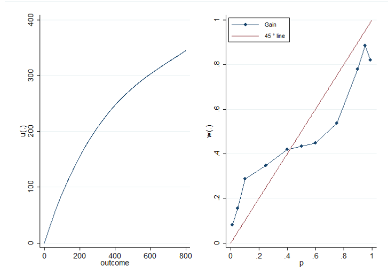
Subject	1	2	3	4	5	6	7	8	9	10
					Utility function					
a_2^+	-0.0005	-0.0007	-0.0006	-0.0045	-0.0219	-0.0008	-0.0067	-0.0067	-0.0053	-0.0085
a_3^+	-5×10^{-6}	-7×10^{-4}								
b_1^+	7×10^{-6}	4×10^{-6}	-1.1×10^{-4}	-1.3×10^{-7}	0.0205	-3.4×10^{-4}	9.5×10^{-4}	2.2×10^{-4}	0.0013	0.0038
b_2^+	-6×10^{-7}	6.3×10^{-6}	1.3×10^{-4}	5.25×10^{-5}	0.0008	-0.0001	0.0041	0.0022	0.0020	0.0035
b_3^+	-1.2×10^{-6}	-3×10^{-6}	0.0004	0.0040	0.0006	0.0009	0.0015	0.0040	0.0020	0.0012
					Probability weights					
$w^+(0.01)$	0.137	0.0801	0.340	0.101	0.233	0.0682	0.127	0.162	0.0282	0.196
$w^+(0.05)$	0.177	0.155	0.334	0.175	0.281	0.113	0.237	0.171	0.0689	0.225
$w^+(0.10)$	0.175	0.286	0.428	0.130	0.365	0.133	0.230	0.187	0.121	0.301
$w^+(0.25)$	0.240	0.347	0.485	0.220	0.451	0.280	0.256	0.283	0.245	0.404
$w^+(0.40)$	0.255	0.420	0.507	0.154	0.517	0.440	0.281	0.242	0.359	0.489
$w^+(0.50)$	0.195	0.434	0.523	0.183	0.535	0.542	0.246	0.221	0.442	0.523
$w^+(0.60)$	0.315	0.448	0.489	0.178	0.574	0.623	0.319	0.355	0.492	0.595
$w^+(0.75)$	0.406	0.537	0.563	0.217	0.597	0.795	0.352	0.396	0.687	0.576
$w^+(0.90)$	0.415	0.779	0.640	0.238	0.681	0.931	0.443	0.434	0.799	0.699
$w^+(0.95)$	0.458	0.885	0.612	0.255	0.730	0.960	0.466	0.469	0.896	0.737
$w^+(0.99)$	0.661	0.819	0.738	0.362	0.850	0.968	0.530	0.620	0.769	0.866
					Order of the spline and smooth parameter					
Q_+	3	3	2	2	2	2	2	2	2	2
ρ^+	10000	9000	5100	1700	1	3000	600	3500	500	400
CV	14.00	14.02	18.63	7.23	14.49	6.24	8.31	11.86	11.68	14.01
N	165	165	165	165	165	165	165	165	165	165

Figure 4: Individual estimates: data of GW99

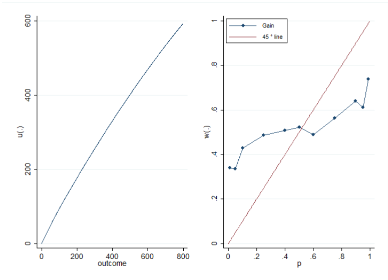
(a) Subject 1



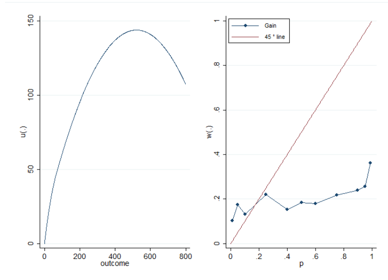
(b) Subject 2



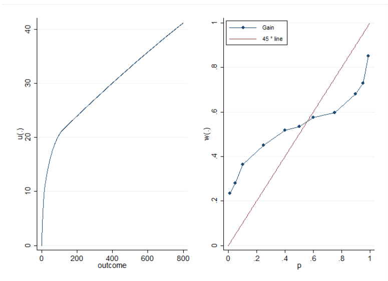
(c) Subject 3



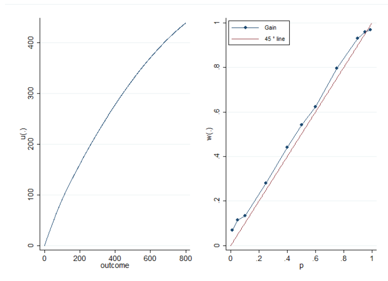
(d) Subject 4



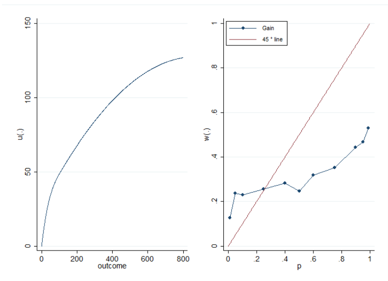
(e) Subject 5



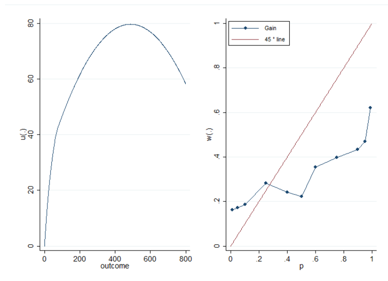
(f) Subject 6



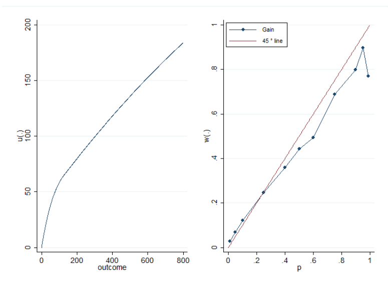
(g) Subject 7



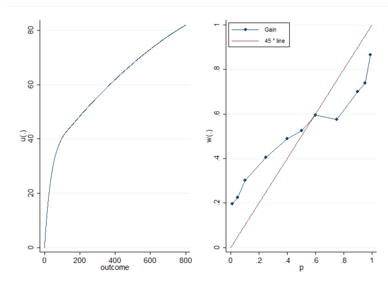
(h) Subject 8



(i) Subject 9



(j) Subject 10



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