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Keywords:
Bubble; Housing; Self-fulfilling fluctuations

JEL codes:
E32, E44, R21

Working Paper

Rational housing demand bubble*

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June 16, 2022

Abstract

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1 Introduction

Financial crises are recurrent events in developed economies with substantial repercussions on real economic activity. There is a large consensus amongst economists and policymakers that housing market plays a crucial role in the emergence of financial crises, as emphasized by Bernanke (2008),

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Reinhart and Rogoff (2009), Brunnermeier and Oehmke (2013) and, more recently, Fischer (2017). The Lost Decades in Japan and the Great recession in the United States are perfect illustrations that show the relation between housing prices and economic activity. As documented by Miao (2014) and Miao *et al.* (2014, 2015), episodes of large boom in housing prices are associated with episodes of large housing price to income ratio and also economic expansions, whereas episodes of large and sudden crash in housing price are associated with episodes of decline in housing price to income ratio and recessions.

A recent and growing literature justifies these large movements in housing price and explains the macroeconomic consequences of such movements by the existence of a rational bubble. In the literature about rational bubble, a *housing price bubble* occurs when the housing price exceeds the fundamental value, which is the present value of future dividends in terms of rents of utility services. A part of the literature considers that houses are pure speculative assets without fundamental value (see, Miao *et al.* (2014), Chen and Wen (2017), Martin *et al.* (2021) and Clain-Chamosset-Yvrard and Seegmuller (2019)). The second line in the literature considers that houses also provide housing services or rents (see, Arce and Lopez-Salido (2011), Basco (2014), Hillebrand and Kikuchi (2015), Zhou (2015) and Graczyk and Phan (2019)). Our contribution is in this second line of the literature. We show that a *housing price bubble* cannot occur when houses generate positive utility services. However, we may observe a *housing demand bubble*. We define a *housing demand bubble* as a situation where the aggregate demand for houses contains a demand driven only by speculative motives in addition to a demand explained by fundamentals like rents or housing services. By speculative motives, we mean that some agents buy houses because they expect that they will be able to sell them at a higher price. The occurrence of such a bubble generates a boom in housing price due to a raise in housing demand. By introducing this new concept of a rational housing demand bubble, our paper aims to explain large housing price fluctuations and the positive correlation between housing price and economic growth.

We develop a baseline three-period overlapping generations exchange economy in which housing plays a crucial role in portfolio choice over the life-cycle (see Cocco (2004) and Causa *et al.* (2019)). In our paper, there are four key elements. First, houses are an asset and a consumption good when they provide housing services. By generating housing services, houses have a fundamental value. Second, households realize a portfolio choice between two assets with different roles: houses and bonds. In addition to provide utility services, houses serve as a store of value and as collateral, whereas bonds are used only to save or borrow. Third, we introduce heterogeneity between households in terms of investment decisions. Young households borrow to purchase houses in order to enjoy housing services, whereas adult households save and may purchase houses only for speculative motives to transfer resources when old. Housing demand of adult individuals corresponds to our definition of a housing demand bubble. Finally, young households face credit market imperfections embodied by a borrowing constraint.

In our economy, housing prices are explained by their fundamentals, and hence, no housing price bubbles exist. The housing price is always equal to the fundamental value. In contrast, a housing demand bubble may occur when the borrowing constraint binds generating a boom in housing price and a drop in the interest rate. More precisely, a housing demand bubble exists as soon as bonds and houses are perfect substitutes for adults, and such a situation may occur only when the borrowing constraint binds. In absence of credit market imperfections, a house held only for speculative purposes is always a dominated asset. Therefore, credit market imperfections are necessary for the existence of a housing demand bubble. Furthermore, the equilibrium exhibits multiple steady states and endogenous housing price fluctuations. We show that three steady states, two without a

housing demand bubble and one with a housing demand bubble, coexist for an intermediate level of credit market imperfections.

We obtain several interesting findings. First, a housing demand bubble may improve the welfare of households at the steady state. Such a bubble implies some transfers of resources from old to adult households through the drop of the interest rate: Adult households benefit from a lower borrowing cost, whereas old households suffer from a lower return of savings. If the gains from a lower interest rate dominate, households are better off with a housing bubble. Second, there exist three sources of fluctuations: local indeterminacy associated with dampened oscillations, global indeterminacy associated with multiplicity of stable steady states and endogenous regime switching. Interestingly, in addition to small fluctuations in the neighborhood of stable steady states, our model can generate large and sudden booms and crashes in housing price that are similar to those observed during the Lost Decades in Japan and the Subprime Crisis. Finally, our economy can endogenously switch between a regime without a housing demand bubble and a regime with a housing demand bubble without requiring any shock on the bubble or fundamentals like income or housing preferences.

In order to study the interplay between housing prices and real economic activity, we then extend our basic framework to an endogenous growth economy with productive capital. A housing demand bubble generating a boom in housing price appears under the same conditions as in the exchange economy. Furthermore, we obtain a similar conclusion as in the literature about rational bubble in that a housing demand bubble has two effects on economic growth: a crowding-out effect and a crowding-in effect. The first effect occurs since a part of the savings are used to purchase houses, whereas the second one occurs because adult individuals can sell their houses at a higher price and get more liquidity to invest in physical capital when a housing demand bubble occurs. This crowding-in effect of the housing demand bubble corresponds to a liquidity effect. When the crowding-in effect dominates, the housing price to income ratio and economic growth are higher. Therefore, our model provides an explanation of the observed patterns in Miao (2014) and Miao *et al.* (2014) that show a positive correlation between housing prices and economic growth.

The rest of the paper is organized as follows. Section 2 discusses the contributions of our paper to the existing literature. Section 3 is devoted to the presentation of the baseline model. In Section 4, we study equilibria in an exchange economy. Global dynamics and endogenous regime switching are studied in Section 5. In Section 6, we extend the basic model to a production economy. A last section provides concluding remarks, whereas some technical details are relegated to an Appendix.

2 Related literature

Our paper is related with the literature that studies the existence of rational housing bubbles, and that also studies how a housing bubble affects welfare and economic growth. It is well-known in the literature about rational bubbles that an asset price bubble cannot exist when the asset provides positive rents (Tirole, 1985). In our OLG (overlapping generations) model in which houses may generate utility, we confirm this result: the market price of a house is always equal to its fundamental value even in presence of financial frictions. Our result contrasts with some recent papers dealing with housing bubbles. Hillebrand and Kikuchi (2015) show that a housing price bubble may emerge in an OLG model with housing services and a perfect credit market. In their paper, households can borrow an unlimited amount from a financial sector which makes possible a sustainable housing

price bubble.[1] However, as debt is the counterpart of the bubble, the housing price bubble is *de facto* a predetermined variable in their model. Unlike Hillebrand and Kikuchi (2015), the housing demand bubble in our model is a non-predetermined variable, which is consistent with the classical view of an asset bubble as in Tirole (1985). To explain boom-bust cycles in housing price, Martin *et al.* (2021) develop a small open OLG economy, producing housing and non-housing goods, in which a housing bubble in the land market may emerge. The authors consider two types of land: new one which is used to produce goods for one period and old one which is intrinsically useless and traded on the land market. In their model, the bubble appears in old land prices, namely an asset which has no fundamental value. Graczyk and Phan (2019) model a housing price bubble in a pure exchange OLG economy in which houses may generate utility to analyze the welfare effect of a housing bubble. However, the bubble appears in a particular case when houses become intrinsically useless assets by generating no additional utility, and thus having no fundamental value. In contrast to Martin *et al.* (2021) and Grazyck and Phan (2019), we explain booms in housing price through the existence of a housing demand bubble attached to houses which have a positive fundamental value.

Some papers in the literature about rational bubble provide alternative explanations for housing bubbles, which do not rely on the existence of a bubble in housing price, such as Arce and Lopez-Salido (2011), Basco (2014) and Zhao (2015). These papers are close to our paper in that they develop an OLG model with housing services, credit market imperfections and heterogeneity among agents in terms of investment decisions and motives for purchasing houses. Zhao (2015) considers heterogeneity among agents born in the same period, whereas Arce and Lopez-Salido (2011), Basco (2014) and our paper consider heterogeneity among agents born in different periods. Zhao (2015) shows the existence of an equilibrium in which some investors held houses only for speculative purposes and not for rents. Referring to the bubbly view of money developed by Tirole (1985), the author considers that this equilibrium depicts a housing bubble even if the housing price is equal to the housing fundamental value. Instead, Arce and Lopez-Salido (2011) and Basco (2014) show that a shortage of asset exists in their model, justifying the existence of a housing bubble. In our paper, we follow Zhao (2015), and derive the optimal housing demand for purely speculative motives which corresponds to the housing demand bubble. Modeling such a housing demand allows us to clarify the result found by Arce and Lopez-Salido (2011) and Basco (2014) and connect it to Zhao (2015). This also allow us to deeply analyze the global dynamics of the exchange economy, and highlight a new result with respect to these papers, namely the existence of regime switching between periods with housing bubbles and periods without. Through this result, our paper overcomes a recurrent criticism about rational bubble models in which a bubble cannot arise after its bursting (see Martin and Ventura (2012, 2021) and Guerron-Quintana *et al.* (2018)). Furthermore, we can explain large fluctuations in housing price without requiring any shock on fundamentals unlike Arce and Lopez-Salido (2011), Basco (2014) and Zhao (2015).

A significant contribution with respect to Arce and Lopez-Salido (2011), Basco (2014) and Zhao (2015) is that we extend our exchange economy to a production one in order to study how a housing bubble positively affects economic activity. Several papers analyze the interplay between housing bubble and capital accumulation, for instance Miao *et al.* (2014), Martin *et al.* (2021), Clain-Chamosset-Yvrard and Seegmuller (2019), and show that bubbles crowd investment and output in.

[1] In their model, the housing price does not contain a bubble component at the steady state. The authors are interested in a particular equilibrium path, namely a divergent path (the bubble) in an OLG framework without economic growth. This result echoes Kamihigashi (2008) who gets a similar result in a Lucas asset pricing model with wealth preferences.

However, all of these studies model the housing bubble as a pure bubble, namely a bubble in an asset without a fundamental value. By contrast, we consider a model in which houses are useful assets by generating utility and have a positive fundamental value. In addition, the crowding-in effect of a housing demand bubble does not require the existence of any exogenous bubble shock like in Martin *et al.* (2021) or altruism like in Clain-Chamosset-Yvrard and Seegmuller (2019). The mechanism relies on the liquidity effect of the housing demand bubble, which is line with many papers in the literature of rational bubbles like Farhi and Tirole (2012), Martin and Ventura (2012), Miao *et al.* (2014) and Hirano and Yanagawa (2017).

Few papers analyze how a housing bubble affects the welfare of households in a framework where houses generate utility. A housing bubble has an ambiguous effect on welfare because it has opposite effects on borrowers and savers in Zhao (2015) and on entrepreneurs and households in Miao *et al.* (2014). In Graczyk and Phan (2019), a housing bubble has a regressive effect on welfare, meaning that it redistributes wealth from lower-income households to higher-income households through the rise of the interest rate. In our paper, we provide a different result and show that a housing demand bubble may improve household welfare by reducing the interest rate and thus easing borrowing.

3 Baseline model

We consider a discrete time ($t = 0, 1, ..., +\infty$) OLG exchange economy with three period-lived households, a consumption good, bonds and houses. Houses are both an asset and a consumption good since they provide housing services.

3.1 Households

The economy is populated by a constant number of households that live for three periods: young, adult and old. The population size of each generation is normalized to 1.

Each household born in t obtains utility from consumption at each period of time, and from houses at period $t + 1$. Preferences are represented by the additively separable life-cycle utility function:

$$\alpha ln(c_{1,t}) + \beta\left[(1-\mu)lnc_{2,t+1} + \mu lnh_{2,t+1}\right] + \gamma lnc_{3,t+2} \tag{1}$$

where $\alpha, \beta, \mu, \gamma$ are positive preference parameters that satisfy $\alpha+\beta+\gamma = 1$ and $0 \leq \mu < 1$. $c_{i,t+i-1}$ and $h_{i,t+i-1}$ represent consumption and housing services of a household born at t, where the index $i = 1, 2, 3$, respectively, identifies young, adult and old individuals.

In the first period of life, the household is young. She is endowed with $\omega > 0$ units of a final good. With her endowments, she can consume an amount $c_{1,t}$ of a final good, invest $a_{1,t}$ in bonds, and buy $h_{2,t+1}$ houses at price p_t to enjoy housing services when adult. Note that when $a_{1,t} < 0$ the household contracts loans and when $a_{1,t} > 0$ she invests in a deposit. In the second period of life, the household is an adult. She receives no endowments, but the returns on bonds $R_{t+1}a_{1,t}$ and sells her houses at price p_{t+1}. Furthermore, she can consume an amount $c_{2,t+1}$ of the final good, invest $a_{2,t+1}$ in bonds, and buy $h_{3,t+2}$ houses at price p_{t+1}. Since we assume that old individuals do not derive utility from housing services, $h_{3,t+2}$ is purely a store of value used to transfer resources to the last period of life. Therefore adult's housing demand is not driven by fundamentals like housing services, but only by speculative motives. Therefore, if $h_{3,t+2} > 0$ at the equilibrium, then a housing demand bubble exists. In the rest of the paper, we define $h_{3,t+2}$ as the housing demand

bubble. Finally, in the third period, she is old. She receives no endowments, but the returns on her savings $R_{t+2}a_{2,t+1}$, sells her houses at price p_{t+2}, and consumes an amount $c_{3,t+2}$.

Let us define the growth factor of the housing price by the variable $R^h_{t+1} \equiv p_{t+1}/p_t$. The budget constraints write:

$$c_{1,t} + a_{1,t} + p_t h_{2,t+1} = \omega \tag{2}$$

$$c_{2,t+1} + a_{2,t+1} + p_{t+1}h_{3,t+2} = R_{t+1}a_{1,t} + R^h_{t+1}p_t h_{2,t+1} \tag{3}$$

$$c_{3,t+2} = R_{t+2}a_{2,t+1} + R^h_{t+2}p_{t+1}h_{3,t+2} \tag{4}$$

This three-period structure aims to replicate the life-cycle portfolio decisions observed in the data. Households borrow to buy a house in the early part of their working life, and they accumulate financial assets after acquiring their house (see, among others, Poterba and Samwick (2001), Ameriks and Zeldes (2004), Fernandez-Villaverde and Krueger (2007)).

When young, a household born at period t faces the following collateral constraint:

$$\theta p_t h_{2,t+1} \geq -a_{1,t} \tag{5}$$

We adopt the borrowing constraint used in Arce and Lopez-Salido (2011), Basco (2014) and Zhao (2015). Because of credit market imperfections, young households can borrow up to a fraction θ of their housing stock value. This parameter measures the degree of pledgeability of houses. Note that $1-\theta$ represents the down-payment rate required for borrowing. The higher θ, the greater the borrowing capacity. Therefore, the parameter θ measures credit market imperfections.[2] Adult households do not face an equivalent borrowing constraint, because they will realize positive deposits ($a_{2,t+1} > 0$) at the equilibrium with $h_{3,t+2} = 0$ to consume in their old age and clear the bond market.

Given her endowment ω and prices (p_t, p_{t+1}, R_{t+1}), a household chooses final goods $c_{1,t}, c_{2,t+1}$ and $c_{3,t+2}$, bond holdings $a_{1,t}$ and $a_{2,t+1}$, housing holdings $h_{2,t+1}$ and $h_{3,t+2}$ to maximize her utility function (1) subject to her budget constraints (2) - (4), the borrowing constraint (5), and the non-negativity constraint $h_{3,t+2} \geq 0$. We use the following variables in the solution of the household's problem:

$$\widetilde{R}_{t+1} \equiv (1-\theta)\frac{\alpha}{\beta(1-\mu)}\frac{c_{2,t+1}}{c_{1,t}} + \theta R_{t+1} \text{ and } div_{t+1} \equiv \frac{\mu}{1-\mu}\frac{c_{2,t+1}}{h_{2,t+1}} \tag{6}$$

The variable $\widetilde{R}_{t+1}$ measures the cost of borrowing for a young individual. The variable div_{t+1} is the marginal rate of substitution between consumption and housing services, which can be interpreted as a dividend of a house obtained when adults. The optimal behavior of a household born at time

[2] Some papers consider an alternative borrowing constraint of the following form $-R_{t+1}a_{1,t} \leq \widetilde{\theta} R^h_{t+1} p_t h_{2,t+1}$ (see for instance Iacoviello (2005)). Results are qualitatively equivalent to our model with the borrowing constraint (5). More details are available upon request.

t is summarized by the following equations (see also Appendix 8.1):

$$\widetilde{R}_{t+1} = \frac{p_{t+1} + div_{t+1}}{p_t} \tag{7}$$

$$\frac{\alpha}{\beta(1-\mu)} \frac{c_{2,t+1}}{c_{1,t}} \geq R_{t+1} \tag{8}$$

$$\frac{\beta(1-\mu)}{\gamma} \frac{c_{3,t+2}}{c_{2,t+1}} = R_{t+2} \tag{9}$$

$$(R_{t+2} - R^h_{t+2}) p_{t+1} h_{3,t+2} = 0 \tag{10}$$

$$\theta p_t h_{2,t+1} + a_{1,t} \geq 0 \tag{11}$$

where the variable R^h_{t+1} represents the growth factor of the housing price, and also measures the return factor of the housing demand bubble. Eq. (7) depicts the house price equation. This equation establishes that the cost of borrowing equals the return of a house purchased by a young individual. The return $\widetilde{R}_{t+1}$ is the sum of the growth of the house price and of the housing services obtained by an adult individual. Since houses purchased by young households provide strictly positive dividends in terms of housing services, the return factor of houses purchased when young $\widetilde{R}_{t+1}$ is strictly greater than the return factor of housing demand bubble R^h_{t+1} ($\widetilde{R}_{t+1} > R^h_{t+1}$).

Using (6) and (7), Inequality (8) rewrites:

$$\widetilde{R}_{t+1} = \frac{p_{t+1} + div_{t+1}}{p_t} \geq R_{t+1} \tag{12}$$

We distinguish two possible solutions of household's problem. When the return factor of a house purchased by young households $\widetilde{R}_{t+1}$ is greater than the return on bonds R_{t+1} ($\widetilde{R}_{t+1} > R_{t+1}$), the borrowing constraint is binding and it is not binding when both returns are equal. Note that the case $\widetilde{R}_{t+1} < R_{t+1}$ cannot be an equilibrium, since nobody would like to contract loans in the economy. When $\widetilde{R}_{t+1} = R_{t+1}$, young households are indifferent between buying houses or investing in deposits and the borrowing constraint never binds at the equilibrium. In this case, (8) holds with equality, and depicts the standard intertemporal trade-off between the first and second-period consumption. When $\widetilde{R}_{t+1} > R_{t+1}$, an arbitrage opportunity exists for young households. They can benefit from a leverage effect, since the return factor of a house held by a young household is greater than the return factor of borrowing. As a result, a young household borrows until the borrowing constraint binds.

Eq. (9) depicts the standard intertemporal trade-off between the second and third period that arises because households can smooth consumption between these two periods by investing in bonds $a_{2,t+1}$ and/or houses $h_{3,t+2}$ to smooth consumption between the second and third periods of life. According to (10), adults do not invest in houses if the return of bonds R_{t+2} is greater than the return of housing demand bubble R^h_{t+2} ($R_{t+2} > R^h_{t+2}$) and may invest in houses if both returns are equal ($R_{t+2} = R^h_{t+2}$). In other words, adult individuals invest in the housing demand bubble when bonds and the housing demand bubble are perfect substitutes.

3.2 Asset markets

In our model, two asset markets coexist: a market for bonds and a market for housing. The market clearing condition on bonds requires:

$$a_{1,t} + a_{2,t} = 0 \tag{13}$$

We do not model a housing construction sector, and assume a constant housing supply, normalized to 1. Moreover, we do not explicitly rationalize a rental market. In any case, if this market is perfect, this does not change our analysis. These assumptions allow us to emphasize the demand effect of housing which seems to play an important role in the Great Recession in the United States (see Mian and Suffi (2011)).[3] The market clearing condition on housing requires:

$$h_{2,t} + h_{3,t} = 1 \tag{14}$$

Before analyzing any equilibrium of this economy, we determine the house price and show that a housing price bubble does not exist. Using Eq. (7), we deduce that:

$$p_t = \sum_{i=1}^{+\infty} \frac{div_{t+i}}{\prod_{s=1}^{i} \widetilde{R}_{t+s}} + \lim_{j \to +\infty} \frac{p_{t+j}}{\prod_{s=1}^{j} \widetilde{R}_{t+s}} \tag{15}$$

This equation decomposes the housing price in two components. The first one is the fundamental component that we denote f_t and it is equal to the discounted value of future services generated by the house. The second one is the bubble component that we denote b_t satisfying:

$$b_t = \frac{b_{t+1}}{\widetilde{R}_{t+1}} \tag{16}$$

Eq. (16) indicates that the bubble component grows by a factor of $\widetilde{R}_{t+1}$. This return of the bubble must be lower or equal to the growth rate of the economy, meaning $\widetilde{R}_{t+1} \leq 1$. Otherwise, the bubble will grow too fast to be sustained. Using this result, we can apply an argument, similar to one developed in Tirole (1985), to show that the house price cannot contain a bubble component. If there is a bubble, then $\widetilde{R}_{t+s} \leq 1$ and $\prod_{s=1}^{i} \widetilde{R}_{t+s} \leq 1$. This implies:

$$\sum_{i=1}^{+\infty} div_{t+i} \leq \sum_{i=1}^{+\infty} \frac{div_{t+i}}{\prod_{s=1}^{i} \widetilde{R}_{t+s}} = f_t$$

Since dividends are always positive, the fundamental value of a house f_t would be infinite when a bubble exists. Therefore, the house price cannot contain a bubble component. Despite demand for resale motives only, no housing price bubble exists. In similar OLG models, Arce and Lopez-Salido (2011), Basco (2014) and Zhao (2015) assert that some agents may buy houses only for speculative motives, which corresponds in our framework to $h_{3,t+2} > 0$. As underlined by Zhao (2015), these authors do not demonstrate the existence of any housing price bubble.[4] The fact that houses serve as collateral for young households and provide liquidity for adults is not sufficient for a housing price bubble to exist. However, as in our paper, a housing demand driven by speculative motives may exist in these papers due to a shortage of assets in the economy. Whereas these authors define this speculative demand as a bubble, we argue that they show that a housing demand bubble exists, but not a housing price bubble.

[3] Basco (2014) models a housing supply which consists of new houses and the stock of undepreciated houses of the previous period. A competitive firm produces new houses using labor only under a decreasing returns to scale technology. However, the existence and the size of housing bubble do not depend on the housing supply elasticity.

[4] See Footnote 2 and the proof of Corollary 1 in Zhao (2015).

4 Equilibria in an exchange economy

In this section, we first study equilibria without a binding borrowing constraint, then equilibria when the borrowing constraint binds. We show that a steady state without a bubble exists in each regime, while a steady state with a housing demand bubble only exists when the borrowing constraint binds. Therefore, when the borrowing constraint binds, both steady states coexist. We also study the local dynamic properties of our model, and we show that the equilibrium exhibits local indeterminacy which means that there exist multiple equilibria path that converges to a steady state. We show that our economy displays fluctuations, either through dampened oscillations or through expectation-driven fluctuations in the neighborhood of the steady state.

4.1 Equilibria without a binding borrowing constraint

We consider the economy when the borrowing constraint is not binding, and young individuals are indifferent between buying houses or contracting loans in each period. Therefore, the return factor of a house purchased by young households $\widetilde{R}_{t+1}$ and the return on one-period bonds R_{t+1} are equal in each period, and greater than the return on housing demand bubble:

$$\widetilde{R}_{t+1} = R_{t+1} > R^h_{t+1},\ \forall t \geq 0$$

This implies that adults prefer investing in bonds rather than in houses. No housing demand bubble may occur when the borrowing constraint does not bind, *i.e.* $h_{3,t+1} = 0\,\forall t \geq 0$. Remind that, in Section 3.2, we claim that if the dividends in terms of utility, *i.e.* $div_{t+1} = \mu c_{2,t+1}/[(1-\mu)h_{2,t+1}]$, do not converge to zero when time goes to infinity, then no housing price bubble exists. As shown in Appendix 8.1, in this equilibrium, $div_{t+1} = R_{t+1}\beta\mu\omega$. Since R_{t+1} is strictly positive along an intertemporal equilibrium, dividends are also strictly positive. In line with Section 3.2, a housing price bubble does not exist. We conclude that the equilibrium exhibits neither a housing price bubble nor a housing demand bubble when the credit constraint does not bind.

From the optimal solutions of a household born at t, namely Eqs. (2)-(10), we deduce that (see also Appendix 8.1):

$$(R_{t+1} - R^h_{t+1})p_t h_{2,t+1} = R_{t+1}\beta\mu\omega \tag{17}$$

$$a_{1,t} + p_t h_{2,t+1} = (\beta+\gamma)\omega \tag{18}$$

$$a_{2,t+1} = R_{t+1}\gamma\omega \tag{19}$$

$$a_{1t} + \theta p_t h_{2,t+1} > 0 \tag{20}$$

Eq. (17) determines the housing demand of young households. Eqs. (18) and (19) respectively measure the savings of young and adult individuals. Note that young households save a constant fraction of their endowments, since preferences are log-linear.

Using the market clearing conditions on bonds (13) and housing (14) at t, equations (17)-(20) rewrite as:

$$(R_{t+1} - R^h_{t+1})p_t = R_{t+1}\beta\mu\omega \tag{21}$$

$$a_{1,t} = (\beta+\gamma)\omega - p_t \tag{22}$$

$$R_t = \frac{p_t - (\beta+\gamma)\omega}{\gamma\omega} \equiv R(p_t) \tag{23}$$

$$p_t < \frac{\beta+\gamma}{1-\theta}\omega \equiv p^*_C \tag{24}$$

If $p_t > p_C^*$, then a young household would face a binding borrowing constraint. Note that the upper bound on housing price p_C^* increases with the degree of pledgeablity θ. It follows that the stronger the credit market imperfections are, the lower the borrowing capacity of a young household is. From Eq. (23), the following inequality must be satisfied at the intertemporal equilibrium:

$$p_t > (\beta + \gamma)\omega \equiv \underline{p} \tag{25}$$

Since adults do not invest in housing assets, they make a deposit in order to consume when old. Market clearing in the credit market implies that young households contract loans, implying $a_{1,t} < 0$.

Substituting Eq. (23) taken one period later into Eq. (21), we derive the intertemporal equilibrium, which is a sequence $\{p_t\}_{t=0}^{+\infty}$, given $p_0 > \underline{p}$, which satisfies for all $t \geq 0$:

$$p_{t+1} = \underline{p}\frac{p_t - \overline{p}}{p_t - \widetilde{p}} \equiv g(p_t) > \underline{p}, \tag{26}$$

with $\widetilde{p} \equiv \omega(\beta\mu + \gamma)$ and $\overline{p} \equiv \omega\beta\mu < \widetilde{p}$. Note that $g(p_t)$ is a decreasing function of p_t.

Note that the dynamics of the equilibrium reduce to a first-order recursive equation in one non-predetermined variable p_t. Furthermore, along an intertemporal equilibrium, inequalities (24) and (25) must hold, namely:

$$\underline{p} < p_t < p_C^* \text{ and } \underline{p} < p_{t+1} = g(p_t) < p_C^*$$

Note that there exists a value $p_a = \dfrac{\beta\mu\theta + \gamma}{\theta}\omega > 0$ such that $p_C^* = g(p_a)$. Figure 1 qualitatively illustrates the dynamics of the economy when the borrowing constraint does not bind.[5]

Proposition 1. *Let*

$$\theta_1 \equiv \frac{-[\beta(1-\mu) + 2\gamma] + \sqrt{[\beta(1-\mu) + 2\gamma]^2 + 4\beta\mu\gamma}}{2\beta\mu}, \quad and\, \widetilde{\widetilde{p}} = max\{\underline{p}, p_a\} \tag{27}$$

If $\theta > \theta_1$, then the intertemporal equilibrium defined by $p_{t+1} = g(p_t)$ satisfies inequalities (24) and (25) for all $p_t \in (\widetilde{\widetilde{p}}, p_C^*)$. Moreover, there exists a unique steady state $p_{NC}^* \in (\widetilde{\widetilde{p}}, p_C^*)$. For each $p_t \in (\widetilde{\widetilde{p}}, p_C^*)$, there exists an equilibrium path $\{p_t\}_{t=0}^{+\infty}$ converging to the locally stable steady state with dampened oscillations, along which neither a housing price bubble nor a housing demand bubble exist.

Proof. See Appendix 8.2 □

Proposition 1 asserts that an intertemporal equilibrium along which the collateral constraint is not binding exists for all $p_t \in (\widetilde{\widetilde{p}}, p_C^*)$ with $\widetilde{\widetilde{p}} = max\{\underline{p}, p_a\}$. If agents initially coordinate p_t such that $p_t < \underline{p}$, then young households buy houses and save in bonds (see Eq. (18)). Since adult individuals also save, the asset market cannot be in equilibrium. If, instead, agents initially coordinate $p_t > p_C^*$, then the house price is so high that young household will borrow as much as possible to buy houses for housing services til the collateral constraint binds. Finally, if $\underline{p} < p_a$ and agents initially coordinate $p_t \in (\underline{p}, p_a)$, then $p_{t+1} > p_C^*$ and young households of the next generation will borrow as much as possible to buy houses for housing services til the collateral constraint binds.

[5]The construction of Figure 1 is given in Appendix 8.2.

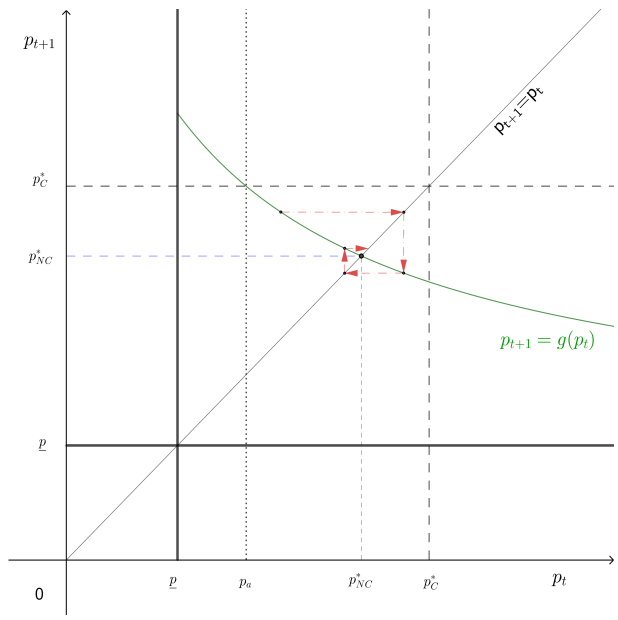

Figure 1: Dynamics when the borrowing constraint does not bind

These two last configurations $p_t > p_C^*$ or $p_t \in (\underline{p}, p_a)$ cannot be an equilibrium without a binding borrowing constraint.

From Proposition 1, we also claim that a unique bubbleless steady state without a binding borrowing constraint exists when financial frictions are not too strong ($\theta > \theta_1$). In this case, the intertemporal equilibrium is such that the price at the next period is strictly lower than p_C^* for all $p_t \in (\widetilde{p}, p_C^*)$. In other words, if the current young households are not borrowing constrained because housing price is low enough, then the next young households will do not face a binding borrowing constraint as well, and housing price will also be low enough. When the financial frictions are severe ($\theta \leq \theta_1$), young household's borrowing capacity reduces. We show in Appendix 8.2 that when $\theta \leq \theta_1$, the intertemporal equilibrium is such that $p_{t+1} = g(p_t) \geq p_C^*$ for all $p_t \in (\widetilde{p}, p_C^*)$. This means that the next young generation will be borrowing constrained. Therefore, no steady state without a binding collateral constraint can exist under a high degree of financial frictions ($\theta \leq \theta_1$).

Proposition 1 shows that the steady state p_{NC}^* is locally stable, and dynamic equilibrium is locally indeterminate. This means that there exists multiple equilibria which converge to a steady state, and that the economy can jump from one equilibrium path to another one due to a change in agents' expectations. Therefore, housing price fluctuations driven by self-fulfilling changes in expectations can emerge. The intuition relies on the three-period structure and is the following. Because preferences are log-linear, consumption $c_{1,t}$ and total savings of young households $s_t = p_t h_{2,t+1} + a_{1t}$ are a positive constant fraction of endowments and do not depend on asset returns:

$$c_{1,t} = \alpha\omega \text{ and } s_t = (\beta + \gamma)\omega$$

On the other hand, consumption when adult $c_{2,t+1}$, savings when adults $a_{2,t+1}$ and consumption

when old $c_{3,t+2}$ depend on asset returns:

$$c_{2,t+1} = R_{t+1}\beta(1-\mu)\omega,\ \ a_{2,t+1} = R_{t+1}\gamma\omega \text{ and } c_{3,t+2} = R_{t+2}R_{t+1}\gamma\omega$$

Assume that households face a higher bond return between t and $t+1$ at adult age. In $t+1$, adults will get a higher income. Since consumptions are normal goods, an increase in adult individuals' income leads to an increase in adult individuals' savings in $t+1$. To understand what happens in $t+2$, let us rewrite the equilibrium condition on the consumption good market:

$$\omega = c_{1,t+2} + c_{2,t+2} + c_{3,t+2} = \alpha\omega + R_{t+2}\beta(1-\mu)\omega + R_{t+2}R_{t+1}\gamma\omega$$

To clear the consumption good market, the remunerated savings of old households in $t+2$ must decrease, which imply a decrease in the return R_{t+2}. This explains oscillations in R_{t+1}.

Now, let us explain the oscillations of housing price. We know that adult individuals' savings in $t+1$ increase if households face a high bond return between t and $t+1$. Since adult individuals save only through deposits, there is a raise in deposits, and as a consequence, the credits made by young individuals also increase. With more credits available, the demand for houses by young households in $t+1$ increases. Since housing supply is fixed, the housing price in $t+1$ increases. These effects reverse at t+2, since bond return decreases in $t+2$. As a consequence, the savings of adults, the amount of deposits and the amount of credits decrease in $t+2$. Young households in $t+2$ will reduce their demand for houses. As housing supply is fixed, the housing price in $t+2$ will decrease as well. This mechanism explains housing price oscillations.

We now consider the case in which young households face a binding borrowing constraint.

4.2 Equilibria with a binding borrowing constraint

Remind that young households face a binding borrowing constraint when $\widetilde{R}_{t+1} > R_{t+1}$. In such an economy, young households borrow to buy houses till the borrowing constraint binds. Furthermore, when the borrowing constraint binds, two cases appear. In one case, bonds provide a strictly higher return than housing demand bubble, $R_{t+1} > R^h_{t+1}$, and housing demand bubble does not occur, $h_{3,t+1} = 0$. In the second case, bonds and housing demand bubble are perfect substitutes and provide the same return R_{t+1}. In this case, a housing demand bubble occurs, $h_{3,t+1} > 0$. Therefore, when the borrowing constraint binds, the following inequalities hold:

$$\widetilde{R}_{t+1} > R_{t+1} \geq R^h_{t+1} > \theta R_{t+1}\ \forall t \geq 0$$

Facing a binding borrowing constraint, a young household borrows θp_t to buy one unit of houses at price p_t. When adult, the household must repay her borrowing, *i.e.* $\theta p_t R_{t+1}$. The returns of selling a house must be greater than the cost of borrowing to ensure a positive income when adult and old, *i.e.* $p_{t+1} > \theta R_{t+1} p_t$ or equivalently $R^h_{t+1} > \theta R_{t+1}$. From the optimal solutions of a household born at t, namely Eqs. (2)-(10), we deduce that (see also Appendix 8.1):

$$p_t h_{2,t+1} = \frac{\beta+\gamma}{1-\theta}\omega \equiv p^*_C > 0 \tag{28}$$

$$(R_{t+2} - R^h_{t+2})p_{t+1}h_{3,t+2} = 0 \tag{29}$$

$$a_{1,t} = -\theta p_t h_{2,t+1} < 0 \tag{30}$$

$$a_{2,t+1} + p_{t+1}h_{3,t+2} = \overline{\theta}(R^h_{t+1} - \theta R_{t+1})p_t h_{2,t+1} \tag{31}$$

with $\overline{\theta} \equiv \gamma/[\beta(1-\mu)+\gamma]$.

Savings of young households are given by $s_t \equiv p_t h_{2,t+1} + a_{1,t}$. Using Eqs. (28) and (30), we obtain that $s_t = (\beta+\gamma)\omega$. Note that a credit constrained young household saves the same amount as a non credit constrained young household. However, the portfolio choice depends on the borrowing constraint being binding. Furthermore, because of credit market imperfections, adults can now invest in bonds $a_{2,t+1}$ and houses $h_{3,t+2}$ if $R_{t+2} = R^h_{t+2}$ (see Eq. (29)). This means that a housing demand bubble can exist. To clarify the conditions under which a housing demand bubble can exist and the mechanisms behind such a phenomenon, we consider first the economy without a housing demand bubble ($h_{3,t+1} = 0$), and second with a housing demand bubble ($h_{3,t+1} \geq 0$).

4.2.1 Equilibrium without a housing demand bubble ($h_{3,t+1} = 0$)

We analyze the equilibrium when $R_{t+1} > R^h_{t+1}$ in each period, implying that housing demand bubble is a dominated asset and $h_{3,t+1} = 0 \ \forall t \geq 0$. As discussed in Section 3.2, the positive dividends exclude the possibility of a housing price bubble as well.

Using the market clearing conditions on housing (14), $h_{2,t+1} = 1$ for all $t \geq 0$. Therefore, using Eq. (28) we deduce that the housing price p_t is constant over time in this economy:

$$p_t \quad = \quad p^*_C > \underline{p} \tag{32}$$

In addition, Eqs. (30) and (31) give:

$$a_{1,t} = -\theta p^*_C \tag{33}$$

$$a_{2,t+1} = \overline{\theta}(R^h_{t+1} - \theta R_{t+1})p^*_C \tag{34}$$

with $a_{2,t+1} > 0$ since $R^h_{t+1} > \theta R_{t+1}$. Since the price is constant, $R^h_{t+1} = 1$ and using the bond market clearing condition, we deduce that the bond return R_{t+1} is constant over time as well:

$$R_{t+1} \quad = \quad \frac{1}{\theta} - \frac{1}{\overline{\theta}} \equiv R^*_C \tag{35}$$

Note from Eq. (35) that $1 > \theta R^*_C$ is always satisfied, whereas $R^*_C > 1$, and thus $h_{3,t+1} = 0$, introduces an upper bound on θ, *i.e.*:

$$\theta < \frac{\gamma}{\beta(1-\mu)+2\gamma} \equiv \theta_2 \quad < \overline{\theta} \tag{36}$$

Proposition 2. *If $\theta \in (\theta_1, \theta_2)$, then there exists a unique equilibrium $p_t > \underline{p}$ with $p_0 = p^*_C$ such that $p_t = p^*_C$ for all $t \geq 0$ satisfying the binding borrowing constraint. A higher degree of pledgeability θ implies a higher housing price p^*_C at the steady state.*

Proof. See Appendix 8.3. □

The higher θ, the greater the borrowing capacity. Therefore, the demand for houses increases. Because of the housing supply is fixed, the housing price rises.

The next section aims to prove the existence of a housing demand bubble in the economy when young households face a binding borrowing constraint.

4.2.2 Equilibria with a housing demand bubble ($h_{3,t+1} > 0$)

We now analyze the equilibrium when $R_{t+1} = R^h_{t+1}$ in each period t. The demand for assets when young are given by Eqs. (28) and (30), and when adults by:

$$a_{2,t+1} + p_{t+1}h_{3,t+2} \quad = \quad \overline{\theta}(1-\theta)R^h_{t+1}p^*_C > 0 \tag{37}$$

Using the market clearing conditions on bonds $a_{1,t+1} + a_{2,t+1} = 0$ with Eqs. (28), (30) and (37), we have:

$$p_{t+1}h_{3,t+2} \quad = \quad \left[\overline{\theta}(1-\theta)R_{t+1} - \theta\right] p^*_C \tag{38}$$

Note that a housing demand bubble occurs ($h_{3,t+2} > 0$) when $R_{t+1} > \theta[\beta(1-\mu)+\gamma]/[\gamma(1-\theta)]$. This inequality ensures that the economy experiences a shortage of assets when bonds and housing demand provide the same return ($R_{t+1} = R^h_{t+1}$). In other words, this condition corresponds to a situation where the savings from adult individuals is larger than loans to be financed. Speculative demand for houses allows clearing the bond market $a_{2,t} = -a_{1,t}$. Moreover, the dividends are given by $div_{t+1} = p_{t+1}\beta\mu(1-\theta)/[\beta(1-\mu)+\gamma]$. Since the house price is strictly positive, the dividends are also strictly positive and do not tend to zero along an intertemporal equilibrium. In line with Section 3.2, a housing price bubble does not exist in an economy characterized by a binding borrowing constraint and a housing demand bubble.

The market clearing condition on housing, $h_{2,t+2} + h_{3,t+2} = 1$, implies:

$$p_{t+1}h_{3,t+2} = p_{t+1} - p^*_C \tag{39}$$

Using Eqs. (38) and (39), we deduce that the intertemporal equilibrium, which is a sequence $\{p_t\}_{t=0}^{+\infty}$, given $p_0 > \underline{p}$, satisfies for all $t \geq 0$:

$$p_{t+1} \quad = \quad \underline{p}\frac{p_t}{p_t - \overline{\overline{p}}} \equiv h(p_t) \tag{40}$$

$$\text{with } \overline{\overline{p}} \quad = \quad \underline{p}\overline{\theta} > 0 \tag{41}$$

The dynamics reduce to a first-order recursive equation in a non-predetermined variable p_t. Note that $h(p_t)$ is a decreasing function of p_t. Since young households buy houses for housing services, houses must be valued at the equilibrium (see Eq. (28)), and thus $p_t > 0$ at t. As regards Eqs. (39) and (40), a housing demand bubble occurs at each period only when:

$$p_t > p^*_C \text{ and } p_{t+1} = h(p_t) > p^*_C \tag{42}$$

Note that the second inequality, which corresponds to $R_{t+1} > \theta[\beta(1-\mu)+\gamma]/[\gamma(1-\theta)]$ (see Eq. (38)), is also equivalent to $p_t < p_d$ with $p_d > 0$ is given by $p^*_C = h(p_d)$. Figure 2 gives a qualitative illustration of the dynamics when the borrowing constraint binds and a housing demand bubble occurs.[6] We deduce the following proposition:

Proposition 3. *Let*

$$p^{**}_C = \frac{\overline{\theta}}{\theta_2}\underline{p} > \quad \underline{p} \tag{43}$$

[6]The construction of Figure 2 is given in Appendix 8.4.

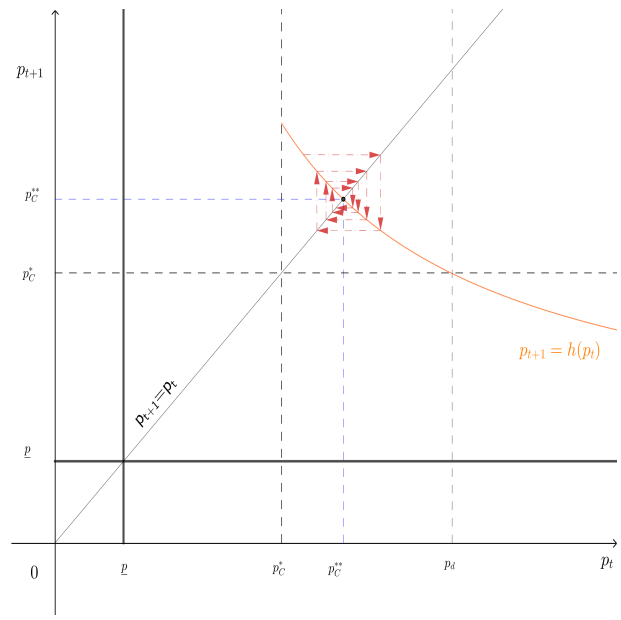

Figure 2: Dynamics when the borrowing constraint binds and a housing demand bubble exists

If $\theta < \theta_2$ and $p_0 \in (p_C^*, p_d)$, then there exists a continuum of equilibria depending on agents' expectations along which a housing demand bubble exists, but not a housing price bubble. For each $p \in (p_C^*, p_d)$, there exists an equilibrium path $\{p_t\}_{t=0}^{+\infty}$ converging to the locally stable steady state p_C^{**} with dampened oscillations. The housing price p_C^{**} does not depend on the degree of pledgeability θ.

Proof. See Appendix 8.4. □

Proposition 3 indicates that if agents coordinate initially in a price p_0 such that $p_0 \in (p_C^*, p_d)$, then a bubble housing demand bubble exists, and housing price endogenously fluctuates around the steady state value p_C^{**}. Obviously, the steady state with a housing demand bubble features a higher housing price than the steady state without a housing demand bubble. When a housing demand bubble occurs, the aggregate housing demand increases. Since the housing supply is fixed, the housing price increases as well. If agents, and thus adults, initially choose p_0 such that $p_0 \geq p_d$, the house price at the next period will be so low that next young households will not face a binding borrowing constraint. Therefore, this cannot be an equilibrium with a binding borrowing constraint.

According to Proposition 3, the threshold θ_2 is a critical value for the existence of a steady state with a housing demand bubble. Such a steady state may occur when credit market imperfections are severe, *i.e.* $\theta < \theta_2$. If they are not ($\theta > \theta_2$), the borrowing capacity is sufficiently large to allow young households to not be credit constrained and contract enough loan to absorb all savings. In this case, housing demand bubbles are ruled out. Interpreting θ as the loan-to-value ratio, which is a macroprudential tool aimed to control mortgage loan creation, we can claim that a restrictive macroprudential policy, namely implementing a low θ such that $\theta < \theta_2$, promotes the occurrence of the housing demand bubble all other things being equal. We can also note that $\theta < \theta_2$ is equivalent

to a bond return, R, greater than one at the bubbleless steady state ($R_C^* > 1$). Remind that, in our paper, a steady state with a housing demand bubble may occur when the bond return is equal to one ($R = 1$) and the bond demand is larger than bond supply ($a_2 + a_1 > 0$). Since in a credit constrained economy the bond demand decreases with the bond return, whereas the supply does not depend on the return, a steady state with a housing demand bubble exists only if $R_C^* > 1$. This results contrasts with the results in the classical rational bubble models as of Samuelson (1958) and Tirole (1985) in which a return lower than one is necessary at the bubbleless steady state for the occurrence of a bubble.

Furthermore, we can note that the existence of a steady state with a housing demand bubble corresponds to a zero lower bound environment, namely a bond return equal to one (or, equivalently, an interest rate equal to zero) in absence of inflation. In contrast to Bonchi (2022), a housing demand bubble does not allow a central bank to escape from the zero lower bound.[7] Our result also means that an expansionary monetary policy where a central bank lowers interest rates to zero to stimulate the economy will create an excess of bond demand and makes room for the existence of a housing demand bubble. This result contrasts with Gali (2014), who argues that the monetary authority should lower the nominal interest rates when the bubble grows to stabilize the economy.

Finally, Proposition 3 also shows that the steady state p_C^{**} is locally indeterminate as in the economy without a binding borrowing constraint. Therefore, endogenous housing price fluctuations driven by self-fulfilling changes in expectations can emerge. In presence of credit market imperfections, a change in housing price or in bond return only affects the portfolio choice and not the savings of young agents ($p_t h_{2,t+1} + a_{1t}$). Indeed, savings of young households are constant, positive and only depend on endowments ω because preferences are log-linear. Moreover, consumption and savings of adults, $c_{2,t+1}$ and $a_{2,t+1} + p_{t+1} h_{3,t+2}$, are linear in bond return R_{t+1}. Therefore, the intuition to explain oscillations of housing price is similar to the one of the model without a binding borrowing constraint.

4.3 Some remarks about welfare

Depending on the degree of pledegeability θ, the economy can feature only one steady state or several. Remind that $\theta_1 < \theta_2$. When credit market imperfections are minor ($\theta \geq \theta_2$), there is a unique steady state p_{NC}^* such that the borrowing constraint does not bind and a housing demand bubble does not exist $h_3 = 0$. When credit market imperfections are severe ($\theta \leq \theta_1$), there is a unique steady state p_C^{**} such that the borrowing constraint binds and a housing demand bubble exists $h_3 > 0$. For intermediate degrees of credit market imperfections ($\theta \in (\theta_1, \theta_2)$), three steady states coexist: p_{NC}^*, p_C^{**} and p_C^*. In this last steady state, the borrowing constraint is binding and $h_3 = 0$. We next focus on the case $\theta \in (\theta_1, \theta_2)$ and compare the level of welfare attained in these three steady states to determine the effect on welfare of a housing demand bubble.

At a steady state, the welfare level W a is given by:

$$W = \alpha ln(c_1) + \beta \left[(1-\mu) lnc_2 + \mu lnh_{2,}\right] + \gamma lnc_3 \tag{44}$$

where c_i with $i = 1, 2, 3$ and h_2 are evaluated at a steady state. Because preferences are log-linear, young households consume the same constant fraction of endowments, namely $\alpha\omega$, whatever

[7] In a three-period OLG model, Bonchi (2022) studies how a pure asset bubble affects monetary policy by relaxing the zero lower bound constraint. Despite financial frictions, the author shows that a pure bubble occurs when the interest rate is lower than the growth rate of the economy in the bubbleless steady state as Tirole (1985), and raises the interest rate.

the steady state. The housing demand bubble and credit market imperfections do not affect the behavior of young households. The following lemma summarizes the consumption of adults and old households at the different steady states.

Lemma 1. *Define $c^*_{2,NC}$, $c^*_{2,C}$, $c^{**}_{2,C}$ $c^*_{3,NC}$, $c^*_{3,C}$ and $c^{**}_{3,C}$ as the levels of adult and old households' consumption at the steady states p^*_{NC}, p^*_C and p^{**}_C, respectively. If $\theta \in (\theta_1, \theta_2)$, $c^*_{2,NC} < c^*_{2,C} < c^{**}_{2,C}$ and $c^*_{3,NC} > c^*_{3,C} > c^{**}_{3,C}$.*

Proof. See Appendix 8.5. □

Housing demand bubbles hurt old individuals, have no effect on young individual and an ambiguous effect on adult individuals since they reduce housing services but increase consumption. As a result, housing demand bubbles increase welfare only when adult individuals care little about housing services (μ small) and old individuals are not so important (γ small). Using Lemma 1, we can derive the following proposition:

Proposition 4. *Define W^*_{NC}, W^*_C and W^{**}_C as the levels of welfare at the steady states p^*_{NC}, p^*_C and p^{**}_C respectively. If $\theta \in (\theta_1, \theta_2)$ and μ and γ are sufficiently small, then $W^*_{NC} < W^*_C < W^{**}_C$.*

Proof. Use Eq. (44) and Lemma 1. □

Proposition 4 indicates that the steady states are Pareto-ranked, and households are better off in the steady state with a housing demand bubble. The intuition is the following. From the analyses of each type of equilibrium, we know that the housing price is higher when the borrowing constraint binds and even higher when a housing demand bubble exists, that is $p^*_{NC} < p^*_C < p^{**}_C$. In contrast, the bond return is lower when the borrowing constraints binds and even lower when a housing demand bubble exists, that is $R^*_{NC} > R^*_C > R^{**}_C = 1$, where R^*_{NC} is the bond return when the borrowing constraint does not bind and R^{**}_C when a housing demand bubble occurs. In other words, adult households sell at a higher price the house and face a lower borrowing cost, and thus are richer, when the borrowing constraint binds and even more when a housing demand bubble occurs. In contrast, old households obtain a lower return on savings, and thus are poorer, when the borrowing constraint binds and even more when a housing demand bubble occurs. As a consequence, adult individual consume more when the borrowing constraint binds and even more when a housing demand bubble occurs, whereas old individuals consume less when the borrowing constraint binds and even less when a housing demand bubble occurs. To a certain extent, the fall of the interest rate, generated by the existence of credit market imperfections and of a housing demand bubble, indirectly induces transfers of resources from old to adult households. If the weight of housing services μ in the utility function and the weight of consumption when old γ are sufficiently small, then the housing demand bubble has a positive welfare effect.

5 Global dynamics and endogenous regime switching

The existence of multiple steady states makes the analysis of equilibrium global dynamics relevant. We show the equilibrium may exhibit global indeterminacy, which is a source of expectation-driven housing price fluctuations. In contrast to local indeterminacy, global indeterminacy means that a change in agents' expectations may make the equilibrium converge to a different steady state. Furthermore, some equilibria display fluctuations characterized by endogenous transitions between

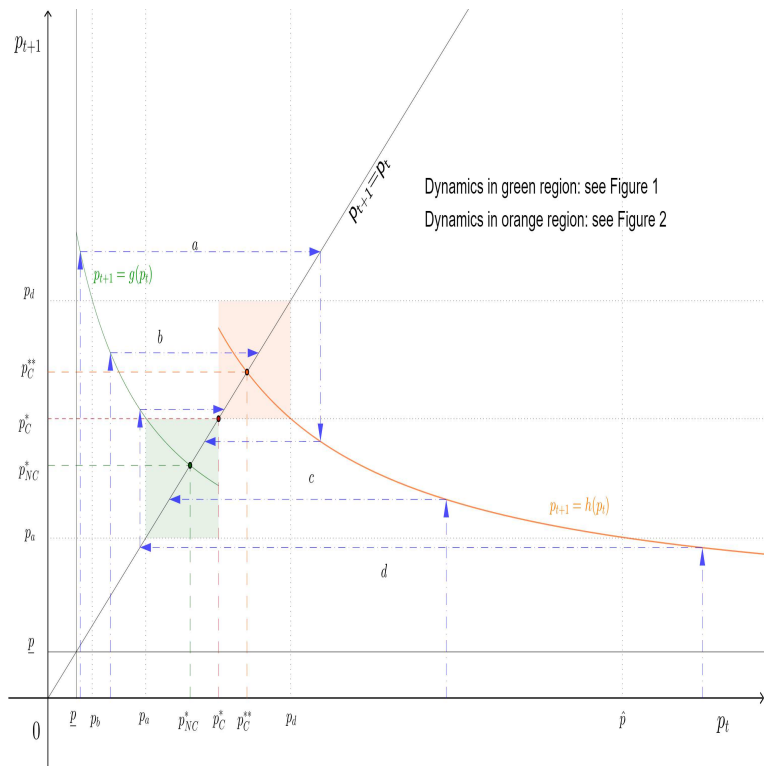

Figure 3: Global dynamics

the two regimes according to agents' expectations about the initial housing price. We show that along the transition periods with a housing demand bubble alternate with periods without such a bubble.

Using the dynamic analysis of each regime, we can now study global dynamics. We restrict our attention to the case $\theta \in (\theta_1, \theta_2)$ in which Propositions 1-3 apply. Assuming $\theta \in (\theta_1, \theta_2)$, the dynamics of the economy is summarized by this dynamic equation:

$$p_{t+1} \quad = \quad \begin{cases} g(p_t) & \forall p_t \in (\underline{p}, p_C^*) \\ p_t & \text{when } p_t{=}p_C^* \\ h(p_t) & \forall p_t \in (p_C^*, +\infty) \end{cases} \tag{45}$$

where $g(p_t)$ and $h(p_t)$ are given respectively by Eqs. (26) and (40).

Assumption 1. *Let* $\theta_3 = max\{\theta_1, \widehat{\theta}\}$, where $\widehat{\theta}$ is defined in Appendix 8.6. We assume $\theta \in (\theta_3, \theta_2)$.

Figure 3 gives a qualitative illustration of global dynamics under Assumption 1. Using Propositions 1, 2 and 3 and Figure 3, we deduce the following proposition[8]:

Proposition 5. *Under Assumption 1, the following generically holds:*

[8]The construction of Figure 3 is given in Appendix 8.6. Figure 3 does not depict case 8 of Proposition 5 .

1. If $p_t \leqslant \underline{p}$, there is no equilibrium.

2. If $p_t \in (\underline{p}, p_b)$, then $p_{t+1} > p_d$ and the long run equilibrium is as in cases 6, 7 or 8 (case a in Figure 3).

3. If $p_t \in (p_b, p_a)$, then $p_{t+i} \in (p_C^*, p_d)$ for all $i \geqslant 1$ and p_{t+i} converges to p_C^{**} (case b in Figure 3).

4. If $p_t \in (p_a, p_C^*)$, then $p_{t+i} \in (p_a, p_C^*)$ for all $i \geqslant 1$ and p_{t+i} converges to p_{NC}^* (See Figure 1).

5. If $p_t \in (p_C^*, p_d)$, then $p_{t+i} \in (p_C^*, p_d)$ for all $i \geqslant 1$ and p_{t+i} converges to p_C^{**} (See Figure 2).

6. If $p_t \in (p_d, \widehat{p})$, then $p_{t+i} \in (p_a, p_C^*)$ for all $i \geqslant 1$ and p_{t+i} converges to p_{NC}^* (case c in Figure 3).

7. If $p_t \in (\widehat{p}, p_B)$, then $p_{t+1} \in (p_b, p_a)$ and $p_{t+i} \in (p_C^*, p_d)$ for all $i \geqslant 2$ and p_{t+i} converges to p_C^{**} (case d in Figure 3).

8. If $p_t > p_B$, then $p_{t+1} \in (\underline{p}, p_b)$, and case 2 determines the long run equilibrium.

Proof. See Appendix 8.6 □

Proposition 5 shows that our economy displays global indeterminacy. This result is in connection with the existence of multiple equilibrium paths converging with dampened oscillations to different steady states. Indeed, two stable steady states - one without a housing demand bubble p_{NC}^* and one with a housing demand bubble p_C^{**} - exist. Our economy may switch between equilibrium paths that converge to these two different steady states p_{NC}^* and p_C^{**} depending on agents' expectations (sunspots). In particular, this global indeterminacy implies that regime-switching sunspot equilibria may occur.[9]

Our economy can endogenously switch from a regime without credit constrained young households to a regime with credit constrained households, and *vice versa*. Therefore, endogenous fluctuations may also occur due to regime changes. For instance, if agents initially coordinate on $p_0 \in (\underline{p}, p_b)$ (case 2), then the initial housing price p_0 is so low that the initial young households do not face a binding borrowing constraint and adults do not purchase houses for speculative motives $h_{3,0} = 0$. No housing demand bubble emerges. At the next period, p_1 is so high that the next generation of young households now face a binding borrowing constraint and the adult individuals purchase houses for speculative motives $h_{3,1} > 0$. A housing demand bubble appears. At time $t = 1$, the dynamics of the economy is given by the dynamics under a binding borrowing constraint regime with housing demand bubbles. The price at the next period p_2 will be so low that the next young households do not face a binding borrowing constraint and the next adults do not purchase houses for speculative motives $h_{3,2} = 0$. The housing demand bubble endogenously bursts. At time $t = 2$, the dynamics of the economy are given by the dynamics under an unconstrained regime without housing demand bubbles. We conclude that transitions between a regime without credit constrained households nor housing demand bubble and a regime with credit constrained households and housing demand bubble can occur in this economy. Our basic model can generate

[9] More precisely, we could construct regime-switching sunspot equilibria in which the borrowing constraint is not binding and no housing demand bubble occurs in one state and the borrowing constraint is binding and a housing demand bubble occurs in the other state, assuming that agent's expectations about future housing price (a sunspot variable) follow a two-state Markov process.

small endogenous housing price fluctuations in the neighborhood of steady states, but also large and sudden booms and crashes in housing price when the equilibrium exhibits transition from a regime with a housing demand bubble to a regime without housing demand bubbles. These large fluctuations in house prices may explain the observed fluctuations that occur around financial crisis (see Miao (2014) and Miao *et al.* (2015)).

This result, showing that our economy can endogenously switch from a regime without a housing demand bubble to a regime with a housing demand bubble, is in contrast to models with rational price bubbles. The difference relies on the nature of the bubble: housing price bubble *versus* housing demand bubble. From the rational bubble literature, an asset price bubble exists if it will persist with a positive probability at the next period. Moreover, if an asset price bubble bursts, such a bubble cannot reappear. In contrast, a housing demand bubble can reappear after its burst. This result overcomes a criticism about rational bubble models (see Martin and Ventura (2012, 2021) and Guerron-Quintana *et al.* (2018)). Our result also echoes the results of Michel and Wigniolle (2003, 2005) where a rational bubble may arise after its bursting. Michel and Wigniolle (2003, 2005) develop a monetary OLG economy in which the demand of money is rationalized by a cash-in-advance constraint. When the cash-in-advance constraint binds, money is held to finance transactions. When the constraint does not bind, money is held for speculative motives only. In the second case, money depicts a pure bubble. The authors show that the economy may oscillate from periods where the cash-in-advance constraint binds, and periods where the constraint does not bind, and a bubble emerges on money.

Now, we turn to study whether the housing demand bubble is growth-enhancing. To do this, we extend our basic framework to endogenous growth in the next section.

6 Extension to a production economy

We extend the model to an endogenous growth framework. We assume that there exists a production sector with a simple Ak type technology. Aggregate output is produced by a continuum of firms, of unit size, using labor, l_t, and capital, k_t, as inputs. A production externality exists that embodies a learning-by-doing process, and generates sustained growth. Following Frankel (1962) or Ljungqvist and Sargent (2004, chapter 14), this externality depends on the average capital-labor ratio.

Let $z_t \equiv k_t/l_t$, and $\overline{z}_t$ represents the average capital to labor ratio. Firms produce the final good using the following technology:

$$y_t = F(k_t, \bar{z}_t l_t)$$

The technology $F(k_t, \bar{z}_t l_t)$ has the usual neoclassical properties: a strictly increasing and concave production function satisfying the Inada conditions, and homogeneous of degree one with respect to its two arguments.

Profit maximization under perfect competition implies that the wage w_t and the return of capital q_t are given by[10]:

$$\omega_t = F_2(k_t, \bar{z}_t l_t)\bar{z}_t \tag{46}$$

$$q_t = F_1(k_t, \bar{z}_t l_t) \tag{47}$$

All equilibria we will consider are symmetric ones, *i.e.* $z_t = \overline{z}_t$. Let us define $s \equiv F_1(1,1)/F(1,1) \in (0,1)$ the capital share in total production and $A \equiv F(1,1) > 0$. Using (46) and (47), we deduce

[10] We denote by $F_i(.,.)$ the derivative with respect to the ith argument of the function.

that:

$$\omega_t = (1-s)Az_t \equiv \omega(z_t) \tag{48}$$

$$q_t = sA \tag{49}$$

which give the wage and the return of capital at an equilibrium.

As regards the household's problem, young households now work. They supply one unit of labor, earn the wage ω_t, consume, purchase houses to enjoy housing services when adult and save or borrow through bonds. Adult individuals do not work, consume and can now save through a diversified portfolio, which consists of deposits $\widetilde{a}_{2,t+1}$, houses $h_{3,t+2}$, and physical capital k_{t+2}.[11] As in Clain-Chamosset-Yvrard and Seegmuller (2019), we assume that young individuals cannot invest in physical capital. Interpreting k_{t+2} as the value of firms, our assumption is in line with some empirical studies. For instance, Poterba and Samwick (2001) show that stock market participation rates are very low when young, increase during middle-aged to reach a peak around age 50 and may decrease during retirement. Furthermore, Heaton and Lucas (2000) show that households who finance their houses through a mortgage contract have a limited financial wealth to participate in stock market.

A household maximizes the utility (1) under the following budget and borrowing constraints:

$$c_{1,t} + a_{1,t} + p_t h_{2,t+1} = \omega_t \tag{50}$$

$$c_{2,t+1} + \widetilde{a}_{2,t+1} + k_{t+2} + p_{t+1}h_{3,t+2} = R_{t+1}a_{1,t} + R^h_{t+1}p_t h_{2,t+1} \tag{51}$$

$$c_{3,t+2} = R_{t+2}\widetilde{a}_{2,t+1} + q_{t+2}k_{t+2} + R^h_{t+2}p_{t+1}h_{3,t+2} \tag{52}$$

$$\theta p_t h_{2,t+1} \geq -a_{1,t} \tag{53}$$

Solving the household problem, we deduce that deposits and physical capital are perfect substitutes, *i.e.* $q_{t+2} = R_{t+2}$. Therefore, the problem is equivalent to the exchange economy with $\omega = \omega_t$, $a_{2,t+1} = \widetilde{a}_{2,t+1} + k_{t+2}$ and $q_{t+2} = R_{t+2}$.

From the analysis of the exchange economy, we know that a housing demand bubble occurs when the borrowing constraint binds. As the household problem with a production sector is equivalent to the exchange economy, it seems interesting to focus only on equilibria for which the borrowing constraint binds. Young households face a binding borrowing constraint when $\widetilde{R}_{t+1} > R_{t+1}$. Using Eqs. (28)-(31), we deduce that:

$$p_t h_{2,t+1} = \frac{\beta+\gamma}{1-\theta}\omega_t > 0 \tag{54}$$

$$(R_{t+2} - R^h_{t+2})p_{t+1}h_{3,t+2} = 0 \tag{55}$$

$$a_{1,t} = -\theta p_t h_{2,t+1} < 0 \tag{56}$$

$$\widetilde{a}_{2,t+1} + k_{t+2} + p_{t+1}h_{3,t+2} = \overline{\theta}(R^h_{t+1} - \theta R_{t+1})p_t h_{2,t+1} \tag{57}$$

where $\overline{\theta}$ is defined in Eq. (31).

Like in the exchange economy, deposits $\widetilde{a}_{2,t+1}$ are the counterpart of loans made by young adults ($a_{1,t} < 0$). Therefore, adults have to make strictly positive deposits $\widetilde{a}_{2,t+1} > 0$ to clear the bond market, which implies $R^h_{t+1} > \theta R_{t+1}$. Like in the exchange economy, two types of equilibria exist

[11] For tractability, we assume that adults do not work. Relaxing this assumption will not change qualitatively our results.

when the borrowing constraint binds: an equilibrium without a housing demand bubble ($h_{3,t+1} = 0$) when $R^h_{t+1} < R_{t+1} = q_{t+1}$ and an equilibrium with a housing demand bubble ($h_{3,t+1} > 0$) when $R^h_{t+1} = R_{t+1} = q_{t+1}$. Therefore, the following inequalities must hold:

$$\widetilde{R}_{t+1} > R_{t+1} = q_{t+1} \geq R^h_{t+1} > \theta R_{t+1} \; \forall t \geq 0 \tag{58}$$

Since we consider an endogenous growth framework, we must define $x_t \equiv p_t/k_t$ and $g_{t+1} \equiv k_{t+1}/k_t$. Taking into account that the population size of each generation is constant and normalized to one, the equilibrium in the labor market requires $l_t = 1$. Hence, we have $k_{t+1} = z_{t+1}l_{t+1} = z_{t+1}$. Using the definition of dividends in Eq. (6), the optimal consumption when adult in Eq. (94), the optimal behavior of the firm in Eqs. (48) and (49) and market clearing conditions on bond $a_{1,t} + \widetilde{a}_{2,t} = 0$ and housing $h_{2,t} + h_{3,t} = 1$, Inequality (58) and Eqs. (54)-(57) rewrite:

$$g_{t+1}\frac{x_{t+1}}{x_t} + \frac{\beta\mu}{\beta(1-\mu)+\gamma}(g_{t+1}\frac{x_{t+1}}{x_t} - \theta sA) > sA \geq g_{t+1}\frac{x_{t+1}}{x_t} > \theta sA \; \forall t \geq 0 \tag{59}$$

$$x_t - x_t h_{3,t+1} = \frac{\beta+\gamma}{1-\theta}(1-s)A \equiv x_C > 0 \tag{60}$$

$$(sA - g_{t+2}\frac{x_{t+2}}{x_{t+1}})x_{t+1}h_{3,t+2} = 0 \tag{61}$$

$$x_C\theta + g_{t+2} + x_{t+1}h_{3,t+2} = \overline{\theta}(g_{t+1}\frac{x_{t+1}}{x_t} - \theta sA)x_C\frac{1}{g_{t+1}} \tag{62}$$

From Inequality (59), we observe that a housing demand bubble occurs when $g_{t+1}x_{t+1}/x_t = sA$ and does not occur when $g_{t+1}x_{t+1}/x_t < sA$. Since the most interesting case is the equilibrium with a housing demand bubble, we will deeply analyze this type of equilibria and study its dynamics.[12]

Remind that in presence of a housing demand bubble, the borrowing constraint is always binding. We deduce from Eq. (60) that a housing demand bubble exists when $x_t > x_C$. Using Eqs. (60) and (62), the equilibrium when $h_{3,t+1} > 0$ is driven by a two-dimensional dynamic system:

$$g_{t+1} + x_t - x_C = x_C\frac{sA\overline{\theta}(1-\theta) - \theta g_t}{g_t} \equiv \Gamma(g_t) \tag{63}$$

$$g_{t+1}\frac{x_{t+1}}{x_t} = sA \tag{64}$$

with one predetermined variable, $g_t = k_t/k_{t-1}$. Note that $g_{t+1} + x_t - x_C > 0$, which is equivalent to $\Gamma(g_t) > 0$, implies an upper bound on growth: $\widetilde{g} \equiv sA\overline{\theta}(1-\theta)/\theta > 0$. If $g_t > \widetilde{g}$, then the income of an adult individual in t will be so low that they cannot save enough to clear the bond market. Therefore, $g_t > \widetilde{g}$ cannot be an equilibrium.

From Eqs. (63) and (64), we deduce that a balanced growth path (BGP), $g_t = g^*$ and $x_{t+1} = x_t = x^*$, with a housing demand bubble ($x^* > x_C$), is given by:

$$\begin{cases} g^* &= sA \\ x^* &= \Omega - sA + x_C \end{cases} \tag{65}$$

[12]We abstract from the analysis of equilibria without a housing demand bubble. In this case, we have a unique unstable BGP with a predetermined variable. Therefore, there is a nul probablilty that we reach the BGP without a housing demand bubble. More details are available upon request.

where $\Omega \equiv x_C \left(\overline{\theta} - \theta\right)(sA - \overline{g})/sA$ with $\overline{g} \equiv sA\theta\overline{\theta}/(\overline{\theta} - \theta_2)$ and $\overline{g} < sA$ if $\theta < \theta_2$.[13] Moreover, if $s < (\beta + \gamma)\overline{\theta}(\theta_2 - \theta)/[\theta_2(1 - \theta)] \equiv \overline{s}$ and $\theta \in (\theta_1, \theta_2)$, then $\Omega(sA, x_C) > sA$.

Denote by g_C the economic growth factor along the BGP without a housing demand bubble. We deduce, from Inequality (59), that growth is lower than the interest factor along a BGP without a housing demand bubble ($g_C < sA$), whereas it equals the interest factor along a BGP with a housing demand bubble ($g^* = sA$). Therefore, a BGP with a housing demand bubble is characterized by a higher economic growth than the BGP without a housing demand bubble ($g^* > g_C$). We deduce the following proposition:

Proposition 6. *If $\theta \in (\theta_1, \theta_2)$ and $s < \overline{s}$, then there exists a unique BGP with a housing demand bubble that satisfies $x^* > x_C$ and $g^* \in (g_C, \widetilde{g})$. This BGP is a stable saddle path. Therefore, in the vicinity of the BGP, there exists a unique equilibrium path converging to the BGP.*

Proof. See Appendix 8.7 □

To understand the crowding-in effect of a housing demand bubble, let us write the total savings of the economy, namely the sum of the savings of young households and adults in period t, using Eqs. (48), (49), (60), (61) and (62):

$$k_{t+1} + p_t = (\beta + \gamma)\omega_t + \overline{\theta}(R_t^h - \theta sA)\frac{\beta + \gamma}{1 - \theta}\omega_{t-1} \tag{66}$$

The left-hand side of Eq. (66) is the total savings in the economy which consist of physical capital k_{t+1} and the value of houses p_t. The right-hand side of Eq. (66) is the sum of two terms. The first one corresponds to a share of young households' income ω_t. The second one depicts a share of adult individuals' income $(R_t^h - \theta sA)(\beta + \gamma)\omega_{t-1}/(1 - \theta)$. As in the rational bubble literature, the housing demand bubble have two effects on economic growth. The first one is the *crowding-out effect* on future physical capital, since part of the savings are used to purchase houses. The second one is the *crowding-in effect* through adult individuals' income. When a housing demand bubble occurs, the growth factor of housing price R_t^h increases, which has a positive effect on adult individuals' income. Since adult individuals invest in physical capital, a higher income allows them to invest more in physical capital.[14] Note that the crowding-in effect of the housing demand bubble is akin to a liquidity effect, because adult individuals have more liquidity to buy physical capital.

Furthermore, Proposition 6 shows that in the presence of a housing demand bubble, the housing price-physical capital ratio x^* is higher than the one without a housing demand bubble x_C. This result means, using Eq. (48), that the housing price to output ratio $(p/y)_C^*$ in the presence of the housing demand bubble is higher than the one without the housing demand bubble $(p/y)_C$.

By extending our framework to a production economy, our concept of housing bubble demand explains the crowding-in effect of a boom in housing price on economic growth. These results are consistent with the stylized facts mentioned in Miao (2014) or in Miao *et al.* (2014), namely episodes of large boom in housing prices are associated with episodes of economic expansions, whereas episodes of large and sudden crash in housing price are associated with episodes of recessions without requiring any exogenous shocks on fundamentals or bubbles in contrast to Martin and Ventura (2012), Zhao (2015) and Martin *et al.* (2021).

[13] Remind that $\theta_2 \in (\theta_1, \overline{\theta})$ is defined in inequality (36).

[14] Note that if adult individuals work, their labor income will reinforce this crowding-in effect.

7 Concluding remarks

Introducing the concept of a housing demand bubble in an OLG model, we can explain large fluctuations in housing prices and co-movements between housing price and economic growth, as observed in the data.

We develop a basic three-period OLG exchange economy with three ingredients: housing, heterogeneity in terms of investment decisions and credit market imperfections embodied in a borrowing constraint. Housing plays three roles: It is a store of value, it can be used as a collateral, and it generates utility services. Households make different investment decisions at different periods of life. Young households borrow to purchase houses for enjoying housing services and face a borrowing constraint. Adults may invest in deposits and a housing asset. The housing demand of adults only results from speculative motives. If such a demand exists in the equilibrium, then a housing demand bubble occurs generating a boom in housing prices.

We show that housing prices are explained by their fundamentals, and hence, no housing price bubble occurs in this economy. However, a housing demand bubble may occur when the borrowing constraint binds. In addition to explain the occurrence of housing demand bubble, credit market imperfections promote multiplicity of steady states and endogenous fluctuations. For not too strict and not too weak credit market imperfections, we show that three steady states may coexist: one stable without a housing demand bubble when the borrowing constraint does not bind and two -one unstable without a housing demand bubble and one stable with a housing demand bubble- when the borrowing constraint binds. Because of this multiplicity of stable steady states, our model can generate large and sudden booms and crashes in housing price due to a change in agents' expectations in addition to small fluctuations in the neighborhood of stable steady states. Furthermore, we also show that the deterministic dynamics allow this economy to switch from a regime without a housing demand bubble to a regime with a housing demand bubble.

We then extend the basic framework to a production economy in order to study the crowding-in effect of the housing demand bubble. Focusing on the case where the borrowing constraint always binds, we show that a housing demand bubble boosts the housing price and economic growth. Based on our results, we can conjecture that a monetary policy or a fiscal policy designed to prevent a financial crisis by ruling out a housing bubble, that is "leaning against the wind" such as increasing the interest rates or taxing the housing value, would be detrimental for economic growth.

8 Appendix

8.1 Proof of optimal household's behavior

We maximize the Lagrangian function:

$$\begin{aligned}
&\alpha lnc_{1,t} + \beta[(1-\mu)lnc_{2,t+1} + \mu lnh_{2,t+1}] + \gamma lnc_{3,t+2} \\
&+\lambda_{1,t}\left(\omega - c_{1,t} - a_{1,t} - p_t h_{2,t+1}\right) \\
&+\lambda_{2,t+1}\left[R_{t+1}a_{1,t} + R^h_{t+1}p_t h_{2,t+1} - c_{2,t+1} - a_{2,t+1} - p_{t+1}h_{3,t+2}\right] \\
&+\lambda_{3,t+2}\left[R^h_{t+2}p_{t+1}h_{3,t+2} + R_{t+2}a_{2,t+1} - c_{3,t+2}\right) \\
&+\lambda_{4,t+1}\left(\theta p_t h_{2,t+1} + a_{1,t}\right) \\
&+\lambda_{5,t+2}h_{3,t+2}
\end{aligned}$$

with respect to: $c_{1,t}, c_{2,t+1}, c_{3,t+2}, a_{1,t}, a_{2,t+1}, h_{2,t+1}, h_{3,t+2}, \lambda_{1,t}, \lambda_{2,t+1}, \lambda_{3,t+2}, \lambda_{4,t+1}, \text{and} \lambda_{5,t+2}$. First-order conditions are given by:

$$c_{1,t} \quad : \quad \frac{\alpha}{c_{1,t}} = \lambda_{1,t} \tag{67}$$

$$c_{2,t+1} \quad : \quad \frac{\beta(1-\mu)}{c_{2,t+1}} = \lambda_{2,t+1} \tag{68}$$

$$c_{3,t+2} \quad : \quad \frac{\gamma}{c_{3,t+2}} = \lambda_{3,t+2} \tag{69}$$

$$h_{2,t+1} \quad : \quad \frac{\beta\mu}{h_{2,t+1}} - p_t\lambda_{1,t} + R^h_{t+1}p_t\lambda_{2,t+1} + \theta p_t\lambda_{4,t+1} = 0 \tag{70}$$

$$h_{3,t+2} \quad : \quad -p_{t+1}\lambda_{2,t+1} + R^h_{t+2}p_{t+1}\lambda_{3,t+2} + \lambda_{5,t+2} = 0 \tag{71}$$

$$a_{1,t} \quad : \quad \lambda_{1,t} = R_{t+1}\lambda_{2,t+1} + \lambda_{4,t+1} \tag{72}$$

$$a_{2,t+1} \quad : \quad \lambda_{2,t+1} = R_{t+2}\lambda_{3,t+2} \tag{73}$$

We distinguish two cases: the optimal behavior when the borrowing constraint is not binding, meaning $\lambda_{4,t+1} = 0$, and the optimal behavior when the borrowing constraint is binding, meaning $\lambda_{4,t+1} > 0$.

Consider first that the borrowing constraint is not binding, *i.e.* $\lambda_{4,t+1} = 0$. Using the first-order conditions, we get:

$$R_{t+1}p_t = \frac{\mu}{1-\mu}\frac{c_{2,t+1}}{h_{2,t+1}} + R^h_{t+1}p_t \tag{74}$$

$$(R_{t+2} - R^h_{t+2})p_{t+1}h_{3,t+2} = 0 \tag{75}$$

$$c_{2,t+1} = \frac{\beta}{\alpha}(1-\mu)R_{t+1}c_{1,t} \tag{76}$$

$$c_{3,t+2} = \frac{\gamma}{\beta}\frac{1}{1-\mu}R_{t+2}c_{2,t+1} \tag{77}$$

We note from Eq. (74) that

$$R_{t+1} > R^h_{t+1} \equiv \frac{p_{t+1}}{p_t} \tag{78}$$

Note that if young households of the next generation do not face a binding borrowing constraint, then $R_{t+2} > R^h_{t+2}$. This means that $h_{3,t+2} = 0$. Let $h_{3,t+2} = 0$. Using the budget constraints (2)-(4) and Eqs. (74)-(77), we obtain the following optimal solutions:

$$c_{1,t} = \alpha\omega \tag{79}$$

$$c_{2,t+1} = R_{t+1}\beta(1-\mu)\omega \tag{80}$$

$$c_{3,t+2} = R_{t+2}R_{t+1}\gamma\omega \tag{81}$$

$$p_t h_{2,t+1}(R_{t+1} - R^h_{t+1}) = R_{t+1}\beta\mu\omega \tag{82}$$

$$a_{1,t} + p_t h_{2,t+1} = (1-\alpha)\omega \tag{83}$$

$$a_{2,t+2} = R_{t+1}\gamma\omega \tag{84}$$

As the borrowing constraint is not binding, the following inequality should be satisfied:

$$\theta p_t h_{2,t+1} + a_{1,t} \quad > \quad 0 \tag{85}$$

Using Eq. (83), Inequality (85) becomes:

$$p_t h_{2,t+1} \quad < \quad \frac{(1-\alpha)\omega}{1-\theta} \tag{86}$$

A binding borrowing constraint means $\lambda_{4,t+1} > 0$. Using the first-order conditions, we get:

$$p_t \left[(1-\theta) \frac{\alpha}{\beta(1-\mu)} \frac{c_{2,t+1}}{c_{1,t}} + \theta R_{t+1} \right] \quad = \quad \frac{\mu}{1-\mu} \frac{c_{2,t+1}}{h_{2,t+1}} + p_{t+1} \tag{87}$$

$$(R_{t+2} - R^h_{t+2}) p_{t+1} h_{3,t+2} \quad = \quad 0 \tag{88}$$

$$c_{3,t+2} \quad = \quad \frac{\gamma}{\beta} \frac{1}{1-\mu} R_{t+2} c_{2,t+1} \tag{89}$$

$$\theta p_t h_{2,t+1} + a_{1,t} \quad = \quad 0 \tag{90}$$

Using Eqs. (67), (68) and (72), the borrowing constraint when young is binding if

$$\frac{\alpha}{\beta(1-\mu)} \frac{c_{2,t+1}}{c_{1,t}} \quad > \quad R_{t+1} \tag{91}$$

Using Eq. (87), Inequality (91) is equivalent to:

$$R^h_{t+1} + \frac{\mu}{1-\mu} \frac{c_{2,t+1}}{p_t h_{2,t+1}} \quad > \quad R_{t+1} \tag{92}$$

As regards (88), one has $R_{t+2} \geq R^h_{t+2}$. Using the budget constraints (2)-(4) and Eqs. (87)-(90), we obtain the following optimal solutions:

$$c_{1,t} \quad = \quad \omega - (1-\theta) p_t h_{2,t+1} \tag{93}$$

$$c_{2,t+1} \quad = \quad \frac{\beta(1-\mu)}{\beta(1-\mu)+\gamma} (R^h_{t+1} - \theta R_{t+1}) p_t h_{2,t+1} \tag{94}$$

$$c_{3,t+2} \quad = \quad \frac{\gamma}{\beta(1-\mu)+\gamma} R_{t+2} (R^h_{t+1} - \theta R_{t+1}) p_t h_{2,t+1} \tag{95}$$

$$p_t h_{2,t+1} \quad = \quad \frac{(1-\alpha)\omega}{1-\theta} \tag{96}$$

$$(R_{t+2} - R^h_{t+2}) p_{t+1} h_{3,t+2} \quad = \quad 0 \tag{97}$$

$$a_{1,t} \quad = \quad -\theta p_t h_{2,t+1} \tag{98}$$

$$a_{2,t+1} + p_{t+1} h_{3,t+2} \quad = \quad \frac{\gamma}{\beta(1-\mu)+\gamma} (R^h_{t+1} - \theta R_{t+1}) p_t h_{2,t+1} \tag{99}$$

8.2 Proof of Proposition 1

We analyze the function $g(p)$ given by Eq. (26) over the set $(\underline{p}, p^*_C)$. Recall that $g(p)$ must satisfy the following inequality $g(p) < p^*_C$.

As $\beta+\gamma > \beta\mu+\gamma > \beta\mu$, $\underline{p} < p$ implies $g(p) > 0$ for all $p > \underline{p}$. Note that $g'(p) < 0$, $g''(p) > 0$ and $lim_{p\to+\infty} g(p) = \underline{p}$. We deduce that $g(p) > \underline{p}$. We determine the following values of $g(p)$. From Eq. (26), we obtain:

$$g(\underline{p}) = \frac{\beta(1-\mu)+\gamma}{\beta(1-\mu)}\underline{p} \; and \; g(p_C^*) = \frac{\beta(1-\mu)+\gamma+\beta\mu\theta}{\beta(1-\mu)+\beta\mu\theta+\gamma\theta}\underline{p} < g(\underline{p})$$

One has :

$$g(\underline{p}) - p_C^* = \underline{p}\frac{\beta(1-\mu)+\gamma}{\beta(1-\mu)}\frac{\overline{\theta}-\theta}{1-\theta}, \qquad with \; \overline{\theta} \equiv \frac{\gamma}{\beta(1-\mu)+\gamma}$$

We deduce that if $\theta < \overline{\theta} \equiv \gamma/[\beta(1-\mu)+\gamma]$, then $g(\underline{p}) > p_C^*$ and if $\theta \geq \overline{\theta}$, then $g(\underline{p}) \leq p_C^*$. We assume that $\theta < \overline{\theta}$. Since $g(p)$ is a decreasing function, we deduce that there exists a threshold $p_a > \underline{p}$ such that $g(p_a) = p_C^*$. Note that

$$p_a = \omega\frac{\beta\mu\theta+\gamma}{\theta}$$

One also has:

$$\begin{aligned} g(p_C^*) - p_C^* =& \underline{p}\frac{G(\theta)}{[\beta(1-\mu)+\beta\mu\theta+\theta\gamma](1-\theta)}, \\ with \; G(\theta) \equiv & \quad -\beta\mu\theta^2 - \theta[\beta(1-\mu)+2\gamma]+\gamma \end{aligned}$$

$G(\theta)$ is a concave function with $G(0) > 0$. As a consequence, there is a threshold $\theta_1 > 0$ such that $\forall\theta > \theta_1$, $G(\theta) < 0$. Thus, if $\theta \leq \theta_1$, then $g(p_C^*) \geq p_C^*$, and if $\theta > \theta_1$, then $g(p_C^*) < p_C^*$. Since $g(p)$ is a decreasing function, we deduce that $g(p) \geq p_C^* \; \forall p \in (\underline{p}, p_C^*)$ when $\theta \leq \theta_1$. As the following inequality $g(p) < p_C^*$ must be satisfied, we restrict our attention to the case $\theta > \theta_1$.

We also derive that:

$$g(p_C^*) - p_a = -\underline{p}\frac{\beta(1-\mu)G(\theta)}{[\beta(1-\mu)+\beta\mu\theta+\theta\gamma]\theta(\beta+\gamma)}$$

Hence, if $\theta > \theta_1$, then $g(p_C^*) > p_a$.

Note that one has $\overline{\theta} > \gamma/[\beta(1-\mu)+2\gamma] \equiv \theta_2$. The set (θ_1, θ_2) is a non-empty set because

$$G(\theta_2) = -\beta\mu\theta_2^2 < 0 \tag{100}$$

Therefore, the set $(\theta_1, \overline{\theta})$ is a non-empty set.

To sum up, if $\theta \in (\theta_1, \overline{\theta})$, then we have $p_a < g(p_C^*) < p_C^* = g(p_a)$. If $\theta \in [\overline{\theta}, 1)$, then we have $\underline{p} < g(p_C^*) < g(\underline{p}) \leq p_C^*$. To conclude, if $\theta \in (\theta_1, \overline{\theta})$, then the intertemporal equilibrium is such that $p \in [p_a, p_C^*)$ and $p_a \leq g(p) \leq p_C^*$. If $\theta \in [\overline{\theta}, 1)$, then the intertemporal equilibrium is such that p belongs to $(\underline{p}, p_C^*)$ and $p_a < \underline{p} < g(p) \leq p_C^*$.

From this proof, we also deduce that a steady sate p_{NC}^*, solution of $p = g(p)$ exists. Indeed, if p belongs to (p_a, p_C^*) then $g(p_a) > p_a$ and $g(p_C^*) < p_C^*$ and if p belongs to $(\underline{p}, p_C^*)$, then $g(\underline{p}) > \underline{p}$ and $g(p_C^*) < p_C^*$. Since $g(p)$ is strictly decreasing, the steady state p_{NC}^* is unique.

We now study the dynamics of this economy. We know that $g(p_t)$ is a strictly decreasing and convex function on $p_t \in (\widetilde{\widetilde{p}}, p_C^*)$. We also know that the phaseline $p_{t+1} = g(p_t)$ intersects the 45° line, $p_{t+1} = p_t$ at a unique point p_{NC}^*. Note that $g'(p_{NC}^*)$ is given by:

$$g'(p_{NC}^*) = -\frac{p_{NC}^*}{p_{NC}^* - \omega\beta\mu}\frac{\gamma\omega}{p_{NC}^* - \omega(\beta\mu+\gamma)}$$

Since $p_{NC}^* > \underline{p} > (\beta\mu+\gamma)\omega$, $g'(p_{NC}^*) > -1$ if and only if

$$0 < p_{NC}^{*2} - 2\omega(\beta\mu+\gamma)p_{NC}^* + \omega^2(\beta\mu)(\beta\mu+\gamma) \equiv H(p_{NC}^*)$$

Note that $p_{NC}^* = g(p_{NC}^*)$ is equivalent to

$$F(p_{NC}^*) \equiv p_{NC}^{*2} - \omega[\beta(1+\mu)+2\gamma]p_{NC}^* + \omega^2(\beta\mu)(1-\alpha) = 0$$

Thus, $g'(p_{NC}^*) > -1$ if and only if $H(p_{NC}^*) > F(p_{NC}^*) = 0$, which is equivalent to

$$p_{NC}^* > \beta\mu\omega$$

Since $p_{NC}^* > \underline{p} > \beta\mu\omega$, we conclude that $g'(p_{NC}^*) \in (0,-1)$. The steady state p_{NC}^* is stable. Furthermore, $g : [p_a, p_C^*] \to [g(p_C^*), p_C^*] \subset [p_a, p_C^*]$, with $g'(p_t) < 0$ and $g''(p_t) > 0$. $g(p_a) = p_C^*$ gives the maximum value, which is in the interval $[p_a, p_C^*]$. Since $g'(p_a)$ is the derivative with the greatest absolute value on this interval, we will have a contraction for all $p_t \in (p_a, p_C^*)$. Proposition 1 follows.

8.3 Proof of Proposition 2

Using Eqs. (35) and (94), Inequality (92) is equivalent to $G(\theta) < 0$ where $G(\theta)$ is given in the proof of Proposition 1. From the proof of Proposition 1, we know that there is a threshold $\theta_1 > 0$ such that $\forall\theta > \theta_1$, $G(\theta) < 0$. Remind that $R_C^* > 1$ must be satisfied and implies an upper bound on θ, *i.e.*:

$$\theta < \frac{\gamma}{\beta(1-\mu)+2\gamma} \equiv \theta_2 \tag{101}$$

From the proof of Proposition 1, we also know that the set (θ_1, θ_2) is a non-empty set.

8.4 Proof of Proposition 3

We analyze the function $h(p)$ given by Eq. (40) over the set (p_C^*, p_d). Since $p_t > 0$ $\forall t \geq 0$, then $p_t > \overline{\overline{p}}$ with $\overline{\overline{p}} = \underline{p}\overline{\theta} > 0$. One has $h'(p) < 0$, $h''(p) > 0$ and $lim_{p\to+\infty}h(p) = \underline{p}$. We deduce that $p_{t+1} = h(p_t) > \underline{p} > \overline{\overline{p}}$. Remind that the intertemporal equilibrium must satisfy the following inequality $p_t > p_C^*$ and $p_{t+1} = h(p_t) > p_C^*$. Note that there exists a value $p_d > 0$ such that $p_C^* = h(p_d)$:

$$p_d = \underline{p}\frac{\overline{\theta}}{\overline{\theta}}.$$

We now determine the following value of $h(p)$:

$$h(p_C^*) = \underline{p}\frac{1}{1-(1-\theta)\overline{\theta}}$$

Note that

$$h\left(p_C^*\right) - p_C^* = \underline{p}\frac{\overline{\theta}}{\theta_2}\frac{\theta_2 - \theta}{(1-\theta)\left[1-(1-\theta)\overline{\theta}\right]}$$

If $\theta < \theta_2$, then $h(p_C^*) > p_C^*$. As the function $h(p_t)$ is strictly decreasing on $(p_C^*, +\infty)$, we deduce that $p_C^* < p_d$, and there exists a unique solution $p_C^{**} \in (p_C^*, p_d)$ satisfying $p = h(p)$ given by (43). Note that,

$$h'(p_C^{**}) = -\frac{\gamma}{\beta(1-\mu)+\gamma} \equiv \overline{\theta} \in (-1, 0) \tag{102}$$

We deduce that the steady state is stable. Furthermore, $h : [p_C^*, p_d] \to [p_C^*, h(p_C^*)] \subset [p_C^*, p_d]$, with $h'(p_t) < 0$ and $h''(p_t) > 0$. $h(p_C^*) < p_d$ gives the maximum value we can obtain on the interval $[p_C^*, p_d]$. Since the more negative derivative is $h'(p_C^*)$, we will have a contraction on (p_C^*, p_d), which proves stability for all $p_t \in (p_C^*, p_d)$. Proposition 3 follows.

8.5 Proof of Lemma 1

$c_{2,NC}^*$, $c_{2,C}^*$, $c_{2,C}^{**}$ $c_{3,NC}^*$, $c_{3,C}^*$ and $c_{3,C}^{**}$ are respectively the levels of adult households and old households' consumption at the steady states p_{NC}^*, p_C^* and p_C^{**}. One has:

$$c_{2,NC}^* = \beta(1-\mu)R_{NC}^*\omega \tag{103}$$

$$c_{2,C}^* = \frac{\beta(1-\mu)}{\gamma}\overline{\theta}(1-\theta R_C^*)p_C^* \tag{104}$$

$$c_{2,C}^{**} = \frac{\beta(1-\mu)}{\gamma}\overline{\theta}(1-\theta)p_C^* \tag{105}$$

$$c_{3,NC}^* = \gamma R_{NC}^{*2}\omega \tag{106}$$

$$c_{3,C}^* = \overline{\theta}R_C^*(1-\theta R_C^*)p_C^* \tag{107}$$

$$c_{3,C}^{**} = \overline{\theta}(1-\theta)p_C^* \tag{108}$$

$$\text{with } R_{NC}^* = \frac{p_{NC}^* - \underline{p}}{\gamma\omega} \text{ and } R_C^* = \frac{1}{\theta} - \frac{1}{\overline{\theta}} \tag{109}$$

We deduce from Eqs. (103) and (104) that:

$$\frac{c_{2,NC}^*}{c_{2,C}^*} = \frac{p_{NC}^* - \underline{p}}{p_C^* - \underline{p}}$$

Since $\underline{p} < p_{NC}^* < p_C^*$ when $\theta \in (\theta_1, \theta_2)$, one has $c_{2,NC}^* < c_{2,C}^*$. From Eqs. (106) and (107), we derive

$$\frac{c_{3,NC}^*}{c_{3,C}^*} - 1 = \frac{F(p_{NC}^*)}{(p_a - \underline{p})(p_C^* - \underline{p})}$$

where $F(p_{NC}^*) = p_{NC}^{*2} - 2\underline{p}p_{NC}^* + (p_a + p_C^*)\underline{p} - p_a p_c$. At the steady state p_{NC}^*, one has $p_{NC}^* = g(p_{NC}^*)$ which is equivalent to $L(p_{NC}^*) = p_{NC}^{*2} - (\widetilde{p} + \underline{p})p_{NC} + \underline{p}\overline{p} = 0$. Note that

$$F(p_{NC}^*) - L(p_{NC}^*) = \beta(1-\mu)(p_C^* - p_{NC}^*).$$

Since $\underline{p} < p_a < p^*_{NC} < p^*_C$ when $\theta \in (\theta_1, \theta_1)$, we deduce that $c^*_{3,NC} > c^*_{3,C}$.

We deduce from Eqs. (104) and (105) that:

$$\frac{c^*_{2,C}}{c^{**}_{2,C}} = \frac{1 - \theta R^*_C}{1 - \theta}$$

When $\theta \in (\theta_1, \theta_2)$, one has $R^*_C > 1$. Thus, $c^*_{2,C} < c^{**}_{2,C}$. We deduce from Eqs. (107) and (108) that:

$$\frac{c^*_{3,C}}{c^{**}_{3,C}} - 1 = (1 - \theta)\frac{\theta_2 - \theta}{\overline{\theta}\theta_2}$$

If $\theta \in (\theta_1, \theta_2)$, then one has $c^*_{3,C} > c^{**}_{3,C}$.

8.6 Proof of Global Dynamics

If $\theta \in (\theta_1, \theta_2)$, the dynamics of the economy is summarized by theses dynamic equations:

$$p_{t+1} \quad = \quad \begin{cases} g(p_t) & \forall p_t \in (\underline{p}, p^*_C) \\ p_t & \text{when } p_t{=}p^*_C \\ h(p_t) & \forall p_t \in (p^*_C, +\infty) \end{cases} \tag{110}$$

$g(p_t)$ and $h(p_t)$ are given respectively by Eqs. (26) and (40). Recall from previous proofs that both functions are strictly decreasing with p_t and strictly positive over the set $[\underline{p}, +\infty)$. Note that:

$$h\left(p_t\right) - g(p_t) = \underline{p}\beta\mu\omega\overline{\theta}\frac{p_t - \underline{p}}{\left(p_t - \overline{p}\right)\left[p_t - \omega\left(\beta\mu + \gamma\right)\right]}$$

Since $p_t > \underline{p}$ along any equilibrium, we deduce that $h(p_t) > g(p_t)$.

If $\theta \in (\overline{\theta}_1, \theta_2)$, then we know from the Proof of Proposition 8.2 that there exists a value $p_a > \underline{p}$ where p_a is given by $p^*_C = g(p_a)$ and $p_{t+1} = g(p_t) \leq p^*_C$ for $p_t \in (\underline{p}, p^*_C[$. Remind from Proposition 3 that there exists a value $p_d > p^*_C$ such that $p^*_C = h(p_d)$. Moreover, we can easily check that $h(p^*_C) < p_d$ because $\theta < \theta_2$. Therefore, using also the proofs of Propositions 1-3, we have $\underline{p} < p_a < g(p^*_C) < p^*_C = g(p_a) < h(p^*_C) < p_d$. We would like to know now for which value of $p_t \in (\underline{p}, p_a)$, if it exists, one has $g(p_t) > p_d$. Note that:

$$g\left(\underline{p}\right) - p_d = \underline{p}\frac{\beta(1-\mu) + \gamma}{\beta(1-\mu)}\frac{\theta - \widehat{\theta}}{\theta} \; where \; \widehat{\theta} \equiv \overline{\theta}\frac{\beta(1-\mu)}{\beta(1-\mu) + \gamma} < \theta_2 < \overline{\theta}$$

Hence, if $\theta \in (\widehat{\theta}, \theta_2)$, then $g\left(\underline{p}\right) > p_d$. Thus, there exists a value $p_b \in (\underline{p}, p_a)$ such that $p_d = g(p_b)$.

Since $\underline{p} < p_a$, we deduce that there exists a value $\widehat{p} > p_d$ such that $p_a = h(\widehat{p})$ with:

$$\widehat{p} = \underline{p}\frac{\overline{\theta}}{\overline{\theta} - \theta}\frac{\beta\mu\theta + \gamma}{\beta(1-\mu) + \gamma}$$

Therefore, there exists also a value $p_B > \widehat{p}$ such that $h(p_B) = p_b$ when $\theta \in (\widehat{\theta}, \theta_2)$. Using these results, we can derive the figure on global dynamics.

Finally, from the proofs of Propositions 1-3, we know that there is stability in the intervals (p_a, p^*_C) and (p^*_C, p_d).

8.7 Proof of Proposition 6

A condition for the existence of a steady state with a housing demand bubble $x^* > x_C$ if $\Omega > sA$. We know that $\Omega = x_C[\overline{\theta} - \theta(1+\overline{\theta})]$. As $1+\overline{\theta} = \overline{\theta}/\theta_2$, Ω rewrites $\Omega = x_C(\theta_2 - \theta)\overline{\theta}/\theta_2$. Remind that $x_C = (\beta+\gamma)(1-s)A/(1-\theta)$. Hence, $\Omega > sA$ is equivalent to

$$(1-s)(1-\alpha)\frac{\overline{\theta}}{\theta_2}\frac{\theta_2-\theta}{1-\theta} > s \tag{111}$$

If $\theta \in (\theta_1, \theta_2)$, then the LHS of (111) is positive. $\Omega > sA$ is satisfied if $\theta \in (\theta_1, \theta_2)$ and

$$s < \overline{s} \equiv \frac{(\beta+\gamma)\overline{\theta}(\theta_2-\theta)/[\theta_2(1-\theta)]}{1+(\beta+\gamma)\overline{\theta}(\theta_2-\theta)/[\theta_2(1-\theta)]}$$

To determine the stability of the steady state (g^*, x^*), we compute the Jacobian matrix J associated to the dynamic system (63)-(64):

$$J = \begin{pmatrix} -\dfrac{x_C\overline{\theta}(1-\theta)}{sA} & -1 \\ \dfrac{x^*}{sA}\dfrac{x_C\overline{\theta}(1-\theta)}{sA} & 1+\dfrac{x^*}{sA} \end{pmatrix}$$

The characteristic polynomial associated to the Jacobian matrix J is given by:

$$P(\lambda) = \lambda^2 - T\lambda + D$$

where T and D are respectively the trace and the determinant of the Jacobian matrix J:

$$D = -\frac{x_C\overline{\theta}(1-\theta)}{sA} < 0$$
$$T = 1 + D + \frac{x^*}{sA}$$

From Eq. (65), we deduce that:

$$x^* = x_C(1-\theta)(1+\overline{\theta}) - sA$$

Hence, we get:

$$P(1) = -\frac{x^*}{sA} < 0$$
$$P(-1) = 1 + x_C(1-\theta)\frac{1-\overline{\theta}}{sA} > 0$$

One eigenvalue of J is in $(-1, 1)$ and the other in $(1, +\infty)$. Therefore, the steady state (g^*, x^*) is a saddle. As g_t is predetermined variable, we can conclude that there exists a unique equilibrium path converging to this BGP.

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