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WP 2102 – January 2021

Pairwise consensus and Borda rule

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We say that a preference profile exhibits pairwise consensus around some fixed preference relation, if whenever a preference relation is closer to it than another one, the distance of the profile to the former is not greater than its distance to the latter. We say that a social choice rule satisfies the pairwise consensus property if it selects the top ranked alternative in the preference relation around which there is such a consensus. We show that the Borda rule is the unique scoring rule that satisfies this property.

Keywords:
Consensus, Borda rule, scoring rules

JEL codes:
D71, D72

Working Paper

Pairwise consensus and Borda rule*

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November 26, 2020

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1 Introduction

We study a standard setting of social choice, in which there is a set of social alternatives and a group of voters, each of which has a preference relation over this set. A social choice rule selects a subset of alternatives for every preference profile.

*Mahajne acknowledges financial support from Université de Lyon (project INDEPTH Scientific Breakthrough Program of IDEX Lyon) within the program Investissement d'Avenir (ANR-16-IDEX-0005).

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The class of scoring rules is a well-known special class of social choice rules, in which each individual assigns a fixed list of K scores to the set of K alternatives, according to their positions in his preference relation, and select those alternatives with the maximal sum of scores. The Borda rule is an instance of a scoring rule.

A major goal of social choice theory is to examine and characterize social choice rules by means of desirable properties. In this paper, we present a new property, called the pairwise consensus property, and show that the Borda rule is the unique scoring rule that satisfies it. Young (1975) has characterized the class of scoring rules by the axioms of Anonymity, Neutrality, Reinforcement and Continuity. Along with these axioms, the new property characterizes the Borda rule.

Roughly speaking, a preference profile exhibits pairwise consensus around some preference relation $\succ_0$, if whenever a preference relation $\succ$ is closer to $\succ_0$ than another one $\succ'$, the more in agreement are the preferences in the profile with $\succ$ than with $\succ'$. Here, the level of agreement of a preference profile to a given preference relation is measured by the well-known Kemeny distance.[1] Our first result states that the highest ranked alternative in the preference relation around which there is such a consensus, must be a Condorcet winner.

Based on the above concept, we define the pairwise consensus property of social choice rules. A social choice rule satisfies this property if whenever a preference profile exhibits pairwise consensus around a preference relation $\succ_0$, the rule selects the highest ranked alternative in $\succ_0$. Our main result states that the Borda rule is the only scoring rule that satisfies the pairwise consensus property.

Chebotarev and Shamis (1998) survey several existing characterizations of the Borda rule. In particular, we can mention Young (1974) who characterizes the Borda rule as the unique rule that satisfies Neutrality, Consistency, Faithfulness and having the Cancellation property, and the Nitzan and Rubinstein (1981), that axiomatizes the Borda rule, by means of these axioms, but by replacing Faithfulness by Monotonicity.

The paper is organized as follows. Section 2 contains basic definitions. In Section 3 we define the concept of pairwise consensus and the pairwise consensus property, and prove our main result.

[1] See Kemeny and Snell (1962).

2 Definitions

Let $A = \{a_1, ..., a_K\}$ be a set of K alternatives. Let $\mathcal{P}$ be the set of complete, transitive and antisymetric binary relations on A. We will refer to the elements of $\mathcal{P}$ as preference relations. For preference relation $\succ$, when we write $\succ = (a_1, a_2, ..., a_K)$ we mean that a_1 is placed first in $\succ$, and so on. Let $\mathbb{N}$ be the set of non-negative integers, which represent the names of the potential voters. For any finite set $V \subseteq \mathbb{N}$ of voters, a preference profile is an assignment of a preference relation to each voter in V.

A *social choice rule* is a function that assigns a nonempty subset of alternatives to each preference profile. A social choice rule is *anonymous* if it is invariant to the names of the voters. In this paper, we consider only anonymous social choice rules, thus a preference profile can be summarized by a list $\pi = (\succ_1, ..., \succ_N)$ of preference relations where N is the number of voters.

A special class of anonymous social choice rules consists of the *scoring rules*. A scoring rule is characterized by K-tuple $S = (S_1, S_2, ..., S_K)$ of non-negative scores with $S_1 \geq S_2 \geq ... \geq S_K$ and $S_1 > S_K$. Given a preference profile π, each voter $i = 1, ..., N$ assigns S_k points to the alternative that is ranked k-th in his preference relation, for $k = 1, ..., K$. The scoring rule associated with the scores S, denoted by F_S, chooses the alternatives with the maximum total score. Many well-known social choice rules are instances of scoring rules. For example, the *plurality rule* is the scoring rule associated with the scores $(1, 0, ..., 0)$. The *inverse plurality rule* is the scoring rule associated with scores $(1, ..., 1, 0)$. More generally, for $1 \leq t \leq K - 1$, the *t-approval rule* is the scoring rule associated with the scores $(1, ..., 1, 0, ..., 0)$ in which the first t scores equal 1 and the last $(K - t)$ scores equal 0. Lastly, the *Borda rule* is the scoring rule associated with the scores $(K - 1, K - 2, ..., 1, 0)$.

Let $d : \mathcal{P}^2 \to \mathbb{R}$ be the *inversion metric* on $\mathcal{P}$, which is defined as follows: $d(\succ, \succ')$ is the number of pairs of alternatives in A that are ranked differently by $\succ$ and $\succ'$. Formally, the inversion metric is defined by

$$d(\succ, \succ') = |(\succ \setminus \succ')|$$

where $\succ \setminus \succ' = \{(a, b) \in A^2 : a \succ b \text{ and } b \succ' a\}$.

We can use the metric d to compare preference relations according to their "closeness" to some fixed preference relation.

Example 1. Let the set of alternatives be $A = \{a, b, c\}$. The set $\mathcal{P}$ contain six preference relations, given by: $\succ_1 = (a, b, c)$, $\succ_2 = (a, c, b)$, $\succ_3 = (b, a, c)$, $\succ_4 = (c, a, b)$, $\succ_5 = (b, c, a)$, $\succ_6 = (c, b, a)$. Consider the preference $\succ_1$. It can be checked that the distances of each preference in $\mathcal{P}$ to $\succ_1$, according to the inversion metric, are given by

$$\begin{aligned} d(\succ_1, \succ_1) &= 0 \\ d(\succ_2, \succ_1) &= d(\succ_3, \succ_1) = 1 \\ d(\succ_4, \succ_1) &= d(\succ_5, \succ_1) = 2 \\ d(\succ_6, \succ_1) &= 3 \end{aligned}$$

For any preference profile $\pi = (\succ_1, \ldots, \succ_N)$ and any preference relation $\succ \in \mathcal{P}$, we denote by

$$d_\pi(\succ) = \sum_{n=1}^{N} d(\succ_n, \succ)$$

the Kemeny distance of π to $\succ$. It is the sum of the distances to $\succ$ of the voters' preferences.

3 Pairwise consensus

We now introduce the concept of pairwise consensus of preference profiles around a preference relation. Later, we will define the strong pairwise consensus property of social choice rule.

Given a preference profile $\pi = (\succ_1, \ldots, \succ_N)$ and two alternatives $a, b \in A$, we denote by

$$\mu_\pi(a \to b) = |\{n \leq N : a \succ_n b\}|$$

the number of voters that prefer a to b. Note that the Borda count of alternative $a \in A$ for a preference profile $\pi = (\succ_1, \ldots, \succ_N)$ is given by $BC(a) = \sum_{b \in A} \mu_\pi(a \to b)$. Also note that the Kemeny distance of π to $\succ \in \mathcal{P}$ can be written as

$$d_\pi(\succ) = \sum_{a \succ b} \mu_\pi(b \to a).$$

Definition 1. A preference profile π exhibits pairwise consensus around preference relation a $\succ_0$ if for all pairs of preference relations $\succ, \succ' \in \mathcal{P}$,

$$d(\succ, \succ_0) < d(\succ', \succ_0) \Longrightarrow d_\pi(\succ) \leq d_\pi(\succ')$$

with strict inequality if $\succ = \succ_0$.

This concept of consensus is similar to the concept of level-1 consensus introduced in Mahajne, Nitzan, and Volij (2015). In order to exhibit consensus around a preference relation $\succ_0$, both concepts require from a preference profile the fulfillment of certain condition. This condition says that the closer to $\succ_0$ a preference relation is, the more similar (in some well-defined way) this preference relation should be to the preference profile. Whereas level-1 consensus measure similarity in terms of the number of voters that have the relevant preference relation, pairwise consensus measures it in terms of the Kemeny distance.

It is not the case that every preference profile exhibits pairwise consensus around some preference relation. However, when such consensus exists, it is around one and only one preference relation.

Example 2. Continuing with Example 1, consider the following profile of 3 individuals: $\pi = (abc, abc, cba)$. We obtain that $\mu_\pi(a \to b) = \mu_\pi(a \to c) = \mu_\pi(b \to c) = 2$, and the Kemeny distances $d_\pi(\succ)$, for $\succ \in \mathcal{P}$, are given by

$$\begin{aligned}
d_\pi(\succ_1) &= d(\succ_1, \succ_1) + d(\succ_1, \succ_1) + d(\succ_6, \succ_1) = 3 \\
d_\pi(\succ_2) &= d(\succ_1, \succ_2) + d(\succ_1, \succ_2) + d(\succ_6, \succ_2) = 4 \\
d_\pi(\succ_3) &= d(\succ_1, \succ_3) + d(\succ_1, \succ_3) + d(\succ_6, \succ_3) = 4 \\
d_\pi(\succ_4) &= d(\succ_1, \succ_4) + d(\succ_1, \succ_4) + d(\succ_6, \succ_4) = 5 \\
d_\pi(\succ_5) &= d(\succ_1, \succ_5) + d(\succ_1, \succ_5) + d(\succ_6, \succ_5) = 5 \\
d_\pi(\succ_6) &= d(\succ_1, \succ_6) + d(\succ_1, \succ_6) + d(\succ_6, \succ_6) = 6
\end{aligned}$$

It can also be checked that, there is pairwise consensus around $\succ_1$.

The following claim presents an important feature of consensus.

Claim 1. If $\pi = (\succ_1, \ldots, \succ_N)$ exhibits pairwise consensus around $\succ_0$, then the first-ranked alternative in $\succ_0$ is a Condorcet winner.

Proof. Assume w.l.o.g. that $\succ_0 = (a_1, a_2, \ldots, a_K)$. We need to show that for any alternative $a_k \neq a_1$, we have that $\mu_\pi(a_1 \to a_k) \geq \mu_\pi(a_k \to a_1)$.

For $k = 1, 2, \ldots, K$. Let $\succ^k = (a_2, a_3, \ldots, a_k, a_1, a_{k+1} \ldots, a_K)$ be the preference relation that is obtained from $\succ_0$ by moving alternative a_1 from the first rank to the kth rank. We have that $d(\succ_0, \succ^{k-1}) < d(\succ_0, \succ^k)$ for $k = 2, \ldots, K$. Since $\pi \in \mathcal{P}^n$ exhibits pairwise consensus around $\succ_0$, we must have that $d_\pi(\succ^{k-1}) \leq d_\pi(\succ^k)$. But

$$\begin{aligned} d_\pi(\succ^k) &= \sum_{n=1}^{N} d(\succ_n, \succ^k) \\ &= \sum_{1<j\leq k} \mu_\pi(a_1 \to a_j) + \sum_{\substack{i<j \\ (i,j)\neq(1,2),\ldots,(1,k)}} \mu_\pi(a_j \to a_i) \end{aligned}$$

Therefore,

$$0 \leq d_\pi(\succ^k) - d_\pi(\succ^{k-1}) = \mu_\pi(a_1 \to a_k) - \mu_\pi(a_k \to a_1).$$

□

Definition 2. We say that choice rule F has the strong pairwise consensus property if F selects the top-ranked alternative according to $\succ$ whenever $\pi \in \mathcal{P}^n$ exhibits pairwise consensus around $\succ$.

The following theorem is our main result.

Theorem 1. The Borda rule is the unique scoring rule that satisfies the pairwise consensus property.

Proof. We first show that the Borda rule satisfies pairwise consensus. Recall that if $\pi \in \mathcal{P}^n$ exhibits pairwise consensus around $\succ_0$, then for all $\succ \neq \succ_0$, we have that $d_\pi(\succ_0) < d_\pi(\succ)$. Assume that $\pi \in \mathcal{P}^n$ exhibits pairwise consensus around $\succ_0 = (a_1, \ldots, a_j, \ldots, a_K)$. Fix $a_j \neq a_1$. We will show that $BC(a_1) > BC(a_j)$. Note that

$$BC(a_1) - BC(a_j) = \sum_{a_k \neq a_1} \mu_\pi(a_1 \to a_k) - \sum_{a_k \neq a_j} \mu_\pi(a_j \to a_k)$$

$$
\begin{aligned}
&= \sum_{a_k \neq a_1} \frac{\mu_\pi(a_1 \to a_k) - (n - \mu_\pi(a_1 \to a_k))}{2} - \sum_{a_k \neq a_j} \frac{\mu_\pi(a_j \to a_k) - (n - \mu_\pi(a_j \to a_k))}{2} \\
&= \sum_{a_k \neq a_1} \frac{\mu_\pi(a_1 \to a_k) - \mu_\pi(a_k \to a_1)}{2} - \sum_{a_k \neq a_j} \frac{\mu_\pi(a_j \to a_k) - \mu_\pi(a_k \to a_j)}{2}
\end{aligned}
$$

Therefore, it is enough to show that

$$
X := \sum_{a_k \neq a_1} [\mu_\pi(a_1 \to a_k) - \mu_\pi(a_k \to a_1)] - \sum_{a_k \neq a_j} [\mu_\pi(a_j \to a_k) - \mu_\pi(a_k \to a_j)] > 0
$$

Note that

$$
\begin{aligned}
X &= \sum_{a_k \neq a_j} [\mu_\pi(a_1 \to a_k) - \mu_\pi(a_k \to a_1)] + \sum_{a_k \neq a_j} [\mu_\pi(a_k \to a_j) - \mu_\pi(a_j \to a_k)] \\
&= \Big(\sum_{k=1}^{K} [\mu_\pi(a_1 \to a_k) - \mu_\pi(a_k \to a_1)] + \sum_{k=j+1}^{K} [\mu_\pi(a_k \to a_j) - \mu_\pi(a_j \to a_k)]\Big) \\
&\quad + \Big(\sum_{k=1}^{j-1} [\mu_\pi(a_k \to a_j) - \mu_\pi(a_j \to a_k)]\Big) \qquad (1)
\end{aligned}
$$

Consider the following preference relations:

$$
\begin{aligned}
\succ &= (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_K, a_j) \\
\succ' &= (a_2, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_K, a_1)
\end{aligned}
$$

Preference $\succ$ is obtained from $\succ_0$ by moving a_j to the last place. Preference $\succ'$ is obtained from $\succ_0$ by moving a_1 to the last place. As a result, $d(\succ_0, \succ) < d(\succ_0, \succ')$. Since there is consensus around $\succ_0$, we have that $d_\pi(\succ') - d_\pi(\succ) \geq 0$. Since

$$
d_\pi(\succ') - d_\pi(\succ) = \sum_{k=1}^{K} [\mu_\pi(a_1 \to a_k) - \mu_\pi(a_k \to a_1)] + \sum_{k=j+1}^{K} [\mu_\pi(a_k \to a_j) - \mu_\pi(a_j \to a_k)]
$$

we have that the the expression within the first pair of brakets in (1) is non-negative. Consider now the following preference relation:

$$
\succ'' = (a_j, a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_K)
$$

Preference relation $\succ''$ is obtained from $\succ_0$ by moving a_j to the top. It is clear that $d(\succ_0, \succ_0) < d(\succ_0, \succ'')$. Therefore, since there is consensus around $\succ_0$, we have

$d_\pi(\succ'') - d_\pi(\succ_0) > 0$. Since

$$\begin{aligned} d_\pi(\succ'') - d_\pi(\succ_0) &= [\mu_\pi(a_1 \to a_j) - \mu_\pi(a_j \to a_1)] + \sum_{k=2}^{j-1} [\mu_\pi(a_k \to a_j) - \mu_\pi(a_j \to a_k)] \\ &= \sum_{k=1}^{j-1} [\mu_\pi(a_k \to a_j) - \mu_\pi(a_j \to a_k)] \end{aligned}$$

we have that the the expression within the second pair of brakets in (1) is positive. Therefore,

$$X = \big(d_\pi(\succ') - d_\pi(\succ)\big) + \big(d_\pi(\succ'') - d_\pi(\succ_0)\big) > 0.$$

We now show that any scoring rule other than the Borda rule fails to satisfy pairwise consensus. It is enough to show this for $K = 3$. So, let now $K = 3$ and let $S = (S_1, S_2, S_3)$ be a scoring rule distinct from Borda. Without loss of generality assume that the scores associated with S are 1, p and 0, for some $0 \le p \le 1$ such that $p \neq 1/2$. Assume first that $p < 1/2$ and consider the following preference profile π, along with the associated d_π function.

$\succ$	# of voters	$d_\pi(\succ)$
abc	1	$3n$
acb		$1+3n$
bac	n	$1+3n$
bca		$2+3n$
cab	n	$2+3n$
cba		$3+3n$

It can be seen that there is consensus around abc and that the score awarded by S to the three alternatives are given by

	1	p	S
a	1	$2n$	$1+2np$
b	n	1	$n+p$
c	n	0	n

However, for $p < 1/2$, we have that $S(b) > S(a)$ whenever

$$n > \frac{1-p}{1-2p}$$

Similarly, assume that $p > 1/2$ and consider the following preference profile π, with the associated d_π function.

$\succ$	# of voters	$d_\pi(\succ)$
abc	n	$3(n-1)$
acb	0	$n+2(n-1)$
bac	0	$n+2(n-1)$
bca	0	$2n+(n-1)$
cab	0	$2n+(n-1)$
cba	$n-1$	$3n$

It can be seen that there is consensus aroun abc and that the score awarded by S to the three alternatives are given by

	1	p	S
a	n	0	n
b	0	$2n-1$	$p(2n-1)$
c	$n-1$	0	$n-1$

However, for $p > 1/2$, we have that $S(b) > S(a)$ whenever

$$n > \frac{p}{2p-1}$$

□

Young (1975) has characterized the class of scoring rules as the only social choice rules that satisfy the axioms of anonymity, neutrality, reinforcement and continuity. Consequently, as a corollary of our result we obtain that the Borda rule is the only social choie rule that satisfies Young's axioms and the pairwise consensus property. It would be interesting to see if other rules can be characterized by means of some similar notions of the consensus property.

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