Principle of Minimum Differentiation Revisited

N. Hanaki[1], E. Tanimura[2], N. Vriend[3]

[1]Aix Marseille University (Aix Marseille School of Economics), GREQAM, IUF
[2]CES, Université Paris 1
[3]Queen Mary University of London

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What is the question we try to answer?

- Does a noisy myopic best reply restore the “principle of minimum differentiation” in a simple linear location model (*a la* Hotelling 1929, EJ) ?

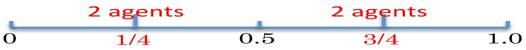
Background: “Principle of minimum differentiation” in a linear location model

- Consumers with a unit demand located over an interval, e.g, $[0, 1]$.
- n agents simultaneously choose a location in the interval.
- Agents charge the same price.
- Consumers buy from the closest agent.
- $n = 2$, under NE, both agents locate at the center
- Applications:
    - political science : parties locating in an ”ideological space”
    - product differentiation by competing firms
    - (spatial location of competing firms)

Background 2: Pure strategy NE when $n > 2$?

- $n = 3$. Does not exist.
- $n > 3$. Eaton and Lipsey (1975, RES)

$n = 4$



$n = 5$

$n = 6$ (multiple eq.)

- Very different behavior: no longer minimum differentiation. Agents spread out on the line.

Background 3: Properties of the equilibrium

It results from the analysis of Eaton and Lipsey (1975) that

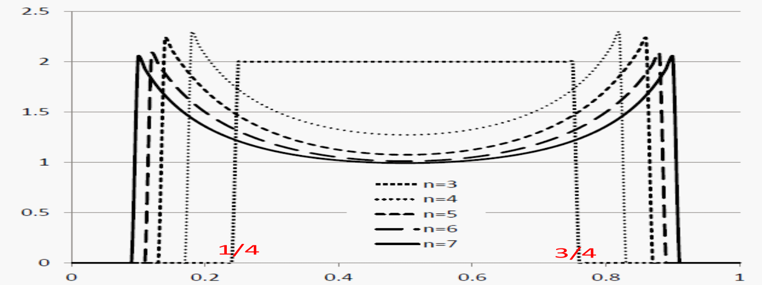
- The extremal agents are at most $1/n$ from the edges of the interval.
- The interior agents are spread out
- There is multiplicity: for $n > 5$ there is an infinite number of equilibria
- For $n > 3$, all equilibria are weak.

We note for later on that in a model with discrete locations, and the possibility of locating in the same position, some details of the equilibrium analysis are slightly modified but essentially results remain the same.

Background 3: Doubly symmetric mixed strategy NE when $n > 2$?

- $n = 3$. Shaked (1983, JIndE).
- $n > 3$. Ewerhart (2014, mimeo, UZurich)

Prob. density of mixed strategy in NE



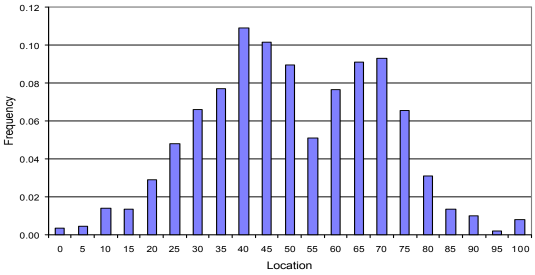
(Source: Ewerhart 2014, Fig. 1)

- We don't see minimum differentiation, either...

Background 4: Experimental evidences Part 1

- Collins and Shestyuk (2000, Econ Inq):
  - n=3. 35 (un-known) repetitions with re-matching. Cumulative payoff.

Relative frequency of chosen locations



(Source: Collins and Shestyuk 2000, Fig 2. T=1785 (35×51))

Background 5: Experimental evidences Part 2

- Huck et al (2002, Econ Inq):
  - n=4. 50 repetitions. Cumulative payoff (24 subjects)
  - The relative frequency of chosen locations is not well explained by pure nor mixed Nash equilibrium.
  - Over time, subjects play central locations more frequently.
  - Neither pure strategy NE nor doubly symmetric mixed strategy NE does good job of explaining the experimental results.
  - A noisy myopic best response seems to explain the observed behavior quite well (Huck et al 2002).

To summarize

Observations:

- High multiplicity of equilibria gives low predictive value.
- All equilibria are weak.
- In experiments, test subjects do not easily find (one of the) Nash equilibria but instead tend to pick locations in the center.

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Our question: Does a noisy myopic best reply restore the "principle of minimum differentiation" in a simple linear location model?

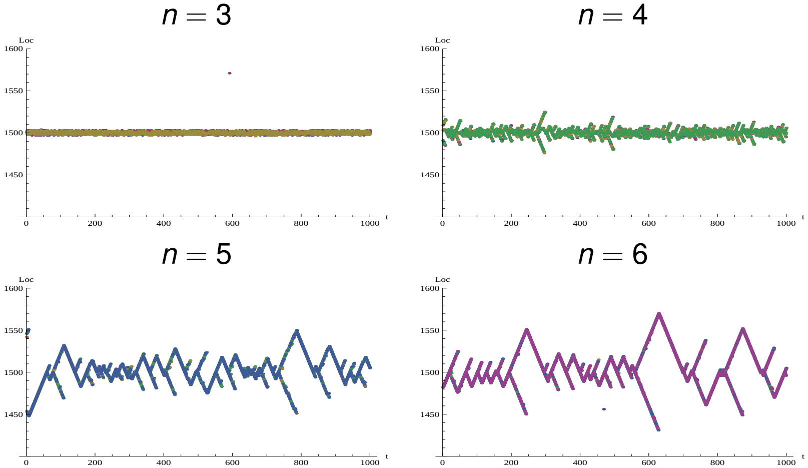
Approach

- Numerical simulations : allows us to verify the robustness of the results when some aspects of the model vary : sequential moves, inertia, players with preferences for close locations
- Analytical proof for $n = 3$, letting the level of random noise go to zero.
- Analytical proof for $n = 4$, letting random noise go to zero and the number of locations grow to infinity.
- Some results and comments about the general case $n > 4$.

Numerical simulations: set up

- *n* agents, initially randomly located.
- Discretize the interval into *N* (uneven) locations.
- All the agents move simultaneously according to a noisy myopic best reply.
  - Assume others stay in the same location.
  - With prob. $1 - \varepsilon$ pick the location that maximizes the payoff. Randomize in case of a tie.
  - With prob $\varepsilon = 0.001$ choose a location uniformly randomly from all the possible locations.
- Drop the result from the first 10,000,000 steps.

Results: Time series $n \in \{3, 4, 5, 6\}$, $N = 3001$

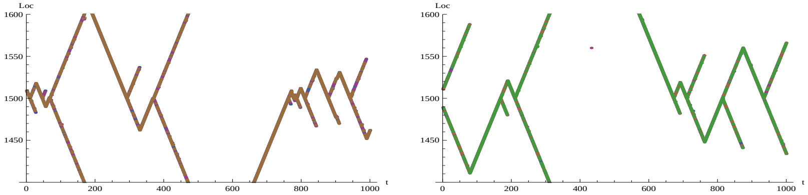


200 locations around the center.
1000 steps (after 10,000,000 initial steps).

Results: Time series $n \in \{7.8\}$, $N = 3001$

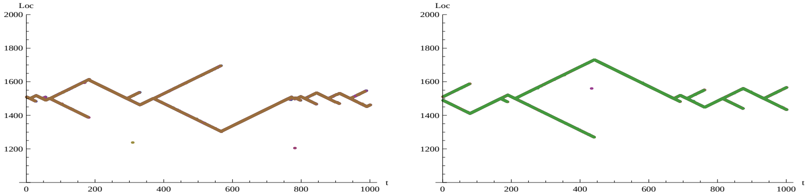
200 locations around the center

$n = 7$ $n = 8$



1000 locations around the center

$n = 7$ $n = 8$

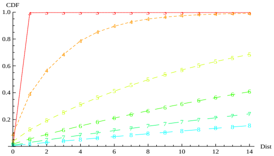


1000 steps (after 10,000,000 initial steps).

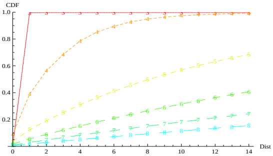
Results: Distribution of frequencies of realization with mean distance of agents from the center

Mean distance less than 15 locations from the center.

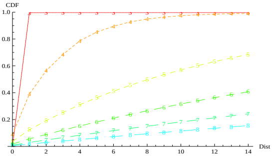
N=3001



N=30001



N=300001



Out of of 100,000,000 steps (after dropping initial 10,000,000 steps).

Red: $n = 3$. Orange: $n = 4$. Light green: $n = 5$.
Green: $n = 6$ and $n = 7$. Light blue: $n = 8$

Results: Mean distance of agents from the center

Mean distance under which the simulation spend 99% of the 100,000,000 steps (after 10,000,000 initial steps).

	n=3	n=4	n=5	n=6	n=7	n=8
$N = 3001$	2.0	15.0	65.9	139.9	247.4	409.3
	(0.0)	(0.0)	(0.35)	(0.25)	(0.82)	(1.44)
NE	540 (90%)	750	850	666.67	803.57	750
$N = 30001$	2.0	15.0	67.2	152.0	291.5	481.9
	(0.0)	(0.0)	(0.38)	(0.45)	(1.38)	(2.37)
NE	5393 (90%)	7500	8500	6667	8035.7	7500
$N = 300001$	2.0	15.0	67.9	154.0	302.0	525.6
	(0.0)	(0.0)	(0.31)	(0.45)	(1.40)	(3.76)
NE	53918 (90%)	75000	85000	66667	80357	75000

Based on 30 simulations.

- Note the difference in the scaling property between the NEs and model simulation

Summary of what is observed in the simulations

- When $n = 3$ agents spend most of their time around the center.
- As n increase, agents spend more time away from the center.
- But the mean distance (compared to the total number of locations) decreases very quickly.

Asymptotic analysis: the model

The model is essentially the same as in the simulations:

- $N = 2M + 1$ locations.
- Agents move simultaneously according to a noisy myopic best reply, assuming others stay in their previous location.
- With prob. $1 - \varepsilon$ pick a best reply. If it is not unique, each best reply is chosen according to a uniform probability.
- With prob ε choose a location uniformly at random among all the possible locations.
- The agents' locations evolve according to an ergodic Markov chain whose long run behavior is described by an invariant measure $\mu_{\varepsilon,N}$

Asymptotic analysis: cases

- It is standard to analyze the states on which the system concentrates as the noise becomes vanishingly small, i.e. the support of $lim_{\varepsilon \to 0} \mu_{\varepsilon}$. These states belong to the long run stochastically stable set (henceforth LRSS). This can be used as a form of equilibrium selection (which equilibria can be learned by boundedly rational players) e.g. Kandori et al (1993), Ellison (2000)
- Only states that belong to a recurrent class in the Markov chain without noise can belong to the long run stochastically stable set.
- Being weak, the Nash equilibria are not recurrent classes.
- In the hotelling model with $n > 2$, there are cycles that are recurrent classes : Let C_k be the union of two states: $\{((M-k)*(n-1), M+k-1), (M-k+1, (M+k)*(n-1))\}$. C_k is a recurrent class for $1 \leq k \leq C$, where $C =: max\{k | 2M - (M+k-1) > \frac{2k-1}{2}\}$ and $a * b$ denotes the location a repeated b times.

Asymptotic analysis: case n=3

Proposition

When $n = 3$, the long run stochastically stable set consists only of states whose distance to the center is at most 3.

We apply a result by Ellison (2000) characterizing LRSS based on the "radius" and "coradius" of a recurrent class.

- Coradius of S : number of mutations required to reach S from outside the basin of attraction of S.
- Radius of S: mutations required to leave basin of attraction of S.

Theorem

In a model of evolution with noise, let Ω be a union of limit states. If $R(\Omega) > CR(\Omega)$ the long run stochastically stable set is contained in Ω (Ellison, 2000, RES)

Agents are close to the center in a strong sense: For any N, agents concentrate at locations at most 3 steps from center as noise vanishes.

Asymptotic analysis: case n=4

- As before the Nash equilibria are not recurrent classes (nor in fact included in a recurrent class). (also true for for $n > 4$)
- The cycles (C_k) are recurrent classes. (also true for for $n > 4$)
- It does not appear to be true (although we have no proof of the contrary) that $lim_{\varepsilon \to 0} \mu_\varepsilon$ concentrates only on the cycles close to the center (in the sense of putting zero weight on distant locations)
- However we can show that when the number of locations is large agents spend most of their time close to the center in a weaker sense: We consider $lim_{N \to \infty} \mu_{N,\varepsilon(N)}$, imposing a condition on the speed that $\varepsilon(N)$ goes to zero. The limit invariant measure is concentrated on a set around the center that is not bounded independently of N but is order of magnitude smaller than N.

The case $n = 4$

Theorem

In the Hotelling model with N locations, let $I_N = [M - l_N, M + l_N$ *be an interval centered at* M*, of length* $2l_N$*. Let* S_N *be a subset of locations such that* $\omega \in S_N \iff \omega \in [M-3, M+3] \bigcup \Omega_{AC}^{l_N/2} \subset I_N$*. Let* B_N *be any set of states* $B_N \subset I^c$ *such that* $card(B_N) \leq card(S_N)$*. Suppose that the Condition* **??** *below is verified, then* $lim_{N \to \infty} \frac{\mu^N(B_N)}{\mu^N(S_N)} = 0$*.*

Condition

There exist constants $\alpha \in]0, \frac{1}{7}[$*,* $\beta_M < \infty$ *and* N_0 *such that for every* $N \geq N_0$*,* $l_N = N^\alpha$ *and the level of random noise* ε_N *verifies* $\varepsilon_N = N^{-\beta}$*, where* $\beta \in [1, \beta_M]$*.*

Outline of Proof

We use the fact that the invariant distribution of a Markov chain verifies: $\mu(S) = \mu(b)E[V(S, b, b)]$, where $V(S, b, b)$ is a random variable that counts the number of times that the process reaches an element in S before it reaches b, starting from state b. To bound the expectation

- we compute a lower bound on the probability of returning to S starting from S (main part of proof). This part can be generalized to $n > 4$.
- a lower bound on the probability of reaching S from a state outside of S (consists in checking a number of cases). Difficult (or at least tedious) to check for $n > 4$

Outline of ideas for obtaining the first bound: With high probability random locations are rare and do not occur consecutively. Conditioning on this, any state in S is such that

Outline of Proof - cntd

1. either best reply brings agents closer to the center in some steps ("contraction")
2. IF a state is such that with positive probability best reply moves (at most one step) farther from the center, THEN, also positive probability that all agents en up on the same side (triggering a return to the center)
3. We can also reach an absorbing class: we need to analyze what happens when we exit the absorbing class. In fact this is similar to the case 2.

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Moving away from the center requires passing through states of type 2 a large number of times. The probability of doing so without ever having all agents end up on the same side becomes very small.

Concluding remarks

- Research question: Does a noisy myopic best reply restore the "principle of minimum differentiation" in a simple linear location model?
- Method: simulation and analytical proof
- Results:
  - The weak Nash equilibria do not survive selection by long run stochastic stability
  - Instead the invariant measure focuses on some other recurrent classes
  - for $n = 3$, these classes contain only states close to the center.
  - For $n = 4$ the measure concentrates on central states in a weaker sense as the number of locations grows.
  - For $n > 4$: we conjecture behavior similar to $n = 4$.
- Thus this work suggests that in some sense the "Principle of minimum differentiation" is restored for $n > 2$ agents under a noisy myopic best reply. Simulations show that this results seems robust to varying details of the model.

THANK YOU VERY MUCH