Axiomatization and Implementation of a Class of Solidarity Values

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Outline of the presentation

- We introduce a (convex) class of values for TU-games which combine marginalist and egalitarian principles.
- This class of values belongs to the class of Efficient, Anonymous and Linear values characterized by Ruiz et al. (1998) and Radzik and Driessen (2013), among others.
- We remark that this class of values constitutes in fact a subclass of the Procedural values (Malawski, 2013), and contains the Egalitarian Shapley values (Joosten, 1996) and the Solidarity value (Novak and Radzik 1994), but excludes the δ-Shapley values (Joosten, 1996) and the Consensus value (Ju et al., 2007).
- We provide a characterization of this class of values in terms of the coefficients given by Radzik and Driessen (2013) to express the Efficient, Anonymous and Linear values via the Shapley value (1953).

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- We also provide an axiomatic characterization of the full class of values, which is comparable to both the characterization of the Procedural values (Malawski) and the characterization of the Egalitarian Shapley values provided by Casajus and Huettner (2013).
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TU-games and values

- The set N represents a fixed and finite set of agents of size n.
- A coalition function on N is a function $v : 2^N \longrightarrow \mathbb{R}$ which assigns a worth $v(S) \in \mathbb{R}$ to each coalition $S \in 2^N$, and where $v(\emptyset) = 0$;
- A TU-game v is monotone if $S \supseteq T$ implies $v(S) \geq v(T)$; and positive if, for each $S \in 2^N$, $v(S) \geq 0$.
- Let V_N be the set of such coalition functions or TU-games v on N. A value on V_N is a function $\Phi : V_N \longrightarrow \mathbb{R}^n$ which assigns a payoff vector $\Phi(v) \in \mathbb{R}^n$ to each $v \in V_N$.

Some values

- The Egalitarian Division rule, ED, is defined on V_N as:

$$\forall i \in N, \quad \mathrm{ED}_i(v) = \frac{v(N)}{n}.$$

- For any permutation σ on N and any agent $i \in N$, consider the coalition containing i and the set of his/her predecessors in σ as $P_i^\sigma = \{j \in N : \sigma(j) \leq \sigma(i)\}$. The Shapley value, Sh, is defined on V_N as follows:

$$\begin{aligned}\forall i \in N, \quad \mathrm{Sh}_i(v) &= \frac{1}{n!} \sum_{\sigma \in \Sigma_N} \big(v(P_i^\sigma) - v(P_i^\sigma \setminus i)\big) \\ &= \sum_{S \in 2^N : S \ni i} \frac{(n-s)!(s-1)!}{n!} \big(v(S) - v(S \setminus i)\big).\end{aligned}$$

Some values

Some values which incorporate a solidarity principle in two different ways.

- The Egalitarian Shapley values are the convex combinations of the Shapley values and the Egalitarian values:

$$\exists \alpha \in [0,1] : \forall i \in N, \quad \mathrm{EDSh}_i^{\alpha}(v) = \alpha \mathrm{Sh}_i(v) + (1-\alpha)\mathrm{ED}_i(v).$$

- The Solidarity value (Nowak and Radzik, 1993), which takes into account the average marginal contribution of all members of a coalition:

$$\forall i \in N, \quad \mathrm{NR}_i(v) = \sum_{S \in 2^N : S \ni i} \frac{(n-s)!(s-1)!}{n!} \sum_{j \in S} \left(\frac{v(S) - v(S \setminus j)}{s} \right).$$

EAL_N values

- All the above-mentioned values belongs to the class of values on V_N which are Efficient, Anonymous and Linear, and denoted by EAL_N.

Proposition

(Radzik and Driessen, 2013)
A value Φ on V_N belongs to EAL_N if and only if there exists a unique vector of constants $B^\Phi = (b_s^\Phi : s \in \{0, 1, \ldots, n\})$ such that $b_0^\Phi = 0$, $b_n^\Phi = 1$, and

$$\Phi(v) = Sh(B^\Phi v),$$

where $(B^\Phi v)(S) = b_s^\Phi v(S)$ for each coalition S of size s, $s \in \{0, 1, \ldots, n\}$,

Examples

For $1 \leq s \leq n-1$:

- $b_s^{\mathrm{Sh}} = 1$;
- $b_s^{\mathrm{ED}} = 0$;
- $b_s^{\mathrm{EDSh}^{\alpha}} = \alpha$;
- $b_s^{\mathrm{NR}} = \dfrac{1}{s+1}$.

Procedural values

- Malawski (2013) introduces the subclass $\text{PV}_N \subset \text{EAL}_N$ of Procedural values which incorporate a solidarity principle among the agents.
- A procedure specifies how the (marginal) contribution of each agent is shared with his or her predecessors in each ordering σ of the agent set.
- Formally, a procedure ℓ on N is a collection of nonnegative coefficients $((\ell_{p,q})_{q=1}^{p})_{p=1}^{n}$ s.t. for each $p \in \{1, \ldots, n\}$, $\sum_{q=1}^{p} \ell_{p,q} = 1$. The coefficient $\ell_{p,q}$ specifies the share of agent at position $q \leq p$ in the marginal contribution of agent at position p in the ordering:

$$\forall i \in N, \quad r_i^{\sigma,\ell}(v) = \sum_{j \in N : \sigma(j) \geq \sigma(i)} \ell_{\sigma(j),\sigma(i)} \big(v(P_j^\sigma) - v(P_j^\sigma \setminus j) \big).$$

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- A Procedural value is defined as:

$$\forall i \in N, \quad \mathrm{Pv}_i^{\ell}(v) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} r_i^{\sigma,\ell}(v).$$

- Theorem 1 in Malawski (2013) shows that the allocation determined by a procedure ℓ depends only on the numbers ℓ_{ss}, $s \in \{1, \ldots, n\}$, where $\ell_{11} = 1$.
- Expressed in terms of the coefficients given by Radzik and Driessen (2013), a Procedural value induced by the procedure ℓ has the coefficients $b_s^{\ell} = \ell_{s+1,s+1}$ for $s \in \{1, \ldots, n-1\}$ (Lemma 2 in Malawski, 2013).
- In particular, we have:

$$\mathrm{EAL}_N \supset \mathrm{Pv}_N \supset \mathrm{EDSh}_N \supset \{\mathrm{Sh}, \mathrm{ED}\}.$$

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Solidarity values

We introduce a set of solidarity values, denoted by Sol_N, which rely on both marginalist and egalitarian principles. The scenario envisaged to define and compute these values consists of the following steps:

1. Consider any integer p between 0 and $n-1$;
2. Choose any $v \in V_N$ and any $\sigma \in \Sigma_N$ in order to gradually form the grand coalition N;
3. Each agent $i \in N$ arriving at position $\sigma(i) \leq p$ obtains his contribution $c_i^{\sigma,p}(v) = v(P_i^\sigma) - v(P_i^\sigma \setminus i)$ upon entering;
4. Each agent $i \in N$ arriving at position $\sigma(i) > p$ obtains an equal share of the remaining worth $v(N) - v(P^\sigma_{\sigma^{-1}(p)})$, i.e.

$$c_i^{\sigma,p} = \frac{v(N) - v(P^\sigma_{\sigma^{-1}(p)})}{n-p}.$$

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5. Steps 1-4 determine a payoff vector denoted by $c^{\sigma,p}(v) \in \mathbb{R}^n$;
6. Define the payoff vector $\text{Sol}^p(v)$ as the average of the payoff vectors $c^{\sigma,p}(v)$ over the $n!$ orderings $\sigma \in \Sigma_N$;
7. Assume p is drawn according to the probability distribution $\alpha = (\alpha_p : p \in \{0, \dots, n-1\})$. The Solidarity value w.r.t. α is defined as the expected payoff vector $\text{Sol}^\alpha(v)$ given by:

$$\text{Sol}^\alpha(v) = \sum_{p=0}^{n-1} \alpha_p \text{Sol}^p(v).$$

For each $i \in N$, $\text{Sol}_i^p(v)$ can also be expressed as:

$$\begin{aligned} \text{Sol}_i^p(v) &= \sum_{S \ni i, s \leq p} \frac{(n-s)!(s-1)!}{n!} (v(S) - v(S \setminus i)) \\ &+ \sum_{S \not\ni i, s = p} \frac{(n-s-1)!s!}{n!} (v(N) - v(S)). \end{aligned}$$

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- To be more precise, each Sol^p, $p \in \{0, \ldots, n-1\}$, an extreme point of the convex set of values $\mathbf{Sol}_N$, has the following property:

Proposition

Fix any $p \in \{0, \ldots, n-1\}$.

1. *If $p = 0$, then $Sol^0 = ED$; if $p = n-1$, then $Sol^{n-1} = Sh$.*
2. *For each $v \in V_N$, $Sol^p(v)$ coincides with $Sh(B^p v)$, where $B^p = (b^p_s : s \in \{0, 1, \ldots, n\})$ is such that:*

$$b^p_0 = 0,\ b^p_n = 1, \quad b^p_s = \begin{cases} 1 & \text{if } s \in \{1, \ldots, p\}, \\ 0 & \text{if } s \in \{p+1, \ldots, n-1\}. \end{cases}$$

- From the above property, we obtain a characterization of Sol_N using the representation of values in EAL_N provided by Radzik and Driessen (2013).

Proposition

A value Φ on V_N belongs to Sol_N if and only it can be represented by the Shapley value applied to TU-games $B^\Phi v$ with constants $B^\Phi = (b_s^\Phi : s \in \{0, \dots, n\})$ such that:

$$b_0^\Phi = 0,\; b_n^\Phi = 1, \quad \text{and}$$

$$\forall s \in \{1, \dots, n-1\}, \quad 1 \geq b_1^\Phi \geq b_2^\Phi \geq \cdots \geq b_{n-1}^\Phi \geq 0.$$

Furthermore, $\Phi = Sol^\alpha$ where $\alpha = (\alpha_s : s \in \{0, \dots, n-1\})$ is obtained from the transformation $B^\Phi \longmapsto \alpha$ such that:

$$\alpha_0 = 1 - b_1^\Phi,\; \alpha_{n-1} = b_{n-1}^\Phi, \quad \text{and} \;\; \forall s \in \{1, \dots, n-2\},\; \alpha_s = b_s^\Phi - b_{s+1}^\Phi.$$

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- The next result shows that $\mathrm{Sol}_N \subset \mathrm{Pv}_N$.

Proposition

For each probability distribution α, $Sol^{\alpha} = Pv^{\ell^{\alpha}}$, where ℓ^{α} is defined as:

$$\ell^{\alpha}_{k,q} = \begin{cases} \alpha_{q-1} & \text{if } q < k, \\ \sum_{j=k-1}^{n-1} \alpha_j & \text{if } q = k. \end{cases}$$

- The contribution of the agent entering at position k is shared as follows: for $q < k$, the agent entering at position q receives a share α_{q-1} of the contribution, and the agent k keeps the remaining part.

Analysis

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Analysis

Conclusion: It holds that

$$\mathrm{EAL}_N \supset \mathrm{Pv}_N \supset \mathrm{Sol}_N \supset \mathrm{EDSh}_N \supset \{\mathrm{Sh}, \mathrm{ED}\}.$$

- We first introduce a variant of the null agent axiom in order to characterize Sol^p, $p \in \{1, \ldots, n-1\}$. Modifications of the axiom of null agent are numerous in the literature, see e.g. Nowak and Radzik (1994), Ju et al. (2007), Kamijo and Kongo (2012), Chameni Nembua (2012), Casajus and Huettner (2014), Béal et al. (2015), van den Brink and Funaki (2015), and Radzik and Driessen (2016).
- Given $p \in \{1, \ldots, n-1\}$, and $v \in V_N$, we say that $i \in N$ is a p-null agent in v if:

$$\forall S \ni i,\ s \leq p, \quad v(S) = v(S \setminus i) \ \text{ and } \ \forall S \not\ni i,\ s = p,\ v(N) = v(S).$$

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p-null agent axiom A value Φ on V_N satisfies the p-null player axiom if, for each $v \in V_N$ and each p-null agent $i \in N$ in v, it holds that: $\Phi_i(v) = 0$.

Proposition

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A value Φ on V_N satisfies:

- Desirability if for each $v \in V_N$ and each pair $\{i, j\} \subseteq N$ such that, for each $S \subseteq N \setminus \{i, j\}$, $v(S \cup i) \geq v(S \cup j)$, it holds that: $\Phi_i(v) \geq \Phi_j(v)$;
- Monotonicity if for each monotone $v \in V_N$ and each $i \in N$, it holds that: $\Phi_i(v) \geq 0$;
- Null agent in a productive environment if for each $v \in V_N$ such that $v(N) \geq 0$, and each null agent $i \in N$ in v, it holds that: $\Phi_i(v) \geq 0$;
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Axiomatic characterizations

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A value Φ on V_N belongs to Pv_N if and only if it satisfies Efficiency, Desirability, Additivity, and Monotonicity.

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Axiomatic characterizations

Proposition

A value Φ on V_N belongs to Sol_N if and only if it satisfies Efficiency, Additivity, Desirability, Monotonicity, and Null agent in a null environment for positive games.

Implementation

- We design a class of mechanisms which implement the values of Sol_N in subgame perfect Nash equilibrium.
- These mechanisms belong to the class of bidding mechanisms initiated by Demange (1984) and Moulin (1984) for economic environments and further adapted by Pérez-Castrillo and Wettstein (2001) and Ju and Wettstein (2009) to implement solution concepts for cooperative games.

Implementation

- Consider any TU-game $v \in V_N$ and any probability distribution $\alpha = (\alpha_p)_{p=0}^{n-1}$ on $\{0, \dots, n-1\}$ and define A_α as the support of α.

Mechanism (B)
Stage 1: each agent $i \in N$ makes bids $h_p^i \in \mathbb{R}$, one for each position $p \in A_\alpha$, under the following constraint:

$$\sum_{p \in A_\alpha} \alpha_p h_p^i = 0.$$

For each position $p \in A_\alpha$, define the aggregate bid H_p as:

$$H_p = \sum_{i \in N} h_p^i.$$

Denote by Ω_{A_α} the subset of positions with the highest aggregate bid.

Stage 2: each agent $i \in N$ makes bids $h^i_\sigma \in \mathbb{R}$, one for each ordering $\sigma \in \Sigma_N$, under the constraint:

$$\sum_{\sigma \in \Sigma_N} \frac{1}{n!} h^i_\sigma = 0,$$

meaning that the designer values each ordering σ equally, i.e. for each $\sigma \in \Sigma_N$, the weight $\alpha_\sigma = 1/n!$. For each $\sigma \in \Sigma_N$, the aggregate bid, defined in a similar way as in Stage 1, is denoted by H_σ. Finally, denote by Ω_{Σ_N} the subset of permutations with the highest aggregate bid.

Implementation

Stage 3: pick at random any $p \in \Omega_A$ and then any $\sigma \in \Omega_{\Sigma_N}$. Together, position p and ordering σ induce a sequential bargaining (sub)game $G_{p,\sigma}$ whose payoffs are denoted by $(g^i_{p,\sigma})_{i \in N}$. This bargaining game contains the following steps:

1. Agent $i \in N$ in position $\sigma(i) = p + 1$ proposes an offer $x^i_j \in \mathbb{R}$ to each other $j \in N \setminus i$.
2. The agents other than agent i, sequentially, either accept or reject the offer. If at least one agent rejects it, then the offer is rejected. Otherwise the offer is accepted.
3. If the offer is accepted, then the payoffs are given by: $g^i_{p,\sigma} = v(N) - \sum_{j \in N \setminus i} x^i_j$, and $\forall j \in N \setminus i$, $g^j_{p,\sigma} = x^i_j$.
4. If the offer is rejected, then each j in position $\sigma(j) \geq p + 1$ leaves the bargaining procedure with a null payoff, i.e. $g^j_{p,\sigma} = 0$, while the agents j in position $\sigma(j) \leq p$ proceed to the next round to bargain over $v(P^\sigma_{\sigma^{-1}(p)})$.

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Implementation

5. The new proposer is agent i in position $\sigma(i) = p$. Agent i makes an offer $x_j^i \in \mathbb{R}$ to each other j such that $\sigma(j) < p$. If the offer is unanimously accepted by all agents j in position $\sigma(j) < p$, then the payoffs are as follows:

$$g_{p,\sigma}^i = v(P_i^\sigma) - \sum_{\sigma(j)<\sigma(i)} x_j^i \text{ and } \forall j : \sigma(j) < \sigma(i), \quad g_{p,\sigma}^j = x_j^i.$$

If the offer is rejected, then i in position $\sigma(i) = p$ leaves the bargaining procedure with a null payoff. Then, Step 5 is repeated between the agents j in position $\sigma(j) \leq p - 1$, where the new proposer is agent in position $p - 1$. Step 5 is repeated until a proposal is accepted. In case the bargaining procedure reaches the situation where the only active agent i is such that $\sigma(i) = 1$, then his or her payoff in $G_{p,\sigma}$ is equal to $v(i)$.

Stage 4: rewards $(z^i_{p,\sigma})_{i\in N}$ resulting from Stages 1, 2 and 3 in $G_{p,\sigma}$, are defined as:

$$\forall i \in N, \quad z^i_{p,\sigma} = g^i_{p,\sigma} - h^i_p - h^i_\sigma + \frac{H_p + H_\sigma}{n},$$

i.e. each agent pays his or her bids, receives an equal share of the aggregate bids $H(p)$ and $H(\sigma)$ plus the payoff resulting from the bargaining procedure $G_{p,\sigma}$. Finally, since p and σ are chosen randomly in Ω_{A_α} and Ω_{Σ_N}, the expected payoff of each agent playing Mechanism (B) is given by:

$$\forall i \in N, \quad m_i = \frac{\sum_{p\in\Omega_{A_\alpha}} \sum_{\sigma\in\Omega_{\Sigma_N}} z^i_{p,\sigma}}{|\Omega_{A_\alpha}| \times |\Omega_{\Sigma_N}|}.$$

- At Stage 1 and Stage 2, the position $p+1$ of the proposer and the ordering σ are chosen. At Stage 3, the bargaining procedure is schematically as follows:

$$\underbrace{\sigma(1)\cdots\sigma(p)}_{\text{Alternating offer proc.}} \qquad \underbrace{\sigma(p+1)}_{\text{Proposer}} \qquad \underbrace{\sigma(p+2)\cdots\sigma(n)}_{\text{Take-it-or-leave-it proc.}}$$

- A TU-game is zero-monotonic if its zero-normalization is a monotone TU-game.

Proposition

Consider any zero-monotonic TU-game $v \in V_N$ and a probability distribution α with support $A_\alpha \subseteq \{0, \ldots, n-1\}$. Then, Mechanism (B) implements the Solidarity value $Sol^\alpha(v)$ in Subgame Perfect Nash Equilibrium.

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