Consistent values and two player games

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Outline

TU games

- N is the nonempty and finite players set,
- $T \subseteq N$ is a coalition,
- $v : \mathcal{P}(N) \to \mathbb{R}$ with $v(\emptyset) = 0$ is a TU game,
- $\mathcal{G}^N$ is the class of TU games with players set N,
- $v_S \in \mathcal{G}^S$, $v_S = v|_S$ is the subgame of v on S.

Some values

Definition

A value ψ on $A \subseteq \Gamma^N$ is function such that for all $S \in \mathcal{N}$ it holds that $\psi\colon A \cap \mathcal{G}^S \to \mathbb{R}^S$.

Definition

The Shapley value (Shapley, 1953) of a game $v \in \mathcal{G}^N$ is defined as follows, $i \in N$:

$$\mathrm{Sh}(v)_i = \sum_{S \subseteq N \setminus \{i\}} v_i'(S) \frac{|S|!(|N \setminus S| - 1)!}{|N|!} .$$

Definition

The nucleolus (Schmeidler, 1969) of a game $v \in \mathcal{G}^N$ is defined as follows. For each $x \in \mathbb{R}^N$ let $e_v(x) \in \mathbb{R}^{\mathcal{P}(N)}$ be defined as $e = \{v(S) - x(S)\}_{S \in \mathcal{P}(N)}$, where $x(S) = \sum_{i \in S} x_i$, and $S \leq T$ if $e_S \geq e_T$. Then the nucleolus of game v is the following:

$$\{x \in I(v) \colon e_v(x) \leq_{lex} e_v(y),\ y \in I(v)\} ,$$

where $\leq_{lex}$ is the lexicographic ordering.

Davies and Maschler

Definition

Take a game $v \in A \cap \mathcal{G}^N$, a value ψ defined on A, and a coalition $S \in \mathcal{N}$. The *DM-reduced game* (Davis and Maschler, 1965) with respect to S and ψ is the game

$$v_{S,\psi}^{DM}(T) = \begin{cases} 0 & \text{if } T = \emptyset\,, \\ v(N) - \sum\limits_{i \in N \setminus S} \psi(v)_i & \text{if } T = S\,, \\ \max\limits_{Q \subseteq N \setminus S} v(T \cup Q) - \sum\limits_{i \in Q} \psi(v)_i & \text{if } 0 < |T| < |S|\,. \end{cases}$$

Imputation saving reduced game

Definition

Take a game $v \in A \cap \mathcal{G}^N$, a value ψ defined on A, and a coalition $S \in \mathcal{N}$. The *imputation saving reduced game* (Snijders, 1995) with respect to S and ψ is the game

$$v^{IS}_{S,\psi}(T) = \begin{cases} 0 & \text{if } T = \emptyset\,, \\ v(N) - \sum_{i \in N \setminus S} \psi(v)_i & \text{if } T = S\,, \\ \max_{Q \subseteq N \setminus S} v(T \cup Q) - \sum_{i \in Q} \psi(v)_i & \text{if } 1 < |T| < |S|\,, \\ \min\{\psi(v)_i, \max_{Q \subseteq N \setminus S} v(\{i\} \cup Q) - \sum_{i \in Q} \psi(v)_i\} & \text{if } T = \{i\}\,. \end{cases}$$

Hart and Mas-Colell

Definition

Take a game $v \in A \cap \mathcal{G}^N$, a value ψ defined on A, and a coalition $S \in \mathcal{N}$. If for each $T \subseteq S$, $T \neq \emptyset$, the subgame of v on coalition $T \cup (N \setminus S)$, $v^{T \cup (N \setminus S)}$ is in A, then the *HM-reduced game* (Hart and Mas-Colell, 1989) with respect to S and ψ is the game

$$v^{HM}_{S,\psi}(T) = \begin{cases} 0 & \text{if } T = \emptyset\,, \\ v(T \cup (N \setminus S)) - \sum\limits_{i \in N \setminus S} \psi_i(v^{T \cup (N \setminus S)}) & \text{otherwise.} \end{cases}$$

Moulin

Definition

Take a game $v \in A \cap \mathcal{G}^N$, a value ψ defined on A, and a coalition $S \in \mathcal{N}$. The *M-reduced game* (Moulin, 1985) with respect to S and ψ is the game

$$v^M_{T,\psi}(T) = \begin{cases} 0 & \text{if } T = \emptyset\,, \\ v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} \psi_i(v) & \text{otherwise.} \end{cases}$$

Reasonability

Definition

A notion of reduced game is reasonable, if for every game $v \in A \cap \mathcal{G}^N$, value ψ defined on A, a coalition $S \subseteq N$, $|S| = 2$ we have a reduced game that $v_{S,\psi}(S) = v(N) - \sum_{i \in N \setminus S} \psi(v)_i$.

Remark

First of all, regarding an RGP property if the above reasonability property does not hold, then an EFF solution cannot meet the RGP.
Second, all the above listed reduced game properties meet the above reasonability property.

Axioms

Definition

A value ψ defined on $A \subseteq \Gamma^N$ meets

- Aggregate Monotonicity (AM) if for all $S \in \mathcal{N}$, $v, w \in A \cap \mathcal{G}^S$ such that $v(T) = w(T)$, $T \subset S$ and $v(S) \geq w(S)$ we have that $\psi(v) \geq \psi(w)$,
- Core Selection (CS) if for all $v \in A$ such that core $(v) \neq \emptyset$ we have that $\psi(v) \in$ core (v),
- Covariance (COV) if for all $v \in A \cap \mathcal{G}^T$, $T \in \mathcal{N}$, $\alpha \geq 0$ and $\beta \in \mathbb{R}^T$ such that $\alpha v \oplus \beta \in A$ we have that $\psi(\alpha v \oplus \beta) = \alpha\psi(v) + \beta$,
- Covariance on singleton games (COVSING) if for all $v \in A \cap \mathcal{G}^T$, $T \subseteq N$, $|T| = 1$, $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that $\alpha v \oplus \beta \in A$ we have that $\psi(\alpha v \oplus \beta) = \alpha\psi(v) + \beta$.

Axioms II

Definition

A value ψ defined on $A \subseteq \Gamma^N$ meets

- Additive for additive games (ADAG) if for all $v \in A \cap \mathcal{G}^T$, $T \in \mathcal{N}$ and $\beta \in \mathbb{R}^T$ such that $v \oplus \beta \in A$ we have that $\psi(v \oplus \beta) = \psi(v) + \beta$,
- Efficiency (EFF) if for all $v \in A$ we have that $\psi(v)(N) = v(N)$,
- Equal Treatment Property (ETP) if for all $S \in \mathcal{N}$, $v \in A \cap \mathcal{G}^S$ and players $i, j \in S$ such that $v(T \cup \{i\}) = v(T \cup \{j\})$, $T \subseteq S \setminus \{i, j\}$, we have that $\psi_i(v) = \psi_j(v)$,
- Null-player Property (NP), if for all $v \in A$, the null players in v get 0 by $\psi(v)$,
- Zero-game Property (ZGP), if $0 \in A \cap \mathcal{G}^T$, $T \in \mathcal{N}$, then $\psi_i(0) = 0$, $i \in T$.

Axioms III

Definition

A value ψ defined on $A \subseteq \Gamma^N$ meets

- Zero Singleton Game Property (ZSGP), if $0 \in A \cap \mathcal{G}^T$, $T \in \mathcal{N}$, $|T| = 1$, then $\psi_i(0) = 0$, $i \in T$,
- R-Reduced Game Property (R-RGP) if for all $S \in \mathcal{N}$, $v \in A \cap \mathcal{G}^S$ and $T \subseteq S$ such that $v^R_{T,\psi(v)} \in A$ we have that $\psi(v)_T = \psi(v^R_{T,\psi(v)})$.
- R-Weak Reduced Game Property (R-WRGP) if for all $S \in \mathcal{N}$, $v \in A \cap \mathcal{G}^S$ and $T \subseteq S$, $1 \leq |T| \leq 2$ such that $v^R_{T,\psi(v)} \in A$ we have that $\psi(v)_T = \psi(v^R_{T,\psi(v)})$.

Axioms IV

Definition

A value ψ defined on $A \subseteq \Gamma^N$ meets

- R-Very Weak Reduced Game Property (R-VWRGP) if for all $S \in \mathcal{N}$, $v \in A \cap \mathcal{G}^S$ and $T \subseteq S$, $|T| = 2$ such that $v^R_{T,\psi(v)} \in A$ we have that $\psi(v)_T = \psi(v^A_{T,\psi(v)})$.
- R-Extremely Weak Reduced Game Property (R-EWRGP) if for all $S \in \mathcal{N}$, $v \in A \cap \mathcal{G}^S$ and $T \subseteq S$, $|T| = 1$ such that $v^R_{T,\psi(v)} \in A$ we have that $\psi(v)_T = \psi(v^A_{T,\psi(v)})$.

Closed classes of games

Definition

Consider a class of games $A \subseteq \Gamma^N$, a value ψ defined on A and a reduced game notion R-reduced game. Then the class of games A is R-reduced game closed for ψ if for all $v \in A \cap \mathcal{G}^S$, $S \subseteq N$, and $T \subseteq S$, $T \neq \emptyset$ we have that $v^R_{T,\psi(v)} \in A$.

Definition

Consider a class of games $A \subseteq \Gamma^N$. Then the class of games A is *ADAG* closed if for all $v \in A \cap \mathcal{G}^T$, $T \in \mathcal{N}$, and $\beta \in \mathbb{R}^T$ we have that $v \oplus \beta \in A$.

The main result

Theorem

Consider two values ϕ and ψ on $A \subseteq \Gamma^N$ and a reasonable reduced game notion R-reduced game such that the class of games A is R-reduced game closed for both ϕ and ψ, and ADAG closed. If

- *both values meet ADAG and R-VWRGB,*
- *for all $S \subseteq N$, $|S| = 2$, $v \in \mathcal{G}^S$ it holds that $\phi(v) = \psi(v)$ and both meet EFF.*

Then $\phi = \psi$ on $\bigcup_{S \subseteq N,\ |S| \geq 2} A \cap \mathcal{G}^S$.

Proof

Proof.

Let $v \in A \cap \mathcal{G}^T$, $T \subseteq N$. If $|T| = 2$, then by assumption we have that $\psi(v) = \phi(v)$. Suppose that $|T| \geq 3$.
Then there exists $\beta \in \mathbb{R}^T$ such that

$$\psi(v) = \phi(v) + \beta \overset{ADAG}{=} \phi(v \oplus \beta)\,.$$

Let $S \subseteq T$ be such that $|S| = 2$. Then by ADAG, R-VWRGB and by that $\psi = \phi$ on $\mathcal{G}^S$ we have that

$$\phi(v^R_{S,\phi(v)+\beta}) = \phi(v^R_{S,\psi(v)}) = \psi(v^R_{S,\psi(v)}) = \phi(v^R_{S,\phi(v)}) + \beta_S = \phi(v^R_{S,\phi(v)} \oplus \beta_S)\,.$$

□

Proof II

Proof.

By EFF and by the reasonability of the reduced game for all we have that

$$v(N) - \sum_{i \in T \setminus S} \phi_i(v) + \beta_i = \sum_{i \in S} \phi(v^R_{S,\phi(v)+\beta})_i$$

$$= \sum_{i \in S} \phi(v^R_{S,\phi(v)} \oplus \beta_S)_i = v(N) - \sum_{i \in T \setminus S} \phi_i(v) + \sum_{i \in S} \beta_i\,, \quad S \subseteq T,\ |S| = 2\,,$$

that is,

$$\sum_{i \in T \setminus S} \beta_i = \sum_{i \in S} \beta_i\,, \quad S \subseteq T,\ |S| = 2\,.$$

It is ?easy? to see that the above system of linear equalities has a unique solution: $\beta = 0$. □

A corollary

It is easy to see that if we change R-VWRGP for R-WRGP in Theorem 16, then we get that the two values equal on the whole (one player games are included) considered class of games. Formally,

Corollary

Consider two values ϕ and ψ on $A \subseteq \Gamma^N$ and a reasonable reduced game notion R-reduced game such that the class of games A is R-reduced game closed for both ϕ and ψ, and ADAG closed. If

- *both values meet ADAG and R-WRGB,*
- *for all $S \subseteq N$, $|S| = 2$, $v \in \mathcal{G}^S$ it holds that $\phi(v) = \psi(v)$ and both meet EFF.*

Then $\phi = \psi$ on A.

Lemma

Consider a class of games $A \subseteq \Gamma^N$, a value ψ defined on A and a reduced game notion R-reduced game. Suppose that the class of games A is R-reduced game closed for ψ and the R-reduced game notion is reasonable. If the value ψ meets COVSING and R-EWRGP, then it meets EFF.

Proof.

The proof goes by induction on the number of the players. For any game $v \in \mathcal{G}^T$, $|T| = 1$,, by COVSING (by positive homogeneity) we have that for all $\alpha \geq 0$ we have that

$$\psi(0) = \psi(\alpha 0) = \alpha \psi(0) \,,$$

that is, $\psi(0) = 0$.
Moreover, by COVSING (ADAG)

$$\psi(v) = \psi(0 + v(T)) = \psi(0) + v(T) = v(T) \,,$$

that is, ψ meets EFF on any game with one player. □

Proof.

Now take a game $v \in A \cap \mathcal{G}^S$, $S \subseteq T$. Then for each player $i \in S$ by R-EWRGP and by that the R-reduced game notion is reasonable and by the previous point (ψ meets EFF on any game with one player) we have that

$$\psi(v)_i = \psi(v_{\{i\},\psi(v)})_i = v_{\{i\},\psi(v)}(\{i\}) = v(S) - \sum_{j \in S \setminus \{i\}} \psi(v)_j ,$$

that is, $v(S) = \sum_{i \in S} \psi(v)_i$. □

Remark

Notice that in Lemma 18 we need only ZSGP and ADAG on the singleton games instead of COVSING.

Lemma

Consider a class of games $A \subseteq \mathcal{G}^N$, $|N| = 2$ such that it is ADAG closed. Then there exists only one solution on A that meets ADAG, EFF and ETP.

Proof.

Let $v \in A$ and two ADAG, EFF and ETP values ψ and ϕ be defined on A, and let $N = \{i, j\}$. If $v(\{i\}) = v(\{j\})$, then $i \sim^v j$, therefore by that both ψ and ϕ meet EFF and ETP we have that $\psi(v) = \phi(v)$. Otherwise, w.l.o.g. we can assume that $v(\{i\}) > v(\{j\})$. Let $\beta \in \mathbb{R}^N$ be such that $\beta_i = 0$ and $\beta_j = v(\{i\}) - v(\{j\})$. Then $i \sim^{v \oplus \beta} j$, hence $\psi(v \oplus \beta) = \phi(v \oplus \beta)$. Therefore, by ADAG we have that

$$\psi(v) = \psi(v \oplus \beta) - \beta = \phi(v \oplus \beta) - \beta = \phi(v)\,.$$

□

The next proposition shows that our result is a generalization of the characterization results known in the literature.

Proposition

A value defined on Γ^N meets ADAG, ETP, ZSGP and

- *DM-RGP, if and only if it is the prenucleolus,*
- *IS-RGP, if and only if it is the nucleolus,*
- *HM-RGP, if and only if it is the Shapley value,*
- *M-RGP, if and only if it is the Moulin value.*

Proof

Proof.

If: It is well known that these values meet ADAG, ETP, ZSGP and DM-RGP/IS-RGP/HM-RGP/M-RGP respectively.

Only if: Apply Theorem 16 and Lemmata 18 and 20. □

Thank you for the attention!

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