

# Network congestion games with player-specific costs

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JOURNÉE MODMAD

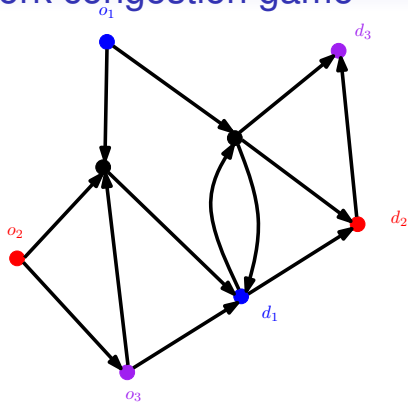
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*Joint work with Thomas Pradeau*

CERMICS, Optimisation et Systèmes

# Model: non-atomic network congestion game

- ★  $D = (V, A)$ : directed graph.
- ★  $\mathcal{L} \subseteq V^2$ : set of **origin-destination pairs**.
- ★  $b^{od} \in \mathbb{R}_+$ : number of players going from  $o$  to  $d$  (the **demand**). Continuum of players: they are negligible.
- ★ On each arc  $a \in A$ , there is a continuous **cost**  $c_a(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ .

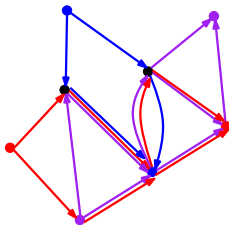
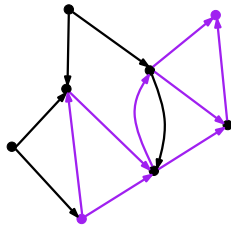
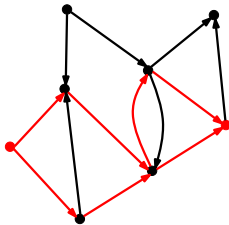
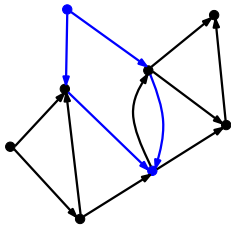


$x_a$  = number of players choosing arc  $a$ .

$\sum_{a \in P} c_a(x_a)$  = **cost** of path  $P$ .

- ★ Each player chooses a path from his origin  $o$  to his destination  $d$ .
- ★ Assumption: the chosen path is of minimal cost.

# Examples



# Equilibrium

<b>Input.</b>	$D = (V, A)$ network	$\mathcal{L} \subseteq V^2$ origin-destination pairs	$(b^{od})_{od \in \mathcal{L}}$ demand	$(c_a(\cdot))_{a \in A}$ costs
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**Output.**

For every  $a \in A$  and every  $od \in \mathcal{L}$ :

- $x_a^{od}$  = number of players choosing arc  $a$  at **equilibrium** among those going from  $o$  to  $d$ .

**Equilibrium:** each player has chosen a path of minimal cost.

# Equilibrium: mathematical description

$(x_a^{od})$  is an **equilibrium** if for every  $(o, d) \in \mathcal{L}$

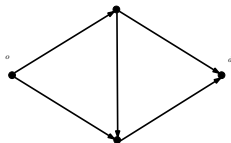
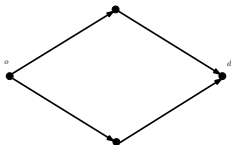
$$\left\{ \begin{array}{l} (x_a^{od})_{a \in A} = o\text{-}d \text{ flow of value } b^{od} \\ x_a = \sum_{(o,d) \in \mathcal{L}} x_a^{od} \\ \sum_{a \in P} c_a(x_a) \leq \sum_{a \in Q} c_a(x_a) \end{array} \right. \quad \begin{array}{l} a \in A \\ P, Q \in \mathcal{P}^{od}, P \text{ is} \\ \text{used} \end{array}$$

$\mathcal{P}^{od}$  = set of  $o$ - $d$  paths

$P$  **used** if  $x_a^{od} > 0$  for all  $a \in P$

# Practical interest

- This model = good approximation of what happens in practice
  - ★ used in transport engineering, telecoms,...
- Useful since the phenomenons are nonintuitive
  - ★ Braess paradox = opening a new road may increase all travel times
  - ★ paradox recovered by the model



# Questions

Does a Nash equilibrium exist?

Is it unique? *(i.e. are the  $x_a$  unique?)*

Is the equilibrium efficiently computable?

How far from **social optimum**?

# Questions

Does a Nash equilibrium exist?

- ★ Yes (fixed point theorem).

Is it unique? *(i.e. are the  $x_a$  unique?)*

- ★  $c_a(\cdot)$  increasing  $\Rightarrow$  uniqueness

Is the equilibrium efficiently computable?

- ★  $c_a(\cdot)$  nondecreasing  $\Rightarrow$  convex optimization

How far from **social optimum**?

- ★ **Price of Anarchy** = cost at equilibrium/optimal social cost



# Existence and uniqueness of the equilibrium

## Theorem (Beckman, 1956)

*An equilibrium always exists and it is “unique” when the cost functions  $c_a(\cdot)$  are increasing.*

“Unique” means there are unique  $x_a$ 's solutions of the system

$$\left\{ \begin{array}{ll} (x_a^{od})_{a \in A} = o\text{-}d \text{ flow of value } b^{od} & (o, d) \in \mathcal{L} \\ x_a = \sum_{(o,d) \in \mathcal{L}} x_a^{od} & a \in A \\ \sum_{a \in P} c_a(x_a) \leq \sum_{a \in Q} c_a(x_a) & \begin{array}{l} P, Q \in \mathcal{P}^{od}, \\ P \text{ is used,} \\ (o, d) \in \mathcal{L} \end{array} \end{array} \right.$$

# Computation of an equilibrium

## Theorem (Beckman, 1956)

When the  $c_a(\cdot)$ 's are nondecreasing,  $(x_a^{od})$  is an equilibrium if and only if it is an optimal solution of

$$\begin{aligned} \min \quad & \sum_{a \in A} \int_0^{x_a} c_a(t) dt \\ \text{s.c.} \quad & \sum_{(o,d) \in \mathcal{L}} x_a^{od} = x_a \quad \forall a \in A \\ & \sum_{a \in \delta^+(o)} x_a^{od} - \sum_{a \in \delta^-(o)} x_a^{od} = b^{od} \quad \forall (o,d) \in \mathcal{L} \\ & \sum_{a \in \delta^+(v)} x_a^{od} = \sum_{a \in \delta^-(v)} x_a^{od} \quad \forall (o,d) \in \mathcal{L}, \forall v \in V \setminus \{o,d\} \\ & x_a^{od} \geq 0 \quad \forall (o,d) \in \mathcal{L}, \forall a \in A. \end{aligned}$$

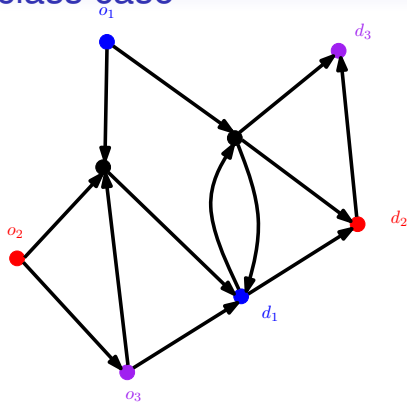
**Convex optimization!**  $\implies$  computation is easy when the  $c_a(\cdot)$ 's are nondecreasing.

# The multiclass case

i.e. with player-specific costs

## Model – multiclass case

- ★  $D = (V, A)$ : directed graph.
- ★  $\mathcal{L} \subseteq V^2$ : set of origin-destination pairs.
- ★  $b^{od,k} \in \mathbb{R}_+$ : number of **class  $k$**  players going from  $o$  to  $d$  (the demand). Continuum of players: they are negligible.
- ★ On each arc  $a \in A$  and for each **class  $k$** , there is a continuous cost  $c_a^k(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ .



$x_a$  = number of players choosing arc  $a$ .

$\sum_{a \in P} c_a^k(x_a)$  = cost of path  $P$  experienced by a **class  $k$**  player.

- ★ Each player chooses a path from his origin  $o$  to his destination  $d$ .
- ★ Assumption: the chosen path is of minimal cost.

# Equilibrium

**Input.**  $D = (V, A)$  |  $\mathcal{L} \subseteq V^2$  |  $(b^{od,k})_{od \in \mathcal{L}, k \in K}$  |  $(c_a^k(\cdot))_{a \in A, k \in K}$   
network | or.-dest. pairs | demand | costs

**Output.**

For every  $a \in A$ , every  $od \in \mathcal{L}$ , and every  $k \in K$ :

- $x_a^{od,k}$  = number of class  $k$  players choosing arc  $a$  at equilibrium, among those going from  $o$  to  $d$ .

**Equilibrium:** each player has chosen a path of minimal cost.

# Equilibrium: mathematical description

$(x_a^{od,k})$  is an equilibrium if for every  $(o, d) \in \mathcal{L}$  and every  $k \in K$

$$\left\{ \begin{array}{l} (x_a^{od,k})_{a \in A} = o\text{-}d \text{ flow of value } b^{od,k} \\ x_a = \sum_{(o,d) \in \mathcal{L}, k \in K} x_a^{od,k} \\ \sum_{a \in P} c_a^k(x_a) \leq \sum_{a \in Q} c_a^k(x_a) \end{array} \right. \quad \begin{array}{l} a \in A \\ P, Q \in \mathcal{P}^{od}, P \text{ is} \\ \text{used by class } k \end{array}$$

## Questions – the multiclass case

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- ★  $c_a(\cdot)$  increasing  $\Rightarrow$  it depends

Is the equilibrium efficiently computable?

- ★  $c_a(\cdot)$  nondecreasing  $\Rightarrow$  it depends

How far from **social optimum**?

- ★ Price of Anarchy



# The multiclass case: uniqueness

# The uniqueness issue

## Theorem (Schmeidler, 1973)

*An equilibrium always exists in the multiclass setting.*

There are examples with several equilibria, with several possible  $x_a$ 's, while all  $c_a^k(\cdot)$  are increasing: uniqueness of the equilibrium flows is not automatically ensured. (It contrasts with the monaclass case).

**Challenge:** Find necessary and/or sufficient conditions ensuring uniqueness.

# Uniqueness: cost-based sufficient conditions

## Proposition (Aashtiani, Magnanti, 1981)

*If the players' cost functions are identical up to additive constants, then, for every two Nash equilibria, the flow on each arc in the network in the first equilibrium is equal to that in the second.*

## Uniqueness property

$G$  = undirected graph,  $\mathcal{L}$  = collection of  $o$ - $d$  pairs.

$(G, \mathcal{L})$  has the **uniqueness property (UP)** if for any classes, demands ( $b^{od,k}$ ), and increasing costs ( $c_a^k(\cdot)$ ), the equilibrium flows are unique.

(on the digraph where each edge has been replaced by two opposite arcs)

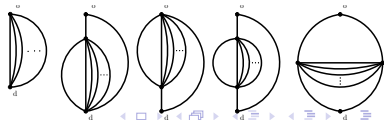
### Theorem (Milchtaich, 2007)

*There is only one  $o$ - $d$  pair:*

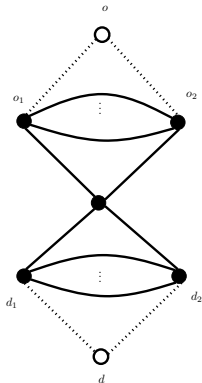
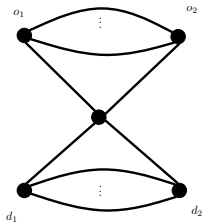
$(G, \{(o, d)\})$  has the UP  $\iff$   $G$  is “nearly-parallel”.

“Nearly-parallel” =

combination in series of



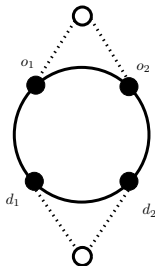
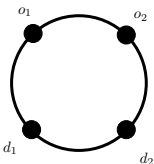
# Uniqueness for general graphs using Milchtaich's theorem



- Add a fictitious origin vertex connected to every origin.
- Add a fictitious destination vertex connected to every destination.

Augmented graph has uniqueness property  $\Rightarrow$  Original graph has uniqueness property.

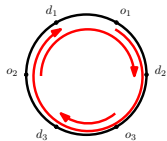
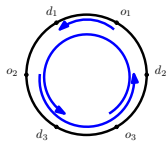
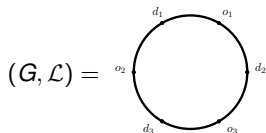
# Uniqueness for general graphs using Milchtaich's theorem



- Add a fictitious origin vertex connected to every origin.
- Add a fictitious destination vertex connected to every destination.

Augmented graph has not the uniqueness property  $\Rightarrow$  ????

# Uniqueness property on a cycle

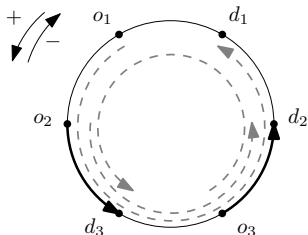


Theorem (M., Pradeau, 2014)

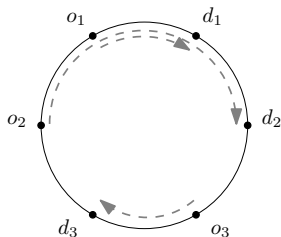
*Assume that  $G$  is a cycle and let  $\mathcal{L}$  be any collection of  $o$ - $d$  pairs.*

*$(G, \mathcal{L})$  has the UP  $\iff$  Each arc belongs to at most two  $o$ - $d$  paths.*

## Example not having the uniqueness property



The positive paths.



The negative paths.

The arcs  $o_2d_3$  and  $o_3d_2$  are contained in three  $o$ - $d$  paths.



# Proof strategy

Step 1.

*Each arc of  $D$  belongs to at most two o-d paths*



*The equilibrium flows are unique whatever are the classes  $K$ , increasing costs ( $c_a^k(\cdot)$ ), and demands ( $b^{od,k}$ )*

Step 2.

*There is an arc of  $D$  belonging to at least three o-d paths*



*There exist classes  $K$ , increasing costs ( $c_a^k(\cdot)$ ), and demands ( $b^{od,k}$ ) leading to two equilibria with distinct flows*

Proof by an explicit construction of costs and demands.

# Step 1:

*Each arc of  $D$  belongs to at most two  $o$ - $d$  paths*



*The equilibrium flows are unique whatever are the classes, costs, and demands*

- Let  $\mathbf{x}$  and  $\hat{\mathbf{x}}$  be two equilibria. Define  $\Delta_{od} = x_{p^+}^{od} - \hat{x}_{p^+}^{od}$ .
- Suppose  $\Delta_{o_0 d_0} \neq 0$  for some  $o_0$ - $d_0$ . There exists an  $o_1$ - $d_1$  s.t.  $\Delta_{o_0 d_0} \Delta_{o_1 d_1} < 0$  and  $\Delta_{o_0 d_0} + \Delta_{o_1 d_1} < 0$ .
- We repeat this argument and get an infinite sequence  $|\Delta_{o_0 d_0}| < |\Delta_{o_1 d_1}| < \dots < |\Delta_{o_j d_j}| < \dots$ .
- Contradiction with finiteness.

## Step 2:

*There is an arc of  $D$  belonging to at least three  $o$ - $d$  paths*



*There exist classes  $K$ , increasing costs ( $c_a^k(\cdot)$ ), and demands ( $b^{od,k}$ ) leading to two equilibria with distinct flows*

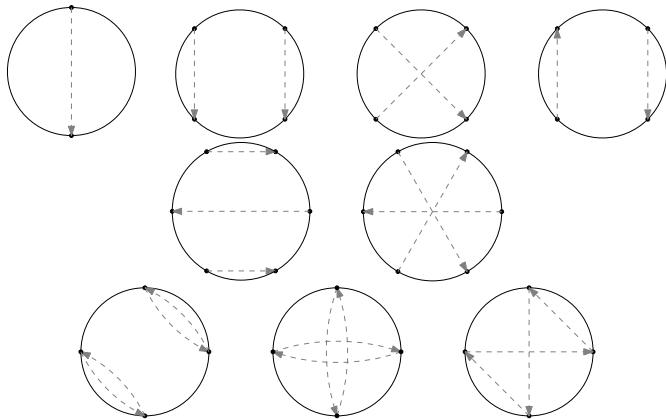
An arc in 3  $o$ - $d$  paths: explicitly building of cost functions and demands leading to two equilibria with distinct flows.

Some features:

- Three **classes**.
- Affine cost functions.
- Explicit construction of two equilibria.
- These equilibria are “single-path”.

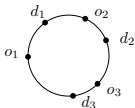
# Structural characterization

Each arc in at most two  $o$ - $d$  paths  $\Leftrightarrow (G, \mathcal{L})$  homeomorphic to a minor of one of



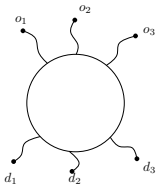
## Corollary for general graphs: examples

If



is in  $(G, \mathcal{L})$ ,  $G$  does not have the UP.

If



is in  $(G, \mathcal{L})$ ,  $G$  does not have the UP.

## Having a minor without the uniqueness property

A **subgraph** of  $(G, \mathcal{L})$  does not have the UP  $\implies (G, \mathcal{L})$  does not have the UP.

A **minor** of  $(G, \mathcal{L})$  does not have the UP:

- If the contractions involve only **bridges**,  $G$  does not have the UP.
- If the counterexamples are obtained via “single-path” flows,  $G$  does not have the UP.
- And in general, **open question**.

## Strong uniqueness property

$G$  has the **strong uniqueness property** (SUP) =  $(G, \mathcal{L})$  has the UP for any collection of  $o-d$  pairs  $\mathcal{L}$

Theorem (M., Pradeau, 2014)

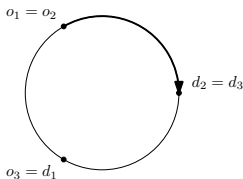
$G$  has the SUP  $\iff$  *No cycles of length 3 or more.*

Graph having the SUP are thus graphs obtained from a forest by replicating some edges.

# Proof

( $\Rightarrow$ )

The graph



has one arc in three  $o$ - $d$  paths: no UP.

( $\Leftarrow$ ) results from two easy statements:

- A graph with two vertices and parallel edges has the SUP.
- Glueing two graphs on a vertex maintains the SUP.





# Equivalence of equilibria

## Equivalence of equilibria

Two equilibria are **equivalent** if the contribution of each  $(od, k) \in \mathcal{L} \times K$  to the flow on each arc is the same in all equilibria.

## Theorem (M.,Pradeau, 2013)

*A ring has the uniqueness property  $\iff$  Generically, for every partition of the population into classes, all equilibria are equivalent.*

## Uniqueness property: a combinatorial sufficient condition for a 'two-sided' game

Consider a nonatomic congestion game with player-specific cost functions, not necessarily played on a graph.

### Proposition

*Suppose that there are finite sets  $A^+$  and  $A^-$  such that every player  $i$  has exactly two available strategies  $r_i^+$  and  $r_i^-$  with  $r_i^+ \subseteq A^+$  and  $r_i^- \subseteq A^-$ . Then, if all triples of pairwise distinct strategies have an empty intersection, the uniqueness property holds.*

# Open questions

## Generalization

UP for general graphs?

## Minor

What if a minor does not have UP?

## Question

What if *one-way* edges are allowed?

# The multiclass case: computation

# Affine costs

## Input.

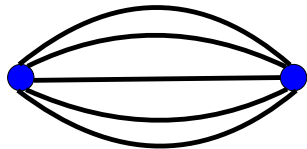
- Directed graph  $D = (V, A)$ ,
- Or.-dest. pairs  $\mathcal{L} \subseteq V^2$
- Demands  $b^{od,k} \in \mathbb{R}_+ \quad \forall od \in \mathcal{L}, \forall k \in K$
- Costs  $c_a^k(x) = \alpha_a^k x + \beta_a^k \quad \forall a \in A, \forall k \in K$

## Theorem (M.,Pradeau, 2014)

*An equilibrium can be computed in polynomial time when the number of vertices and the number of classes are fixed.*

## Corollary

*Problem is polynomial for the parallel-arc graphs when the number of classes is fixed.*



# Polynomial algorithm

Let  $\mathbf{A} = \{(S^k)_{k \in K} : S^k \subseteq A\}$ . Algorithm consists in two steps.

- A Compute a set  $\mathcal{S} \subseteq \mathbf{A}$  of polynomial size such that for any equilibrium multiflow  $(\mathbf{x}^k)_{k \in K}$ , there is a  $(S^k)_{k \in K} \in \mathcal{S}$  with  $\text{supp}(\mathbf{x}^k) \subseteq S^k$  for all  $k$ .
- B Test for every  $(S^k)_{k \in K} \in \mathcal{S}$  whether there exists an equilibrium multiflow  $(\mathbf{x}^k)_{k \in K}$  with  $\text{supp}(\mathbf{x}^k) \subseteq S^k$  for all  $k$ , and compute it if it exists.

## Polynomial algorithm: first step

- A Compute a set  $\mathcal{S} \subseteq \mathbf{A}$  of polynomial size such that for any equilibrium multiflow  $(\mathbf{x}^k)_{k \in K}$ , there is a  $(S^k)_{k \in K} \in \mathcal{S}$  with  $\text{supp}(\mathbf{x}^k) \subseteq S^k$  for all  $k$ .

For fixed  $|K|$  and  $|V|$ , can be done in polynomial time: there is a hyperplane arrangement whose cells corresponds to the possible  $\mathcal{S}$ 's.

## Polynomial algorithm: second step

- B Test for every  $(S^k)_{k \in K} \in \mathcal{S}$  whether there exists an equilibrium multiflow  $(\mathbf{x}^k)_{k \in K}$  with  $\text{supp}(\mathbf{x}^k) \subseteq S^k$  for all  $k$ , and compute it if it exists.

Can be done in polynomial time: once the support of each class is known, amounts to solve a system of linear inequalities (interior point method).

$$\left\{ \begin{array}{l} (x_a^{od,k})_{a \in A} = o\text{-}d \text{ flow of value } b^{od,k} \\ x_a = \sum_{(o,d) \in \mathcal{L}, k \in K} x_a^{od,k} \\ \sum_{a \in P} c_a^k(x_a) \leq \sum_{a \in Q} c_a^k(x_a) \end{array} \right. \quad \begin{array}{l} a \in A \\ P, Q \in \mathcal{P}^{od}, P \text{ is} \\ \text{used by class } k \end{array}$$



# Affine costs

## Input.

- Directed graph  $D = (V, A)$ ,
- Or.-dest. pairs  $\mathcal{L} \subseteq V^2$
- Demands  $b^{od,k} \in \mathbb{R}_+$   $\forall od \in \mathcal{L}, \forall k \in K$
- Costs  $c_a^k(x) = \alpha_a^k x + \beta_a^k$   $\forall a \in A, \forall k \in K$

## Output.

- An equilibrium.

The exact complexity of this computational problem remains to be determined.

PPAD-complete?

# A practically efficient algorithm for affine costs

$$\begin{aligned} \min \quad & \omega \\ \text{s.t.} \quad & \sum_{a \in \delta^+(v)} x_a^k = \sum_{a \in \delta^-(v)} x_a^k + b_v^k && k \in K, v \in V \\ & \alpha_{uv}^k \sum_{k' \in K} x_{uv}^{k'} + \pi_u^k - \pi_v^k - \mu_{uv}^k + \mathbf{e}_{uv}^k \omega = -\beta_{uv}^k && k \in K, (u, v) \in A \\ & x_a^k \mu_a^k = 0 && k \in K, a \in A \\ & \pi_{s^k}^k = 0 && k \in K \\ & x_a^k \geq 0, \mu_a^k \geq 0, \omega \geq 0, \pi_v^k \in \mathbb{R} && k \in K, a \in A, v \in V. \end{aligned}$$

It is a **linear complementary problem**.

# A Lemke-like algorithm

## Proposition (M., Pradeau, 2014)

*There is a Lemke-like algorithm solving this problem.*

Theoretical consequences:

- The problem with affine costs is in the **PPAD** class.
- If all input parameters are rational, then there always exists a rational equilibrium multiflow.

Practical consequence:

Classes	Grid	Vertices	Arcs	Pivots	Algorithm (seconds)
4	6 × 6	36	120	126	0.9
	8 × 8	64	224	249	5.4
10	6 × 6	36	120	322	15
	8 × 8	64	224	638	87
50	2 × 2	4	8	56	0.3
	4 × 4	16	48	636	105

**Thank you**