Formation of coalition structures as a non-cooperative game

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Games and Optimization Workshop
St. Etienne, 22 November

November 22, 2016

Example 1. A corporate dinner game

1. Four players: A - president; B - senior vice-president; C_1, C_2 - two other vice-presidents.
2. A coalition - a group of players at *one* table. Every player may sit only at one table.
3. A coalition structure: allocations of all players over ≤ 4 tables.
4. Individual set of strategies: a set of all coalition structures for the players.
5. A player chooses a coalition structure.
6. A set of all strategies: direct product of 4 individual strategy sets.

Everyone (besides A) would like to have a dinner with A, but A only with B. Everyone wants players outside his table to eat individually, due to possible dissipation of rumors or information exchange. No one can enforce others to form or not to form coalitions.

cont.

1. In every partition any coalition (i.e. a table) is formed only if everybody at the table agrees to have dinner together, otherwise a player eats alone.
2. The game is simultaneous and one shot.
3. A realization of a final partition ($\equiv$ a coalition structure) depends on choices of coalition structures of all players.
4. Example of coalition structure formation.
   - Player A chooses $\{\{A, B\}, \{C_1\}, \{C_2\}\}$; player B chooses $\{\{A, B\}, \{C_1\}, \{C_2\}\}$; player C_1 chooses $\{\{A, C_1\}, \{B\}, \{C_2\}\}$, and player C_2 chooses $\{\{A, C_2\}, \{B\}, \{C_2\}\}$,
   - the final partition is $\{\{A, B\}, \{C_1\}, \{C_2\}\}$.

Players have preferences over coalition structures. Payoff profile should be defined for every final coalition structure (next slide).

Coalition structures and payoffs

Players simultaneously announce choices of a desirable coalition structure, then a final coalition structure is formed according to the rule above.

Table: Strategies and payoffs in the corporate dinner game

num	Best final partitions	Non-cooperative payoff profile $(U_A, U_B, U_{C_1}, U_{C_2})$	Values of coalitions as in cooperative game theory
1	$\{\{A, B\}, \{C_1\}, \{C_2\}\}$	(10,10,3,3)	20_{AB}, 3_{C_1}, 3_{C_2}
2*	$\{\{A, B\}, \{C_1, C_2\}\}$	(8,8,5,5)	16_{AB}, $10_{C_1,C_2}$
3	$\{\{A, C_1\}, \{B, C_2\}\}$	(3,5,10,5)	13_{AC_1}, 10_{BC_2}
4	$\{\{A, C_1\}, \{B\}, \{C_2\}\}$	(3,3,10,3)	13_{AC_1}, 3_B, 3_{C_2}
5	$\{\{A, C_2\}, \{B, C_1\}\}$	(3,5,5,10)	13_{AC_2}, 10_{BC_1}
6	$\{\{A, C_2\}, \{B\}, \{C_1\}\}$	(3,3,3,10)	13_{AC_1}, 3_B, 3_{C_1}
7	all other partitions	(1,1,1,1)	≤ 4

* - an equilibrium strategy profile for all players, A and B have an indeterminacy between two coalition structures, compare options 1 and 2.

Results of the game

- Players A and B can form the coalition $\{A, B\}$, but cannot prevent formation of the coalition $\{C_1, C_2\}$. Players C_1, C_2 anticipate not having the dinner with A or with B, hence maximize their individual payoffs forming the coalition $\{C_1, C_2\}$. There is no enforcement in the game between players.
- Resulting coalition structures are formed from non-cooperative actions of **all** players, where an individual strategy is a coalition structure, but not a coalition.
- Individual preferences over all possible coalition structures do matter.
- An equilibrium in the example exists, is second-best efficient for everyone (with different benchmarks for A and B *vs* C_1 and C_2). The equilibrium contains two coalitions.
- The game is based only on individual rationality.

Used terminology

Coalition structure

1. Coalition structure (for short, a partition) - a collection of disjoint subsets from a set of players such that a union of all elements of a partition makes the original set.
2. Coalition - an element of a partition.

Relation to the Nash Program will be discussed in the final slide.

Research agenda:

- coalition structure construction as a non-cooperative game along with an equilibrium concept for this game, (this paper)
- equilibrium concept for the family of games, (this paper)
- further: non-cooperative stability criterion, cooperation, stochastic games, repeated games from self-interest behavior of players.

Reference to literature (voluminous!): Aumann, Hart, Holt, Maschler, Maskin, Myerson, Peleg, Roth, Serrano, Shapley, Schmeidler, Weber, Winter, Wooders *etc.*

Formal setup for a family of games

1. **Players**: $N = \{i\}$, N - finite integer, $2 \leq N$.
2. **Parameter** for the family of games, K: max coalition size for a game or a maximum number of deviators in the game, $K = 1, N$.
3. **Families of partitions** for every K: $\mathcal{P}(K) = \{P\colon P \subset 2^N, \#P \leq K\}$: $\mathcal{P}(K=1) \subset \ldots \subset \mathcal{P}(K=N)$.
4. Finite **strategies** (as in Nash, 1950) of $\forall i \in N$ for every K: $S_i(K) = \{S_i(P_i)\colon P_i \in \mathcal{P}(K)\}$, $S_i(K=1) \subset \ldots \subset S_i(K=N)$.
5. **Coalition structure formation rules** for every K: $\mathcal{R}(K)$: $\times_{i \in N} S_i(K) \mapsto \cup_{i \in N} S_i(P)$, P - a formed partition, $P \in \mathcal{P}(K)$, $\mathcal{R}(K=1) \subset \ldots \subset \mathcal{R}(K=N)$.

Realization of a partition may differ from one's choice of a partition.
$S(K) = \times_i S_i(K)$: $S(K) = \times_{i \in N} S_i(K)$ in terms of chosen partitions;
$S(K) = \cup_P S(P) = \cup_{P \in \mathcal{P}(K)} \cup_{i \in N} S_i(P)$ in terms of realized strategies.
See example below.

Coalition structure formation mechanism for a given K

The mechanism formally:

$$\mathcal{R}(K)\colon S(K) = \times_{i\in N} S_i(K) \mapsto \colon$$
$$\begin{cases} \forall s = (s_1, \ldots, s_N) \in S(K) \; \exists P \in \mathcal{P}(K)\colon s \in S(P), \\ S(K) = \cup_{P\in\mathcal{P}(K)} S(P), \\ \forall \bar{P}, \tilde{P} \in \mathcal{P}(K), \bar{P} \neq \tilde{P} \Rightarrow S(\bar{P}) \cap S(\tilde{P}) = \emptyset. \end{cases}.$$

Nested property of the mechanism:

$$\mathcal{R}(K=1) \subset \ldots \subset \mathcal{R}(K) \subset \ldots \subset \mathcal{R}(K=N)$$

Partition-specific payoffs

Utility of i for every K:

$$\mathcal{U}_i(K) = \{U_i(P)\colon \times_{i\in N} S_i(P) \mapsto \mathbb{R}, P \in \mathcal{P}(K)\}$$

is a family of partition specific payoffs of i for every final partition P from $\mathcal{P}(K)$. Every partition P is a non-cooperative game, (like state-contingent payoffs) inside a bigger game. Realization of a final partition depends on choices of all players. A game of N players in a partition P has:

1. a set of finite partition-specific strategies, $S(P) = \cup_{i\in N} S_i(P)$,
2. payoffs profiles, $(U_i(P))_{i\in N}\colon U_i(P) < \infty, \forall i \in N$.

Every coalition structure from $\mathcal{P}(K), K \in \{1, \ldots, N\}$, may have it's intra and inter-coalition externalities (interactions without pricing mechanism).

Individual payoffs for different K are nested:

$$\mathcal{U}_i(K=1) \subset \ldots \subset \mathcal{U}_i(K) \subset \ldots \subset \mathcal{U}_i(K=N).$$

Definition of family of games

Definition (a simultaneous coalition structure formation game)

A non-cooperative game for coalition structure formation is $\Gamma(K) = \left\langle N, \left\{K, \mathcal{P}(K), \mathcal{R}(K)\right\}, \left(S_i(K), \mathcal{U}_i(K)\right)_{i \in N} \right\rangle$, where $\left\{K, \mathcal{P}(K), \mathcal{R}(K)\right\}$ - coalition structure formation mechanism (a social norm, a social institute). $\left(S_i(K), \mathcal{U}_i(K)\right)_{i \in N}$ - properties of players in N, strategies and payoffs, such that:
$\times_{i \in N} S_i(K) \overset{\mathcal{R}(K)}{\rightarrow} \{S(P) \colon P \in \mathcal{P}(K)\} \rightarrow \{(\mathcal{U}_i(K))_{i \in N}\}$.

Novelty: Coalition structure formation mechanism shapes a set of strategies of the game.

Definition (family of games)

Family of nested games is: $\Gamma(1) \subset \ldots \subset \Gamma(K) \subset \ldots \subset \Gamma(N)$.

For every fixed K a game is the game with complete information. Players may have pre-game negotiations, but afterwards they act in ones' best interests without any binding obligations.

Mixed strategies for a game $\Gamma(K)$

Game: $\times_{i\in N} S_i(K) \overset{\mathcal{R}(K)}{\rightarrow} \{S(P)\colon P \in \mathcal{P}(K)\} \rightarrow \{(\mathcal{U}_i(K))_{i\in N}\}$.

$\Sigma_i(K) = \{\sigma_i(K)\colon \int_{S_i(K)} d\sigma_i(K) = 1\}$ - a set of all mixed strategies.

Expected utility can be defined in terms of strategies the players choose or in terms of final partition-specific strategies. Expected utility of i:

$$EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}(K)) = \int_{S(K)=\times_{i\in N} S_i(K)} U_i(s_i, s_{-i}) d\sigma_i d\sigma_{-i} \text{ or}$$

$$EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}(K)) = \sum_{P\in\mathcal{P}(K)} \int_{S(P)} U_i(P)(s_i, s_{-i}) d\sigma_i d\sigma_{-i},$$

$S(P)$ - is a set of strategies of all players after a realization of a partition P.

Equilibrium

Definition (An equilibrium in a game $\Gamma(K)$)

$\sigma^*(K) = (\sigma_i^*(K))_{i\in N}$ is an equilibrium strategy profile for a game $\Gamma(K)$ if for every $\sigma_i(K) \neq \sigma_i^*(K)$ the following inequality holds true: $EU_i^{\Gamma(K)}(\sigma_i^*(K), \sigma_{-i}^*(K)) \geq EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}^*(K))$, $\forall i \in N$.

Theorem

The family of games $\mathcal{G} = \{\Gamma(K), K = 1, 2, \dots, N\}$ has an equilibrium in mixed strategies, $\sigma^* = (\sigma^*(K = 1), \dots, \sigma^*(K = N))$.

The theorem expands classic Nash theorem.
Non-cooperative equilibrium for coalition structures always exists, what is different from an empty-core cases in games of coalitional form from cooperative game theory.

Example: family of nested games similar to PD

1. **Players**: $N = 2$.
2. **Parameter**, maximum coalition sizes: $K = 1$, $K = 2$.
3. **Partitions**: $\mathcal{P}(K=1) \equiv P_{separ} = \{\{1\},\{2\}\} \subset \mathcal{P}(K=2) = \{P_{separ}, P_{joint} = \{\{1,2\}\}\}$.
4. **Strategies**, $i = 1, 2$:

$$S_i(K=1) = (H_{i,P_{separ}}, L_{i,P_{separ}}) \subset S_i(K=2) = \{S_i(K=1); H_{i,P_{joint}}, L_{i,P_{joint}})\}$$

5. **Rule**: unanimous formation rule for any K and any $\mathcal{P}(K)$.
6. $\mathcal{U}_i(K=1) = U_i(P_{separ}) \subset \mathcal{U}_i(K=2) = \{U_i(P_{separ}), U_i(P_{joint})\}$:

If both choose P_{separ}, then there is the standard Prisoner's Dilemma game and efficiency $\neq$ equilibrium.

Table: Payoff for the family of games with unanimous formation rules.

Different partitions have payoff-equal efficient outcomes. What means cooperation in these efficient outcomes?

	L_2, P_{separ}	H_2, P_{separ}	L_2, P_{joint}	H_2, P_{joint}
L_1, P_{separ}	(0;0) : $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$	(0;0) : $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$
H_1, P_{separ}	(3;-5): $\{\{1\},\{2\}\}$	**(-2;-2)**: $\{\{1\},\{2\}\}$	(3;-5) : $\{\{1\},\{2\}\}$	**(-2;-2)** : $\{\{1\},\{2\}\}$
L_1, P_{joint}	(0;0) : $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$	$(0;0)$: $\{1,2\}$	$(-5;3)$: $\{1,2\}$
H_1, P_{joint}	(3;-5): $\{\{1\},\{2\}\}$	**(-2;-2)** : $\{\{1\},\{2\}\}$	$(3;-5)$: $\{1,2\}$	**(-2;-2)** : $\{1,2\}$

Cont. K - max coalition size, $K = 1, 2$

Table: Two players, $K = 1$ for P_{separ}, $K = 2$ for P_{separ}, P_{joint}

	L_2, P_{separ}	H_2, P_{separ}	L_2, P_{joint}	H_2, P_{joint}
L_1, P_{separ}	(0;0) : $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$	(0;0) : $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$
H_1, P_{separ}	(3;-5): $\{\{1\},\{2\}\}$	(-2;-2): $\{\{1\},\{2\}\}$	(3;-5) : $\{\{1\},\{2\}\}$	(-2;-2) : $\{\{1\},\{2\}\}$
L_1, P_{joint}	(0;0) : $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$	$(0+\epsilon; 0+\epsilon)$: $\{1,2\}$	$(-5+\epsilon; 3+\epsilon)$: $\{1,2\}$
H_1, P_{joint}	(3;-5): $\{\{1\},\{2\}\}$	(-2;-2) : $\{\{1\},\{2\}\}$	$(3+\epsilon; -5+\epsilon)$: $\{1,2\}$	**(-2+ϵ;-2+ϵ)** : $\{1,2\}$

1. The whole strategy set of the game is partitioned into *coalition structure specific domain*s.
2. **Every coalition structure or a partition is a non-cooperative game with it's own strategy set**.
3. Every cell in the Table above contains a payoff profile and a *final* coalition structure.
4. Final partition may not coincide with an individual choice.
5. Equilibrium $\neq$ efficiency.
6. Games for $K = 1$ and $K = 2$ are nested.
7. Players prefer to be together in $P_{joint} = \{0, 1\}$, $\epsilon > 0$, are extroverts.

NEW FEATURES OF THE GAME

Game in Nash (1950,51): $\times_{i \in N} S_i \to (U_i)_{i \in N}$.
Family of games in this paper
$\times_{i \in N} S_i(K) \overset{\mathcal{R}(K)}{\to} \{S(P) \colon P \in \mathcal{P}(K)\} \to \{(\mathcal{U}_i(K))_{i \in N}\}, K \in \{1, N\}$.
Difference of the research agenda from the Nash program: studying non-cooperative formation of coalition structures with intra/inter externalities between all players, but not formation of **a** coalition.

A number of deviators in this form of the game may be different.

Deviations are described in terms of a number of deviators, no group deviation is required.

We propose a criterion of stability of a game based on a number of deviators.

We can study self-enforcing properties of an equilibrium in coalition structures.

Thank you.

Characterization of game family

1. Game family

$$\mathcal{G} = \{\Gamma(K=1), \dots, \Gamma(K), \dots \Gamma(K=N)\}$$

   such that

$$\Gamma(K=1) \subset \Gamma(K) \subset \Gamma(K=N).$$

2. Every game $\Gamma(K)$ has an equilibrium, $\sigma^*(K) = (\sigma_i^*(K))_{i \in N}$. for every $K \in \{1, \dots, K\}$.
3. The family $\mathcal{G}$ has an equilibrium $\Big(\sigma^*(1), \dots, \sigma^*(K), \dots, \sigma^*(K=N)\Big)$.
4. The family has equilibrium partitions for every $\Gamma(K)$: $\Big(\{P^*\}(1), \dots, \{P^*\}(K), \dots, \{P^*\}(K=N)\Big)$, $\{P^*\}(K) \subset \mathcal{P}(K)$.

Cooperation

In the dinner game above players C_1 and C_2 cooperated against a loss from eating individually.

Definition (perfect cooperation)

In a game $\Gamma(K)$ a set of players g, $g \subset N$, **intentionally perfectly cooperate in an equilibrium** if for every player $i \in g$ there is

1. for every i in g, $s_i \in S_i(P_i)$, P_i - chosen by i and $i \in g$
2. $g \in \forall P^*$, where P^* is a formed equilibrium partition of $\Gamma(K)$
3. $EU_i^{\Gamma(K)}(\sigma^*(K)) \geq EU_i^{\Gamma(K)}(\sigma_i^*(K), \sigma_{-i}^*(K))$, for $\forall \sigma_i(K) \neq \sigma_i^*(K)$,

Research questions for the 2-nd paper:

1. When *ex ante* actions to cooperate and *ex post* outcomes may be different.
2. Can cooperation be imperfect?

Why not to use a "threat", Nash (1953)

In the paper on cooperative behavior Nash (1953) offered to used a "threat "as a basic concept for coalition formation analysis. One way to structure is to construct threats (in a terminology of Nash, 1953). Consider a strategies profile from a subset of players. Let this profile be a threat to someone, beyond this subset. The threatening player**s** may produce externalities for each other (and negative externalities not excluding!). How credible could be such threat? From the other side, there may be some other player beyond the subset of players who may obtain a bonanza from this threat. But this beneficiant may not join the group due to expected intra-group negative externalities for members or from members of this group. Thus a concept of a threat can not serve as an elementary concept.

An example: contemporary alliances in Syria conflict.

Lunch game of identical players

Table: Office lunch game: strategies and payoff profiles for player A

num	Final coalition structure also strategies for A	Payoff profile U_A, U_B, U_C, U_D	Coalition values as in cooperative game theory
1^*	$\{A,B\},\{C\},\{D\}$: $\sigma_A^* = 1/3$	(10,10,3,3)	$20_{A,B}$, 3_C, 3_D
2^*	$\{A,C\},\{B\},\{D\}$: $\sigma_A^* = 1/3$	(10,3,10,3)	$20_{A,C}$, 3_B, 3_D
3^*	$\{A,D\},\{C\},\{B\}$: $\sigma_A^* = 1/3$	(10,3,3,10)	$20_{A,C}$, 3_C, 3_B
4	$\{A\},\{B\},\{C,D\}$	(3,3,10,10)	3_A, 3_B, $29_{C,D}$
5	$\{A\},\{D\},\{C,B\}$	(3,10,10,3)	3_A, 3_B, $29_{C,D}$
6	$\{A\},\{C\},\{B,D\}$	(3,10,3,10)	3_A, 3_B, $29_{C,D}$
3	$\{A\},\{B\},\{C\},\{D\}$	(3,3,3,3)	3_A, 3_B, 3_C, 3_D
4	all other with $K = 3,4$	(0,0,0,0)	$= 0$

1. Equilibrium in mixed strategies exists and is unique.
2. Resulting game is a stochastic game, with coalition structures as states of the game.
3. Every player has imperfect cooperation.

Non-cooperative stability criterion: idea

1. There is a family of games $\mathcal{G} = \langle \Gamma(1), \dots, \Gamma(K) \rangle$ and a game $\Gamma(K_0)$ has an equilibrium $\sigma^*(K_0), \{P^*\}(K_0)$.
2. If we change K_0 to $K_0 + 1$ upto $K = N$, what are the conditions when the equilibrium does not change?
3. Further described only the sufficient criterion, less tight cases are possible.

The criterium should be calculated directly from a definition of a game.

Non-cooperative stability criterium: definition

Definition

Partition stability criterion (a test value K) is a maximum coalition size when an equilibrium $\sigma^*(K_0)$ still holds true, i.e. for all $i \in N$ there is a number K^* such that

$$K^* = \max_{\substack{K=K_0,\dots,N \\ \Gamma(K_0)\dots,\Gamma(K=N)}} \Big\{ EU_i^{\Gamma(K_0)}\Big(\sigma_i^*(K_0), \sigma_{-i}^*(K_0)\Big) \geq EU_i^{\Gamma(K)}\Big(\sigma_i^*(K), \sigma_{-i}^*(K)\Big)\Big\},$$

$$Dom\ \sigma^*(K^*) = Dom\ \sigma^*(K_0)$$

where $\sigma^*(K_0)$ is an equilibrium in the game $\Gamma(K_0)$, $\sigma^*(K)$ is an equilibrium in the game $\Gamma(K)$, $K = K_0, \dots, N$, and Dom is a domain of equilibrium mixed strategies set.

Role of the criterion

1. Nash equilibrium is not self-enforcing (Aumann, 1991). Revisited stag and hare game is further.
2. The criterion may serve as a measure of trust to an equilibrium or as a test for self-enforcing of an equilibrium.
3. Cooperation in a game can be ruined by an increase in K?
4. Opportunistic behavior in coalitions: a coalition is supported as an equilibrium in a game $\Gamma(K_1)$, but may not in a wider game $\Gamma(K_2)$, $K_1 < K_2$, where bigger coalitions appear.
5. Does an uncertainty in K matter for an equilibrium or not?

Example 3: revisited stag and hare game, Aumann (1991)

Aumann uses the game to demonstrate absence of a self-enforcement property of Nash equilibrium. Every hunter independently chooses to hunt alone, P_{separ}, or together, P_{joint}, and a target for hunting, a hare or a stag. They can hunt together, P_{joint}, only if both agree. They can not reach the efficient outcome $(100, 100)$ if both do not choose to hunt together for a stag. This focal point can not be reached for any other combination of individual strategies. The game is a game of non-cooperative coalition formation with $K = 2$. Properties of payoffs: $(8, 8)$ a hare per each, $(4, 4)$ a hare per both, $(100, 100)$ a stag per both.

Table: Expanded stag and hare game

	P_{separ}, hare	P_{separ}, stag	P_{joint}, hare	P_{joint}, stag
P_{separ}, hare	(8;8); P_{separ}	(8;0); P_{separ}	(8;8); P_{separ}	(8;0); P_{separ}
P_{separ}, stag	(0;8); P_{separ}	(0;0); P_{separ}	(0;8); P_{separ}	(0;0); P_{separ}
P_{joint}, hare	(8;8); P_{separ}	(8;0); P_{separ}	(4;4); P_{joint}	(8;0); P_{joint}
P_{joint}, stag	(0;8); P_{separ}	(0;0); P_{separ}	(0;8); P_{joint}	$(100; 100)^*$; P^*_{joint}

Terminology for network games

Networks

1. Network structure - a collection of non-overlapping subgraphs on a set of nodes. Every network structure becomes a game, realized from actions of everybody.
2. An isolated subgraph (a graph for short) is an analogue to a coalition.
3. Network structure is a generalization for network games
4. For a player membership in a graph has two sides: intra-graph and inter-graph externalities (more further)
5. For every player in every graph in every network structure a payoff is defined in terms of a strategy profile from the whole network structure.

Network games introduced by Myerson (1977) and Aumann, Myerson (1978).

Network interpretations of the game similar to PD

Let players may make strategic relations with some special others, or these relations are the most important links for them then relations with the rest of others. For simplicity, one-direction cases are omitted.

1. **Players**: $N = \{1, 2\}$.
2. **Maximum connections of a player:** $K = 1$ and $K = 2$.
3. **Network structures**: $\mathcal{P}(K=1) = \Big\{P_{separ} = \{\{1\},\{2\}\}\Big\}, \subset$ $\mathcal{P}(K=2) = \Big\{P_{separ} = \{\{1\},\{2\}\}, P_{joint}\{1 \leftrightarrow 2\}\Big\}$.
4. **Set of strategies**. Player i chooses a network structure from $\mathcal{P}(K=2)$ and what to do in it is $S_i(P) = \{(H_{i,P_{separ}}, L_{i,P_{separ}}, H_{i,P_{joint}}, L_{i,P_{joint}})\}$, where $H_{i,P_{joint}} = H_{\rightarrow}, L_{i,P_{joint}} = L_{\rightarrow})$
5. **Rule**: if for $K = 2$ both choose any strategies from P_{joint}, then a connection is realized, otherwise realized is P_{separ}.

cont.

Payoffs for the game Notation: $P_{separ} = \{\{1\}, \{2\}\}$, $P_{joint} = \{1 \leftrightarrow 2\}$.

Table: Unanimous link formation game

	L_2, P_{separ}	H_2, P_{separ}	L_2, P_{joint}	H_2, P_{joint}
L_1, P_{separ}	$\underline{(0;0)}$ $\underline{\{\{1\},\{2\}\}}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0;0)}$ $\underline{\{\{1\},\{2\}\}}$	(-5;3) $\{\{1\},\{2\}\}$
H_1, P_{separ}	(3;-5) $\{\{1\},\{2\}\}$	(-2;-2) $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	(-2;-2) $\{\{1\},\{2\}\}$
L_1, P_{joint}	$\underline{(0;0)}$ $\underline{\{\{1\},\{2\}\}}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0+\epsilon; 0+\epsilon)}$ $\underline{\{1 \leftrightarrow 2\}}$	$(-5+\epsilon; 3+\epsilon)$ $\{1 \leftrightarrow 2\}$
H_1, P_{joint}	$(3; -5)$ $\{\{1\},\{2\}\}$	$(-2; -2)$ $\{\{1\},\{2\}\}$	$(3+\epsilon; -5+\epsilon)$ $\{1 \leftrightarrow 2\}$	$(-2+\epsilon; -2+\epsilon)^*$ $\{1 \leftrightarrow 2\}^*$

Table: One-side link formation game, $K = 2$

	L_2, P_{separ}	H_2, P_{separ}	L_2, P_{joint}	H_2, P_{joint}
L_1, P_{separ}	$\underline{(0;0)}$: $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$	$\underline{(0;0)}$: $1 \leftarrow 2$	(-5;3) : $1 \leftarrow 2$
H_1, P_{separ}	(3;-5): $\{\{1\},\{2\}\}$	(-2;-2): $\{\{1\},\{2\}\}$	(3;-5) : $1 \leftarrow 2$	(-2;-2) : $1 \leftarrow 2$
L_1, P_{joint}	$\underline{(0;0)}$: $1 \rightarrow 2$	(-5;3) : $1 \rightarrow 2$	$\underline{(0;0)}$: $1 \leftrightarrow 2$	(-5;3) : $1 \leftrightarrow 2$
H_1, P_{joint}	(3;-5): $1 \rightarrow 2$	(-2;-2) : $1 \rightarrow 2$	(3;-5) : $1 \leftrightarrow 2$	$(-2; -2)$: $1 \leftrightarrow 2$

Results of the network examples

1. Every network structure operates as a non-cooperative game within a larger game.
2. Outcome of a game: a payoff profile and a coalition/network structure.
3. Every structure for every player may have two types of externalities: inter-coalition or intra-coalition.
4. A rule to form network structures transports chosen strategy profile into one of coalition structures prescribed by a construction of a game.
5. An equilibrium and an efficient outcome may be different like in a standard non-cooperative game theory.
6. A step from $K = 1$ to $K = 2$ makes two games be nested.

Definition (network structure formation game)

A non-cooperative game for network structure formation is

$$\Gamma(K) = \left\langle N, \left\{K, \mathcal{P}(K), \mathcal{R}(K)\right\}, \left(S_i(K), \mathcal{U}_i(K)\right)_{i \in N} \right\rangle,$$

where $\left\{K, \mathcal{P}(K), \mathcal{R}(K)\right\}$ - network structure formation mechanism (a social norm, a social institute). $\left(S_i(K), \mathcal{U}_i(K)\right)_{i \in N}$ - properties of players in N, $S_i(K)$ strategies,a set of connections i may choose with an assumption that i always has connection with oneself and no more than K connections, and individual payoffs for every final network structure, such that:
$\times_{i \in N} S_i(K) \overset{\mathcal{R}(K)}{\to} \{S(P) \colon P \in \mathcal{P}(K)\} \to \{(\mathcal{U}_i(K))_{i \in N}\}$.

And the novelty of the paper is the same: network structure formation mechanism shapes a set of strategies of the game.

Definition (family of games)

Family of nested network structure games is:

$\mathbb{G} = \Gamma(1) \subset \ldots \subset \Gamma(K) \subset \ldots \subset \Gamma(N)$.