A coalition formation value for games in partition function form

Michel GRABISCH
Université de Paris I – Panthéon-Sorbonne
Centre d'Economie de la Sorbonne
106-112 Bd. de l'Hôpital, 75013 Paris, France
email michel.grabisch@univ-paris1.fr

Yukihiko FUNAKI
School of Political Science & Economics
Waseda University, Tokyo, Japan
email funaki@waseda.jp

Version of September 10, 2008

1 Introduction

The coalition formation problem is one of the big issues of economics and game theory, both in cooperative and noncooperative games. There are several attempts to analyze this problem. Many papers tried to find stable coalition structures in a cooperative game theoretic fashion (see, e.g., Ray and Vohra [13], Diamantoudi and Xue [7], and Funaki and Yamato [10], or a dynamic process using a noncooperative approach, as Bloch [4].

If we suppose that forming the grand coalition generates the largest total surplus, it is natural to assume that the grand coalition structure will eventually form after some negotiations. Then, the worth of the grand coalition has to be allocated to the individual players. The question is how to do that, taking into account the whole process of coalition formation.

In a coalition formation problem, it is important to consider situations with externalities. Typically, a coalition formation problem in an economy with externalities is related to public goods, public bads and common pool resource games. One of the best way to analyze it in game theory is to use games in partition function form (PFF games for short) introduced in Thrall and Lucas [14] (see also Funaki and Yamato [10]). A partition function assigns a worth to each pair consisting of a coalition and a coalition structure which contains that coalition. Such pairs are called embedded coalitions. Games in partition function form are considered as a useful extension of classical TU games, since they well capture the externalities in an economy.

Then the above problem of allocation of the worth of the grand coalition amounts to defining a suitable solution concept or value for PFF games. For TU games, one of the most well-known solutions is the Shapley value. This solution concept is based on the marginal contribution of players when they enter the game one by one, considering all possible orders. There are already many attempts to define a (Shapley) value for PFF games, e.g., by Myerson [12], Bolger [5], Do and Norde [8], Clippel and Serrano [6], Albizuri et al. [2], Macho-Stadler et al. [11], etc. They proposed several new kinds of null player or dummy player axioms, and carrier axioms, which are extensions of the original axioms in TU games. Then the resulting formulas are averaging of marginal contributions of players when the players enter the game one by one. However, these approaches do not reflect the process of coalition formation, but are still rooted in the spirit of TU games.

Also, it is rather surprising that none of these attempts seem to take into account the mathematical structure of embedded coalitions, or even to define it. Partitions, and a fortiori, embedded coalitions, have a structure much more complicated than the structure of the Boolan lattice of coalitions in a TU game, however, they share similar properties, the most striking one being that all the maximal chains (paths from the minimal element to the maximal element) are of the same length. This mathematical coincidence could be of low interest if they were not a central concept in cooperative game theory. But indeed they are: a maximal chain is a particular way to aggregate (or to call) players, starting from the empty coalition to the grand coalition. They are called "roll call" in voting games, and they clearly correspond to a permutation of the players (the order of their appearance). Maximal chains lead to the notion of marginal worth vectors, which are central for studying the core. Also, they are at the basis of the definition of the classical Shapley value, since it is merely an average of the contribution of a given player i into all possible maximal chains.

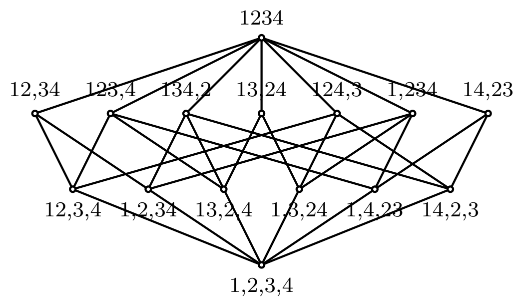
Our idea is to take advantage of the structure of embedded coalitions endowed with a suitable ordering, and to follow as closely as possible the original idea of Shapley based on maximal chains. In this way we can propose a new value of a game in partition function form, which is rooted in the process of coalition formation. We also propose an axiomatization of this value, which is very close in spirit to the original one of Shapley, but very different from the previous approaches.

An original feature of our approach is that we define three notions of value, which are in some sense embedded one into the others. The first one, called scenario-value, consider only one given maximal chain in the set of embedded coalitions, i.e., a given scenario of coalition formation, starting from a single player and arriving to the grand coalition. The second one, called process-value, consider all scenarios which follow the same fixed sequence of partitions of the society of players, starting from a society of individual players and arriving to the grand coalition. The third one, which corresponds to the usual notion of value, considers all possible processes (sequences of scenarios). According to the applicative context, the one which best makes sense can be used.

2 Partitions and embedded coalitions

Let $N := \{1, 2, \ldots, n\}$ be the set of players. We denote by $S, T, \ldots$ subsets of N (coalitions), whose cardinalities are denoted by $s, t, \ldots$. We consider the set $\Pi(N)$ (denoted also by $\Pi(n)$) of all possible partitions of N. A partition $\pi := \{S_1, \ldots, S_k\} \in \Pi(N)$ is a way of sharing the society of n players into disjoint nonempty coalitions $S_1, \ldots, S_k$.

This is usually called a *coalition structure*. Subsets $S_1, \ldots, S_k$ are called *blocks* of π. A partition in k blocks is called a *k-partition*. A natural ordering of partitions is given by the notion of "coarsening" or "refinement". Taking π, π' partitions in $\Pi(N)$, we say that π is a *refinement* of π' (or π' is a *coarsening* of π), denoted by $\pi \leq \pi'$, if any block of π is contained in a block of π' (or every block of π' fully decomposes into blocks of π). Then $(\Pi(N), \leq)$ is a lattice, called the *partition lattice*. With this ordering, the bottom element of the lattice is the finest partition $\pi^\perp := \{\{1\}, \ldots, \{n\}\}$, while the top element is the coarsest partition $\pi^\top := \{N\}$. An example with $n = 4$ is given below.



Details about the partition lattice can be found in the appendix.

An *embedded coalition* is a pair (S, π), where $S \in 2^N \setminus \{\emptyset\}$, and $\pi \in \Pi(N)$ is such that $S \in \pi$. We denote by $\mathfrak{C}(N)$ (or by $\mathfrak{C}(n)$) the set of embedded coalitions on N. For the sake of concision, we often denote by $S\pi$ the embedded coalition (S, π), and omit braces and commas for subsets (example with $n = 3$: $12\{12, 3\}$ instead of $(\{1, 2\}, \{\{1, 2\}, \{3\}\})$). Remark that $\mathfrak{C}(N)$ is a proper subset of $2^N \times \Pi(N)$. We propose the following order relation on embedded coalitions, which is merely the product order on $2^N \times \Pi(N)$:

$$(S, \pi) \sqsubseteq (S', \pi') \Leftrightarrow S \subseteq S' \text{ and } \pi \leq \pi'.$$

Evidently, the top element of this ordered set is $(N, \pi^\top)$ (denoted more simply by $N\{N\}$ according to our conventions). However, due to the fact that the empty set is not allowed in (S, π), there is no bottom element in the ordered structure $(\mathfrak{C}(N), \sqsubseteq)$. Indeed, all elements of the form $(\{i\}, \pi^\perp)$ are minimal elements (i.e., there is no smaller element than them). For mathematical convenience, we introduce an artificial bottom element $\perp$ to $\mathfrak{C}(N)$ (it could be considered as $(\emptyset, \pi^\perp)$), and denote $\mathfrak{C}(N)_\perp := \mathfrak{C}(N) \cup \{\perp\}$. We give as illustration the partially ordered set $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ with $n = 3$ (Fig. 1).

Definition 1 *A* game in partition function form *(PFF-game for short) on N is a mapping $v : \mathfrak{C}(N)_\perp \to \mathbb{R}$, such that $v(\perp) = 0$. The set of all PFF-games on N is denoted by $\mathcal{PG}(N)$.*

To be meaningful, we need to assume that forming the grand coalition generates the largest total surplus, i.e., $v(N\{N\}) \geq \sum_{S \in \pi} v(S, \pi)$, for all $\pi \in \Pi(N)$. Hence, we consider economic environments where doing so is the best way to organize the society.

We recall that in a partially ordered set $(P, \leq)$ with a bottom element $\perp$ and a top element $\top$, a *chain from $\perp$ to $\top$* is a totally ordered sequence of elements of P including $\perp, \top$. The chain is *maximal* if no other chain can contain it (equivalently, if between two consecutive elements x_i, x_{i+1} of the sequence, there is no element $x \in P$ such that $x_i < x < x_{i+1}$). If no ambiguity occurs, we say *maximal chain* instead of *maximal chain*

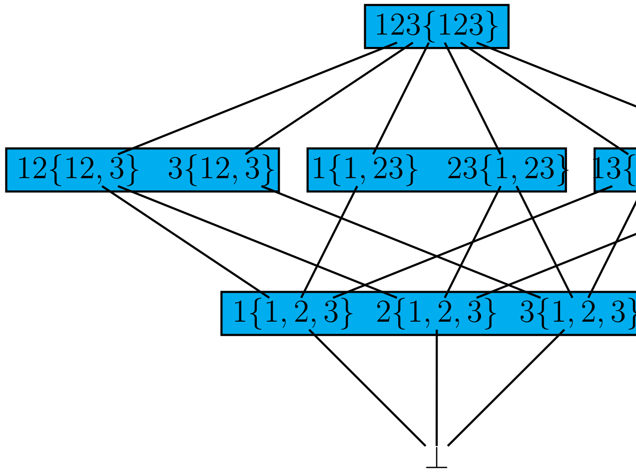


Figure 1: Hasse diagram of $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ with $n = 3$. Elements with the same partition are framed in blue.

from $\perp$ *to* $\top$. The set of maximal chains in P is denoted by $\mathcal{C}(P)$. The *length* of a maximal chain is the number of elements of the sequence minus 1. If all maximal chains have same length, this length is the *height* of the partially ordered set. In Fig. 1, the sequence $\perp$, $1\{1,2,3\}$, $1\{1,23\}$, $123\{123\}$ is a maximal chain, and there are 9 maximal chains in total, all of length 3, hence the height of $(\mathfrak{C}(123)_\perp, \sqsubseteq)$ is 3. Concerning the partition lattice $\Pi(N)$, it is easy to see that its height is $n-1$.

It is proved in the appendix that $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ is a lattice, whose maximal chains have all the same length n, hence the height of this lattice is n. The combinatorial complexity of $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ is far beyond the complexity of the Boolean lattice of coalitions in a TU game (for n players, there are 2^n coalitions and $n!$ maximal chains). The following facts are proven in the appendix, and well illustrate the difficulty of dealing with embedded coalitions:

- The total number of elements is $\sum_{k=1}^{n} kS_{n,k} + 1$, where $S_{n,k}$ is the Stirling number of second kind.

n	1	2	3	4	5	6	7	8
$\lvert\mathfrak{C}(n)_\perp\rvert$	2	4	11	38	152	675	3264	17008

- The number of maximal chains from $\perp$ to $(N, \{N\})$ is $\frac{(n!)^2}{2^{n-1}}$, which is also the number of maximal longest chains in $\mathfrak{C}(N)$.

n	1	2	3	4	5	6	7	8
$\lvert\mathcal{C}(\mathfrak{C}(n)_\perp)\rvert$	1	2	9	72	900	16 200	396 900	12 700 800

For commodity, we put $c := \lvert\mathcal{C}(\mathfrak{C}(n)_\perp)\rvert$. As shown in the appendix, $\lvert\mathcal{C}(\Pi(n))\rvert = \frac{c}{n}$.

3 Processes and scenarios

Definition 2 *(i) A* coalition formation process $\mathcal{P}$ *is any maximal chain in* $\Pi(n)$*, i.e., a sequence of partitions starting from* $\pi^{\perp}$ *(the society of individual players) and ending at* $\{N\}$ *(the grand coalition). The set of processes is* $\mathcal{C}(\Pi(n)) =: \mathfrak{P}$.

(ii) A scenario $\mathcal{S}$ in a process $\mathcal{P}$ *is any maximal chain in* $\mathfrak{C}(n)_{\perp}$ *so that the sequence of partitions corresponds to* $\mathcal{P}$ *(notation:* $\mathcal{S} \leftarrow \mathcal{P}$*). The set of all scenarios, considering all processes, is therefore* $\mathcal{C}(\mathfrak{C}(n)_{\perp})$*, denoted by* $\mathfrak{S}$ *for simplicity.*

We know from Section 2 that $|\mathfrak{S}| = c$ and $|\mathcal{P}| = \frac{c}{n}$. For a given process $\mathcal{P}$, there are n scenarios $\mathcal{S}_i$, $i \in N$, scenario $\mathcal{S}_i$ tracking the history of player i in the coalition formation process. Note that a given scenario belongs to a unique process.

In a scenario $\mathcal{S}$, some elements play a special role. We consider those elements $S\pi$ such that in the sequence of elements of $\mathcal{S}$ from bottom to top, $S\pi$ is the last element with S (*terminal elements*). Specifically, let us denote $\mathcal{S}$ by

$$\mathcal{S} = \{\emptyset\pi^{\perp}, S_1\pi_{1,1}, \ldots, S_1\pi_{1,m_1}, S_2\pi_{2,1}, \ldots, S_2\pi_{2,m_2}, \ldots, S_k\pi_{k,1}, \ldots, S_k\pi_{k,m_k}, N\{N\}\}, \quad (1)$$

with $S_1 \neq \cdots \neq S_k \neq N$. Then the terminal elements are $S_i\pi_{i,m_i}$, $i = 1, k$. We denote by $\mathcal{F}(\mathcal{S})$ this family of elements. The same applies to a process $\mathcal{P}$ since any element has a unique successor, and we denote the corresponding family of terminal elements by $\mathcal{F}(\mathcal{P})$ (note that it forms a binary tree: see Example 2). Note that the definition does not make sense for the whole set $\mathfrak{C}(n)$, since an element has several successors.

Example 1: We consider 4 players and the following process $\mathcal{P}$:

$$\{1, 2, 3, 4\} \rightarrow \{13, 2, 4\} \rightarrow \{13, 24\} \rightarrow \{1234\}.$$

The four different scenarios in $\mathcal{P}$ are (terminal elements are in bold):

$$\mathcal{S}_1 = \mathbf{1\{1, 2, 3, 4\}} \rightarrow 13\{13, 2, 4\} \rightarrow \mathbf{13\{13, 24\}} \rightarrow \mathbf{N\{N\}}$$
$$\mathcal{S}_2 = 2\{1, 2, 3, 4\} \rightarrow \mathbf{2\{13, 2, 4\}} \rightarrow \mathbf{24\{13, 24\}} \rightarrow \mathbf{N\{N\}}$$
$$\mathcal{S}_3 = \mathbf{3\{1, 2, 3, 4\}} \rightarrow 13\{13, 2, 4\} \rightarrow \mathbf{13\{13, 24\}} \rightarrow \mathbf{N\{N\}}$$
$$\mathcal{S}_4 = 4\{1, 2, 3, 4\} \rightarrow \mathbf{4\{13, 2, 4\}} \rightarrow \mathbf{24\{13, 24\}} \rightarrow \mathbf{N\{N\}}$$

4 Definition of the coalition formation value

In this section we motivate our definition of the value for PFF games. We call it indifferently the *coalition formation value* (due to its interpretation) or the *Shapley value* (due to its mathematical definition).

Basically, a value represents the average contribution of players in the game. The contribution of player i is defined as the sum of his contribution in each scenario of each process. In a given scenario, the contribution of i is the "added value" of i during the scenario. However, this notion is not so obvious to define because, unlike TU games, it is not true that in each step of the scenario a single player enters the game. Let us consider scenario $\mathcal{S}_1$ in the previous example:

$$\mathcal{S}_1 = 1\{1, 2, 3, 4\} \rightarrow 13\{13, 2, 4\} \rightarrow 13\{13, 24\} \rightarrow N\{N\}.$$

Three situations can arise:

(i) In one step, a single player enters: this happens for step 1 and 2. This is exactly like TU games, and the contribution goes for the entering player.

(ii) In one step, several players enter together: this happens in the last step where players 2 and 4 enter together. In this case, the principle of insufficient reason tells us to divide the contribution equally among the entering players.

(iii) In one step, no new player enter: this happens in step 3 of $\mathcal{S}_1$. In this case, we wait till a new change occurs (i.e., a new player enters), hence these steps are taken into account.

Note that the third case implies that only steps which correspond to terminal elements are taken into account. Applying this methodology to the above scenario $\mathcal{S}_1$, we obtain:

(i) contribution of player 1: $v(1\{1,2,3,4\}) - 0$

(ii) contribution of player 2: $\frac{1}{2}(v(1234\{1234\}) - v(13\{13,24\}))$

(iii) contribution of player 3: $v(13\{13,24\}) - v(1\{1,2,3,4\})$

(iv) contribution of player 4: $\frac{1}{2}(v(1234\{1234\}) - v(13\{13,24\}))$

Based on the above considerations, we can define the contribution of a player in a scenario.

Definition 3 *The* contribution of player i in a given scenario $\mathcal{S}$ *is given by*

$$\Delta_i^{\mathcal{S}}(v) := \frac{1}{|S' \setminus S|}(v(S'\pi') - v(S\pi)),$$

where $S\pi$ is the last element where i is not present, and $S'\pi'$ is the last element with S', where S' is the subset next to S (note that $S\pi$ and $S'\pi'$ belong to $\mathcal{F}(\mathcal{S})$; see Figure 4).

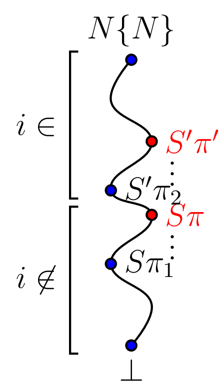


Figure 2: Notations used for snenario $\mathcal{S}$

Definition 4 *(i) A* scenario-value *is a mapping* $\phi : \mathcal{PG} \to \mathbb{R}^{n\times|\mathfrak{S}|}$. *Components of* $\phi(v)$ *are denoted by* $\phi_i^{\mathcal{S}}(v)$ *for scenario* $\mathcal{S}$ *and player* i.

The Shapley scenario-value *is defined by*

$$\phi_i^{\mathcal{S}}(v) := \Delta_i^{\mathcal{S}}(v), \quad i \in N, \mathcal{S} \in \mathfrak{S}.$$

(ii) A process-value *is a mapping* $\phi : \mathcal{PG} \to \mathbb{R}^{n\times|\mathfrak{P}|}$. *Components of* $\phi(v)$ *are denoted by* $\phi_i^{\mathcal{P}}(v)$ *for process* $\mathcal{P}$ *and player* i. *Any scenario-value* ϕ *induces a process-value (denoted with some abuse by the same letter* ϕ*) by:*

$$\phi_i^{\mathcal{P}}(v) := \frac{1}{n}\sum_{\mathcal{S}\leftarrow\mathcal{P}} \phi_i^{\mathcal{S}}(v), \quad i \in N, \mathcal{P} \in \mathfrak{P}.$$

The Shapley process-value *is induced by the Shapley scenario-value.*

(iii) A value *is a mapping* $\phi : \mathcal{PG} \to \mathbb{R}^n$. *Components of* $\phi(v)$ *are denoted by* $\phi_i(v)$ *for player* i. *Any scenario-value or process-value* ϕ *induces a value by:*

$$\phi_i(v) := \frac{n}{c}\sum_{\mathcal{P}\in\mathfrak{P}} \phi_i^{\mathcal{P}}(v) = \frac{1}{c}\sum_{\mathcal{S}\in\mathfrak{S}} \phi_i^{\mathcal{S}}(v).$$

The Shapley value of v *is induced by the Shapley process- (or scenario-) value.*

From the definition, a scenario-value (resp. a process-value) uniquely induces a process-value and a value (resp. a value), but the converse is not true in general: values are not necessarily expressed as sums of scenario-values (just consider $\phi_i(v) := \phi_i'^{\mathcal{S}_1}(v)\phi_i'^{\mathcal{S}_2}(v)$), or if it the case, several scenario-values may induce the same value. We say that a value ϕ *admits a representation* (or *is representable*) by a scenario value ϕ' if ϕ' induces ϕ. Of course, the same remark applies to process-values: one can have values representable by a process-value, and process-values representable by a scenario-value.

Definition 5 *(i) A value is* efficient (E) *if* $\sum_{i\in N}\phi_i(v) = v(N\{N\})$.

(ii) A process-value is process-efficient (PE) *if* $\sum_{i\in N}\phi_i^{\mathcal{P}}(v) = v(N\{N\})$ *for all* $\mathcal{P} \in \mathfrak{P}$.

(iii) A scenario-value is scenario-efficient (SE) *if* $\sum_{i\in N}\phi_i^{\mathcal{S}}(v) = v(N\{N\})$ *for all* $\mathcal{S} \in \mathfrak{S}$.

Be careful that it makes no sense to say that a value representable by a scenario-value is scenario-efficient, since in general the representation is not unique, and all representation need not be scenario-efficient. However, if a value (resp. process-value) is representable by a scenario-value which is scenario-efficient, then the value is efficient (resp. the process-value is process-efficient). With some abuse, we may say that a scenario-value is efficient (resp. process-efficient) if the induced value (resp. process-value) is.

Proposition 1 *The Shapley scenario-value is scenario-efficient. Hence, it is process-efficient and efficient as well.*

Proof: Consider a scenario $\mathcal{S}$, written as

$$\mathcal{S} = \{\emptyset\pi^{\perp}, S_1\pi_{1,1}, \ldots, S_1\pi_{1,m_1}, S_2\pi_{2,1}, \ldots, S_2\pi_{2,m_2}, \ldots, S_k\pi_{k,1}, \ldots, S_k\pi_{k,m_k}, N\{N\}\},$$

with $S_1 \neq \cdots \neq S_k \neq N$.

For any S_j and any $i \in S_j \setminus S_{j-1}$, we have

$$\phi_i^{\mathcal{S}}(v) = \frac{1}{s_j - s_{j-1}}(v(S_j\pi_{j,m_j}) - v(S_{j-1}\pi_{j-1,m_{j-1}})).$$

Hence

$$\sum_{i \in S_j \setminus S_{j-1}} \phi_i^{\mathcal{S}}(v) = v(S_j\pi_{j,m_j}) - v(S_{j-1}\pi_{j-1,m_{j-1}}).$$

So in total,

$$\sum_{i \in N} \phi_i^{\mathcal{S}}(v) = \sum_{j=1}^{k+1}(v(S_j\pi_{j,m_j}) - v(S_{j-1}\pi_{j-1,m_{j-1}})) = v(N\{N\}) - v(\emptyset\pi^{\perp}) = v(N\{N\}),$$

with $S_0 := \emptyset$ and $S_{k+1} := N$. ■

5 Axiomatizations

We give in this section two axiomatizations of the Shapley scenario-value. Since it induces the Shapley process-value and the Shapley value, we have also indirectly in some sense axiomatizations for them. However, since infinitely many scenario-values can induce the Shapley value, our axiomatizations are therefore stronger than necessary to axiomatize the Shapley value. Note also that since the ranges are different for the scenario-value, the process-value and the value, axioms are dedicated to one type of value and cannot always be easily transposed for another type. We believe however that axioms written for the scenario-value are the most natural and close to the intuition of coalition formation.

A scenario-value satisfies **linearity (L)** if it is linear on the set of PFF-games.
We define similarly linearity for process-values and values.

We do not give different names for these three axioms, since their domains are different and should be clear from the context.

Proposition 2 *If ϕ is a linear scenario-value, then for all $v \in \mathcal{PG}(N)$*

$$\phi_i^{\mathcal{S}}(v) = \sum_{S\pi \in \mathfrak{C}(n)} \gamma^i_{\mathcal{S},S\pi} v(S\pi), \quad \forall i \in N, \forall \mathcal{S} \in \mathfrak{S}$$

for some real constants $\gamma^i_{\mathcal{S},S\pi}$ (and similarly for a linear process-value, with constants $\gamma^i_{\mathcal{P},S\pi}$, and for a linear value, with constants $\gamma^i_{S\pi}$).

Proof: As usual, consider the identity PFF-games $e_{S\pi}(S'\pi') := 1$ iff $S\pi = S'\pi'$ and 0 else, for all $S\pi \in \mathfrak{C}(n)$. Then linearity implies, for e.g. a scenario-value:

$$\phi(v) = \phi\Big(\sum_{S\pi\in\mathfrak{C}(n)} v(S\pi)e_{S\pi}\Big) = \sum_{S\pi\in\mathfrak{C}(n)} v(S\pi)\phi(e_{S\pi}),$$

hence the result, letting $\phi_i^{\mathcal{S}}(e_{S\pi})) =: \gamma^i_{\mathcal{S},S\pi}$ for $i \in N$, $\mathcal{S} \in \mathfrak{S}$. ■

Remark 1:

(i) Each linear value ϕ is representable by at least one linear scenario-value ϕ'. Indeed, by linearity we have:

$$\phi_i(v) = \sum_{S\pi\in\mathfrak{C}(n)} \gamma^i_{S\pi} v(S\pi) = \sum_{S\pi\in\mathfrak{C}(n)} v(S\pi) \sum_{\mathcal{S}\ni S\pi} \gamma^i_{\mathcal{S},S\pi} = \sum_{\mathcal{S}\in\mathfrak{S}} \sum_{S\pi\in\mathcal{S}} \gamma^i_{\mathcal{S},S\pi} v(S\pi),$$

taking any set of coefficients $\gamma^i_{\mathcal{S},S\pi}$ solution of the system $\sum_{\mathcal{S}\ni S\pi}\gamma^i_{\mathcal{S},S\pi} = \gamma^i_{S\pi}$, $S\pi \in \mathfrak{C}(n)$. Evidently, this system has in general infinitely many solutions. Then it suffices to take the scenario-value defined by $\phi_i'^{\mathcal{S}}(v) := \sum_{S\pi\in\mathcal{S}} \gamma^i_{\mathcal{S},S\pi} v(S\pi)$, $i \in N$, $\mathcal{S} \in \mathfrak{S}$, which is linear. The same remark applies to process-values as well.

(ii) If a scenario-value is linear, then clearly its induced value is linear too. The converse is not true in general. Take simply $n = 2$ and the two scenarios $\mathcal{S}_1, \mathcal{S}_2$. Define for $i = 1, 2$:

$$\phi_i^{\mathcal{S}_1}(v) := v(1\{1,2\}) + v^2(12\{12\})$$
$$\phi_i^{\mathcal{S}_2}(v) := v(2\{1,2\}) - v^2(12\{12\}).$$

Then clearly the scenario-value is not linear but the induced value is.

Definition 6 *Let us consider $i \in N$, a scenario $\mathcal{S}$, and denote by $S\pi$ the last element in $\mathcal{S}$ not containing i, and $S'\pi'$ its successor in $\mathcal{F}(\mathcal{S})$. Player i is* null in scenario $\mathcal{S}$ for v *if $v(S\pi) = v(S'\pi')$. Player i is* null for v *if i is null for every scenario $\mathcal{S}$.*

Null axiom (N): If i is null in scenario $\mathcal{S}$ for v, then $\phi_i^{\mathcal{S}}(v) = 0$.

Similarly as for linearity, if i is null for every scenario $\mathcal{S}$, then $\phi_i(v) = 0$ (null axiom for the induced value), but the converse does not hold.

Proposition 3 *Notation: for given $i \in N$ and scenario $\mathcal{S} := \{\emptyset\pi^\perp, S_1\pi_1, \dots, S_n\pi_n = N\{N\}\}$, $S\pi$ and $S'\pi'$ above are denoted respectively by $S_k\pi_k$ and $S_l\pi_l$. Under (L) and (N), for every scenario $\mathcal{S}$, every player i, the scenario-value reads*

$$\phi_i^{\mathcal{S}}(v) = \gamma^i_{\mathcal{S},S_l\pi_l}\big(v(S_l\pi_l) - v(S_k\pi_k)\big)$$

with real constants $\gamma^i_{\mathcal{S},S_l\pi_l}$, $i \in N$, $\mathcal{S} \in \mathfrak{S}$.

Proof: Take any scenario $\mathcal{S}$ with the above notation. Define $v(S_j\pi_j) = 1$ for $j = k, l$, and 0 otherwise. Then i is null for v in $\mathcal{S}$. Then by (N) we get $\gamma^i_{\mathcal{S},S_k\pi_k} + \gamma^i_{\mathcal{S},S_l\pi_l} = 0$. Now take $v' = v$, except on a single element $S\pi \in \mathfrak{C}(n)$ different from $S_k\pi_k, S_l\pi_l$. Since i is still null for v' in $\mathcal{S}$, we get by (N) that $\gamma^i_{\mathcal{S},S\pi} = 0$. ■

Symmetry axiom for the scenario-value (SS): For any $i \in N$, any $\mathcal{S} \in \mathfrak{S}$, and any permutation σ on N, it holds

$$\phi^{\mathcal{S}}_i(v) = \phi^{\sigma(\mathcal{S})}_{\sigma(i)}(v \circ \sigma^{-1})$$

with $\sigma(\mathcal{S}), \sigma(S,\pi)$ defined in the obvious way.

Proposition 4 *Under (L) and (SS), the scenario-value takes the form*

$$\phi^{\mathcal{S}}_i(v) = \sum_{\substack{S\pi\in\mathfrak{C}(n)\\ S\ni i}} \gamma_{\tau(\mathcal{S}),\rho(S)} v(S\pi) + \sum_{\substack{S\pi\in\mathfrak{C}(n)\\ S\not\ni i}} \gamma'_{\tau(\mathcal{S}),\rho(S)} v(S\pi)$$

for any scenario $\mathcal{S}$ and player i, where $\tau(\mathcal{S})$ is the nondecreasing sequence of cardinalities of the subsets in $\mathcal{S}$ (length is n), and $\rho(S)$ the rank of S in $\mathcal{S}$ (empty set has rank 0).

Proof: From (L), any scenario-value has the following form, for any $\imath \in N$, $\mathcal{S} \in \mathfrak{S}$

$$\phi^{\mathcal{S}}_i(v) = \sum_{S\pi\in\mathfrak{C}(n)} \gamma^i_{\mathcal{S},S\pi} v(S\pi).$$

Hence, for any permutation σ on N:

$$\phi^{\sigma(\mathcal{S})}_{\sigma(i)}(v \circ \sigma^{-1}) = \sum_{(S,\pi)\in\mathfrak{C}(n)} \gamma^{\sigma(i)}_{\sigma(\mathcal{S}),S\pi} v(\sigma^{-1}(S\pi)).$$

By (SS), we deduce that $\gamma^i_{\mathcal{S},S\pi} = \gamma^{\sigma(i)}_{\sigma(\mathcal{S}),\sigma(S\pi)}$, for all $i \in N$, all $\mathcal{S} \in \mathfrak{S}$, all $S\pi \in \mathfrak{C}(n)$, and all permutation σ. Suppose $S \not\ni i$. Then for any permutation σ leaving i invariant (i.e., $\sigma(i) = i$), we deduce that $\gamma^i_{\mathcal{S},S\pi} = \gamma^i_{\sigma(\mathcal{S}),\sigma(S\pi)}$, so that these coefficients depend only on cardinalities of subsets and number of blocks of partition. Hence we can define $\gamma^i_{\tau(\mathcal{S}),s,\ell(\pi)} := \gamma^i_{\mathcal{S},S\pi}$, for all $i \in N$, all $S\pi$ such that $S \not\ni i$, and all $\mathcal{S} \in \mathfrak{S}$, where $\tau(\mathcal{S})$ is the nondecreasing sequence of cardinalities of subsets in $\mathcal{S}$, and $\ell(\pi)$ the number of blocks of π. Remark that $\gamma^i_{\tau(\mathcal{S}),s,\ell(\pi)}$ can be more compactly denoted by $\gamma^i_{\tau(\mathcal{S}),\rho(S)}$, where $\rho(S) = n - \ell(\pi) + 1$ is the rank of S in $\mathcal{S}$, and s is superfluous because s is the $\rho(S)$-th number in the sequence $\tau(\mathcal{S})$.

Take now any permutation σ such that $\sigma(i) \neq (i)$. We obtain that

$$\gamma^i_{\tau(\mathcal{S}),\rho(S)} = \gamma^i_{\mathcal{S},S\pi} = \gamma^{\sigma(i)}_{\sigma(\mathcal{S}),\sigma(S\pi)} = \gamma^{\sigma(i)}_{\tau(\mathcal{S}),\rho(S)},$$

hence $\gamma^i_{\tau(\mathcal{S}),\rho(S)}$ does not depend on i.

Consider now $S \ni i$. By the same reasoning we get $\gamma^i_{\mathcal{S},S\pi} =: \gamma'_{\tau(\mathcal{S}),\rho(S)}$ for all i, all $\mathcal{S}$, and all $S\pi$ such that $S \ni i$. Observe however that we do not have $\gamma_{\tau(\mathcal{S}),\rho(S)} = \gamma'_{\tau(\mathcal{S}),\rho(S)}$, because no permutation can produce an equality of the form $\gamma^i_{\mathcal{S},S\pi} = \gamma^{\sigma(i)}_{\sigma(\mathcal{S}),\sigma(S\pi)}$ with both $S \ni i$ and $\sigma(S) \not\ni \sigma(i)$. ■

In summary, we have under (L), (N), and (SS):

$$\phi_i^{\mathcal{S}}(v) = \gamma_{\tau(\mathcal{S}),l}(v(S_l\pi_l) - v(S_k\pi_k)), \tag{2}$$

for every scenario $\mathcal{S}$ and every player $i \in N$.

Proposition 5 *The Shapley scenario-value is the unique scenario-value satisfying (L), (N), (SS), and (SE).*

Proof: The fact that the Shapley scenario-value satisfies (L), (N), (SS) and (SE) is easy to check. Indeed, it has the form (2) with $\gamma_{\tau(\mathcal{S}),l} = \frac{1}{|S_l \setminus S_k|}$, and we have proved in Proposition 1 that it satisfies (SE).

Conversely, let $\mathcal{S} = \{\emptyset\pi^{\perp}, S_1\pi_{1,1}, \ldots, S_1\pi_{1,m_1}, S_2\pi_{2,1}, \ldots, S_2\pi_{2,m_2}, \ldots, S_{n-z-1}\pi_{n-z-1,1}, \ldots, S_{n-z-1}\pi_{n-z-1,m_{n-z-1}}, N\{N\}\}$ be fixed, z being the number of zeroes in $\tau(\mathcal{S})$.

From (L), (N), (SS), and (2) we get:

$$\begin{aligned}
\sum_{i\in N} \phi_i^{\mathcal{S}}(v) = \;& \gamma_{\tau(\mathcal{S}),m_1} v(S_1\pi_{1,m_1}) + |S_2 \setminus S_1|\gamma_{\tau(\mathcal{S}),m_1+m_2}\big(v(S_2\pi_{2,m_2}) - v(S_1\pi_{1,m_1})\big) + \cdots \\
& \cdots + |S_j \setminus S_{j-1}|\gamma_{\tau(\mathcal{S}),m_1+\cdots+m_j}\big(v(S_j\pi_{j,m_j}) - v(S_{j-1}\pi_{j-1,m_{j-1}})\big) + \cdots \\
& \cdots + |N \setminus S_{n-z-1}|\gamma_{\tau(\mathcal{S}),n}\big(v(N\{N\}) - v(S_{n-z-1}\pi_{n-z-1,m_{n-z-1}})\big) \\
= \;& \big(\gamma_{\tau(\mathcal{S}),m_1} - |S_2 \setminus S_1|\gamma_{\tau(\mathcal{S}),m_1+m_2}\big) v(S_1\pi_{1,m_1}) + \cdots \\
& \cdots + \big(|S_j \setminus S_{j-1}|\gamma_{\tau(\mathcal{S}),m_1+\cdots+m_j} - |S_{j+1} \setminus S_j|\gamma_{\tau(\mathcal{S}),m_1+\cdots+m_{j+1}}\big) v(S_j\pi_{j,m_j}) + \cdots \\
& \cdots + |N \setminus S_{n-z-1}|\gamma_{\tau(\mathcal{S}),n} v(N\{N\}).
\end{aligned}$$

From the (SE) axiom, we deduce the following linear system of $n-z$ equations and $n-z$ unknowns:

$$\begin{aligned}
\gamma_{\tau(\mathcal{S}),m_1} - |S_2 \setminus S_1|\gamma_{\tau(\mathcal{S}),m_1+m_2} &= 0 \\
\vdots \quad &= 0 \\
|S_j \setminus S_{j-1}|\gamma_{\tau(\mathcal{S}),m_1+\cdots+m_j} - |S_{j+1} \setminus S_j|\gamma_{\tau(\mathcal{S}),m_1+\cdots+m_{j+1}} &= 0 \\
\vdots \quad &= 0 \\
|N \setminus S_{n-z-1}|\gamma_{\tau(\mathcal{S}),n} &= 1.
\end{aligned}$$

Evidently the system is nonsingular, since from the last equation $\gamma_{\tau(\mathcal{S}),n}$ is obtained, then substituting it into the last but one, we get $\gamma_{\tau(\mathcal{S}),m_1+\cdots+m_{n-z-1}}$ and so on. Knowing that the coefficients of the Shapley scenario-value are solutions of the system, it is the unique solution. ■

The scenario-efficiency being a very strong axiom, let us try to use efficiency, which is the weaker form of efficiency among our definitions. We will see that a fifth axiom is then necessary to characterize the scenario-value.

For illustration purpose, we give an example of process.

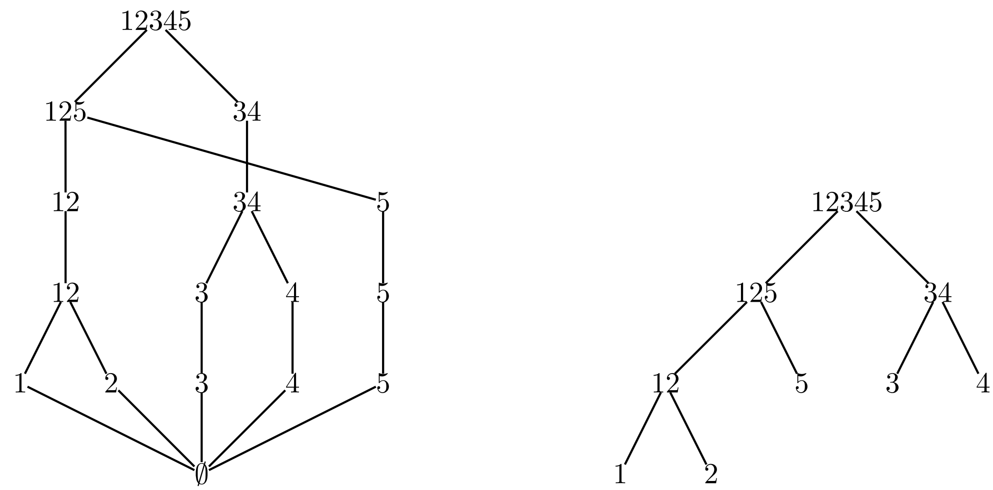


Figure 3: Example of process $\mathcal{P}$: the associated scenarios (only coalitions are given) (left) and the corresponding tree $\mathcal{F}(\mathcal{P})$ (right)

Example 2: Let $n = 5$, and consider the following process:

$$\{1,2,3,4,5\}, \quad \{12,3,4,5\}, \quad \{12,34,5\}, \quad \{125,34\}, \quad \{12345\}.$$

The five scenarios as well as the binary tree $\mathcal{F}(\mathcal{P})$ are depicted on Figure 2.

We now introduce an axiom related to the temporal nature of the coalition formation process. In system theory, where the state of a system evolves with time, fundamental concepts are the fact that the state at time t depends on the state at time $t-1$, but regardless of times $t-2$, etc., and that the evolution of states does not depend on the time origin. In Markov processes, these are known as the Markov property, and ergodicity. We will follow here the same philosophy. Translated into the language of coalition formation, the Markov property means that for a scenario $\mathcal{S}$ and a given $S\pi \in \mathcal{F}(\mathcal{S})$, only its predecessor $S^-\pi^-$ in $\mathcal{F}(\mathcal{S})$ matters, not the whole senario. Concerning ergodicity, time is here simply the step number in the scenario, i.e., exactly the rank ρ of coalitions (or the number of blocks of π). Hence ergodicity means that the rank of the current coalition (or the number of blocks of the current partition) should not matter.

Markovian and Ergodic axiom (ME): Let us consider two coalitions S, S^-, such that $S^- \subset S$, and a game v_{S,S^-} whose worth for S and S^- does not depend on externalities, with $v_{S,S^-}(S\pi) = 1$ and $v_{S,S^-}(S^-\pi^-) = 0$, for all possible π, π^-.

A scenario-value ϕ satisfies (ME) if, for all $S, S^- \in 2^N$, $S^- \subset S$, for all $i \in S \setminus S^-$, we have $\phi_i^{\mathcal{S}}(v_{S,S^-})$ constant for all scenarios $\mathcal{S}$ such that $S\pi, S^-\pi^- \in \mathcal{F}(\mathcal{S})$ for some partitions π, π^- such that $S^-\pi^-$ is the predecessor of $S\pi$ in $\mathcal{F}(\mathcal{S})$.

Note that the Markovian property becomes apparent when one restricts to fixed partitions π, π^-. However, this is not enough to derive the next proposition.

Proposition 6 *Under (L), (N), (SS) and (ME), we have, for any scenario $\mathcal{S}$, any $S\pi \in \mathcal{F}(\mathcal{S})$ and its predecessor $S^-\pi^-$ in $\mathcal{F}(\mathcal{S})$*

$$\phi_i^{\mathcal{S}}(v) = \gamma_{s,s^-}(v(S\pi) - v(S^-\pi^-)) \tag{3}$$

for all $i \in S \setminus S^-$. *If* S *is a singleton,* γ_{s,s^-} *is denoted by* $\gamma_{1,0}$.

Proof: Under (L), (N) and (S), we already know that $\phi_i^{\mathcal{S}}(v)$ has the form (2), with coefficient $\gamma_{\tau(\mathcal{S}),\rho(S)}$. Let S, S^- be fixed, $S^- \subset S$. Then we get for any $i \in S \setminus S^-$, any scenario $\mathcal{S}$ defined as above, $\phi_i^{\mathcal{S}}(v_{S,S^-}) = \gamma_{\tau(\mathcal{S}),s,\rho(S)}$. The result then easily follows by (ME). ■

Theorem 1 *The Shapley scenario-value is the unique scenario-value satisfying (L), (N), (SS), (E) and (ME).*

Proof: Again, the Shapley-scenario value clearly satisfies these axioms.

Conversely, let us assume that the scenario-value satisfies the five axioms, and let us compute $\sum_{i\in N}\sum_{\mathcal{S}\in\mathfrak{S}}\phi_i^{\mathcal{S}}(v)$. From $\sum_{i\in N}\sum_{\mathcal{S}\in\mathfrak{S}}\phi_i^{\mathcal{S}}(v) = v(N\{N\})$, we get a system of linear equations, one per element $S\pi$. We know that there exists at least one solution to this system, since our Shapley scenario-value satisfies the five axioms. Our task will be to prove that this is the only solution. To this aim, we will prove that there are at least as many equations as variables, and that there exists a subsystem which can be made triangular.

First, we determine the form of the equation for element $S\pi := S\{S, S_2, \dots, S_k\}$, assuming $S\pi$ is any element different from $N\{N\}$ and S is not a singleton. There is a negative contribution for $S\pi$ with coefficient $\gamma_{s^+,s}$ for all $S^+\pi^+$ such that $S^+\pi^+ \sqsupset S\pi$, $|S^+| = s^+$, $\pi^+ := \{S^+, S_2^+, \dots, S_{k^+}^+\}$, and $S^+ = S \cup S_j$, for some $j = 2, \dots, k$, for all $i \in S^+ \setminus S$, and all scenarios $\mathcal{S}$ passing through $S^+\pi^+$ and $S\pi$, such that $S^+\pi^+, S\pi \in \mathcal{F}(\mathcal{S})$. Hence, any scenario of the following form, with $S^+ = S \cup S_j$:

$$\bot, \dots, S\pi, \underbrace{S \cup S_j}_{S^+}\pi_{S\cup S_j}, \dots, \underbrace{S \cup S_j}_{S^+}\pi^+, S^+ \cup S_l^+\pi^+_{S^+\cup S_l^+}, \dots, N\{N\}, \quad l = 2, \dots, k^+$$

will lead to a negative contribution with coefficient $\gamma_{s^+,s}$. The notation $\pi_{S\cup S_j}$, etc. is a shorthand for $(\pi \setminus \{S, S_j\}) \cup \{S \cup S_j\}$. The number of such scenarios is:

$$\beta_{s^+,s,\pi} = \mathcal{C}([\bot, S\pi])| \sum_{\substack{\pi^+ > \pi \\ \pi^+ \ni S\cup S_j}} \Big(||\mathcal{C}([\pi_{S\cup S_j}, \pi^+])| \times \sum_{l=2}^{k^+} |\mathcal{C}([S^+ \cup S_l^+\pi^+_{S^+\cup S_l^+}, N\{N\}])| \Big).$$

Although this number seems difficult to compute (!), it depends ultimately only on k, $s, s_2, \dots s_k$ (see the Lemma below), hence on the number of blocks of π and their cardinalities.

Similarly, there is a positive contribution for $S\pi$ with coefficient γ_{s,s^-} for all $S^-\pi^-$ such that $S^-\pi^- \sqsubset S\pi$, $|S^-| = s^-$, and $S = S^- \cup S_1^-$ for some $S_1^- \in \pi^-$, all $i \in S \setminus S^-$, and all scenarios $\mathcal{S}$ passing through elements $S\pi$ and $S^-\pi^-$, so that $S\pi, S^-\pi^- \in \mathcal{F}(\mathcal{S})$. Hence, any scenario of the following form, with $S^-\pi^-$ defined as above

$$\bot, \dots, S^-\pi^-, \underbrace{S^- \cup S_1^-}_{S}\pi^-_{S^-\cup S_1^-}, \dots, S\pi, S \cup S_j\pi_{S\cup S_j}, \dots, N\{N\}, \quad j = 2, \dots, k$$

will lead to a positive contribution with coefficient γ_{s,s^-}. The number of such scenarios is

$$\alpha_{s,s^-,\pi} = \sum_{\substack{S^-\subset S, |S^-|=s^- \\ \pi^-<\pi, \pi^-\ni S^-, S_1^- \\ \text{s.t. } S=S^-\cup S_1^-}} \Big(|\mathcal{C}([\bot, S^-\pi^-])| \times |\mathcal{C}([\pi^-_{S^-\cup S_1^-}, \pi])| \times \sum_{j=2}^{k} |\mathcal{C}([S\cup S_j\pi_{S\cup S_j}, N\{N\}])|\Big)$$

Again, this number depends only on the number of blocks of π and their cardinalities. In summary, the equation for $S\pi \neq N\{N\}$ is

$$\sum_{s^-<s}(s-s^-)\alpha_{s,s^-,\pi}\gamma_{s,s^-} + \sum_{j=2}^{k}(s^+-s)\beta_{s+s_j,s,\pi}\gamma_{s+s_j,s} = 0,$$

Let us address briefly the case of singletons and N. If $S=\{i\}$, the first term is replaced by $\alpha_{1,0,\pi}\gamma_{1,0}$, with

$$\alpha_{1,0,\pi} = \sum_{j=2}^{k} |\mathcal{C}([S\cup S_j\pi_{S\cup S_j}, N\{N\}])|.$$

If $S=N$, the second term does not exist. In summary:

$$\alpha_{1,0,\pi}\gamma_{1,0} + \sum_{j=2}^{k}(s^+-s)\beta_{s+s_j,s,\pi}\gamma_{s+s_j,s} = 0, \qquad (S \text{ is a singleton})$$

$$\sum_{n^-<n}\alpha_{n,n^-,\{N\}}\gamma_{n,n^-} = v(N\{N\}), \qquad (S\pi = N\{N\}).$$

From the above considerations, equations for $S\pi$ and $S'\pi'$ will be identical if and only if π and π' are of the same type (same number of blocks and same cardinalities of blocks). Hence, the number of different equations is the number of integer partitions of $n-s$, denoted by $p(n-s)$, for $s=1,\dots,n$. For example, the numbers of integer partitions of 1, 2, 3, 4 are respectively 1, 2, 3, 5. Hence, for $n=2,3,4,5$ we have respectively 2, 4, 7, and 12 different equations.

The number of variables $\gamma_{s,s'}$ is much easier to compute. For $s=1$, there is only one variable, namely $\gamma_{1,0}$. For $1<s\leq n$ fixed, s' varies from 1 to $s-1$. Hence the total number of variables is:

$$1+\sum_{s=2}^{n}(s-1) = \sum_{s=2}^{n} s - n + 2 = \frac{n^2-n+2}{2}.$$

This gives for $n=2,3,4,5$ players, 2, 4, 7, 11 variables. It is easy to see that this number is less or equal than the number of equations. Indeed, for large n, the following formula is known:

$$p(n) \approx \frac{1}{4n\sqrt{3}}\exp\Big(\pi\big(\frac{2n}{3}\big)^{\frac{1}{2}}\Big)$$

which is clearly exponential (see Andrews [3]).

It remains to find a subsystem of equations which can be made triangular. Let us order the variables as follows: $\gamma_{1,0}, \gamma_{2,1}, \gamma_{3,1}, \gamma_{3,2}, \gamma_{4,1}, \ldots$. For each variable $\gamma_{s',s}$, except for $\gamma_{1,0}$, let us find an equation using only variables up to $\gamma_{s',s}$. It suffices to take the equation for $S\pi$ such that the largest block of $\pi \setminus S$ is of size $s' - s$. Doing so for all $\gamma_{s',s}$, we form a subsystem. If in this subsystem, we put for each equation $\gamma_{1,0}$ on the right side, the subsystem becomes triangular. So it has a unique solution in terms on $\gamma_{1,0}$, which can be determined by substituting all variables in the equation corresponding to $S\pi = N\{N\}$. This proves the uniqueness of the solution. ■

Lemma 1 *(i) Consider the lattice of partitions $\Pi(n)$, and two distincts elements π, π', with $\pi' < \pi$. Then*

$$|\mathcal{C}([\pi', \pi])| = \frac{(k'-k)!}{2^{k'-k}} l_1! l_2! \cdots l_k!$$

with $\pi := \{S_1, \ldots, S_k\}$, $\pi' := \{S_{11}, \ldots, S_{1l_1}, S_{21}, \ldots, S_{2l_2}, \ldots, S_{kl_k}\}$, with $\{S_{i1}, \ldots, S_{il_i}\}$ a partition of S_i, $i = 1, \ldots, k$, and $k' := \sum_{i=1}^{k} l_i$.

(ii) Consider the lattice of embedded coalitions $\mathfrak{C}(n)$, and two distinct elements $S\pi$ and $S'\pi'$, with $S\pi \sqsubset S'\pi'$. Then

$$|\mathcal{C}([S'\pi', S\pi])| = \frac{l_1(k'-k)}{2^{k'-k}} l_1! l_2! \ldots l_k!$$

with $S\pi := S\{S, S_2, \ldots, S_k\}$, $S'\pi' := S'\{S', S_{12}, \ldots, S_{1l_1}, S_{21}, \ldots, S_{2l_2}, \ldots, S_{k1}, \ldots, S_{kl_k}\}$, with $\{S_{i1}, \ldots, S_{il_i}\}$ a partition of S_i, $i = 2, \ldots, k$, $\{S', S_{12}, \ldots, S_{1l_1}\}$ a partition of S, and $k' := \sum_{i=1}^{k} l_i$.

Proof:

(i) Simply consider $S_{11}, \ldots, S_{1l_1}, S_{21}, \ldots, S_{2l_2}, \ldots, S_{kl_k}$, and use the formula for the number of maximal chains from bottom to π in $\Pi(k')$ (see Appendix).

(ii) Same as above.

■

6 Comments on the value for PFF games by Macho-Stadler et al.

In [11], Macho-Stadler et al. propose a value for PFF games, with an axiomatization, different from ours.

In their framework, i is a dummy player iff for every (S, π), it is the case that $v(S, \pi) = v(S', \pi')$, where (S', π') is obtained from (S, π) by changing the affiliation of i (in particular, $v(i, \{i, S_1, \ldots, S_k\}) = 0$). For a dummy player, its value is zero.

The key idea is to consider that the worth assigned to a coalition should be an average of the different worths of this coalition for all possible organizations of other players. In other words we construct an average (classical) game

$$\tilde{v}(S) := \sum_{\pi \ni S} \alpha_{S,\pi} v(S, \pi).$$

Then the Shapley value of v is simply the classical Shapley value of $\tilde{v}$. They show that if the value satisfies linearity, the dummy axiom and efficiency, it is an average value iff it satisfies a strong symmetry axiom we do not detail here. Then the weights depends only on cardinality, and satisfy a certain condition. To ensure uniqueness of the weights, they propose an additional axiom, which is called the similar influence axiom. Players i and j have similar influence in games v, v' if $v = v'$ except for 2 elements $S\{S, i, j, S_2, \ldots, S_k\}$ and $S\{S, \{i, j\}, S_2, \ldots, S_k\}$, and $v(S\{S, i, j, S_2, \ldots, S_k\}) = v'(S\{S, \{i, j\}, S_2, \ldots, S_k\})$, $v(S\{S, \{i, j\}, S_2, \ldots, S_k\}) = v'(S\{S, i, j, S_2, \ldots, S_k\})$. Then the axiom says that in this case, $\phi_i(v) = \phi_i(v')$, and $\phi_j(v) = \phi_j(v')$.

Finally their value is characterized by linearity, the dummy, strong symmetry, similar influence axioms, and efficiency.

Our value is different, and does not satisfy the above dummy axiom, nor the strong symmetry and similar influence axiom. What is interesting is to have a clear view of the difference by looking at the underlying structure. In their paper, there is no explicit mention of the underlying structure, that is, they do not consider any order on the embedded coalitions.

To our opinion, the underlying structure is suggested by the dummy axiom (more exactly the null axiom). In the classical case, the nullity condition is based on the difference between the worths of S and $S \setminus i$, assuming $S \ni i$. These elements are "neighbors" in the lattice $(2^N, \subseteq)$, that is, $S \succ S \setminus i$ (see Appendix A for the covering relation $\succ$). In the PFF case, our dummy axiom is defined along a maximal chain of $\mathfrak{C}(N)_\perp$. The dummy axiom of Wettstein et al. takes the difference of worth between $S\{S, S_2, \ldots, S_k\}$ and $S \setminus i\{S \setminus i, S_2 \cup i, \ldots, S_k\}$, for $S \ni i$. In $\mathfrak{C}(N)_\perp$, these elements are not neighbors because they are on the same level. To recover them as neighbors, one possibility is the following: take the Boolean lattice $(2^N, \subseteq)$. Duplicate each element S as many times they are different possible coalition structures containing S, and indicate these coalition structures. Put all possible links between duplicates of an element S and duplicates of an element T iff these elements are linked in the Boolean lattice. Doing so, the dummy condition of Wettstein appears for neighbors elements. This structure also explains well the average approach: it can almost be seen on the picture.

We finish this presentation by giving $\mathfrak{C}(N)$ and the structure of Wettstein for $n = 3$.

References

[1] M. Aigner. *Combinatorial Theory.* Springer Verlag, 1979.

[2] M. Albizuri, J. Arin, and J. Rubio. An axiom system for a value for games in partition function form. *Int. Game Theory Review*, 7:63–72, 2005.

[3] G. E. Andrews. *The theory of partitions*, volume 2 of *Encyclopedia of mathematics and its applications.* Addison-Wesley, 1976.

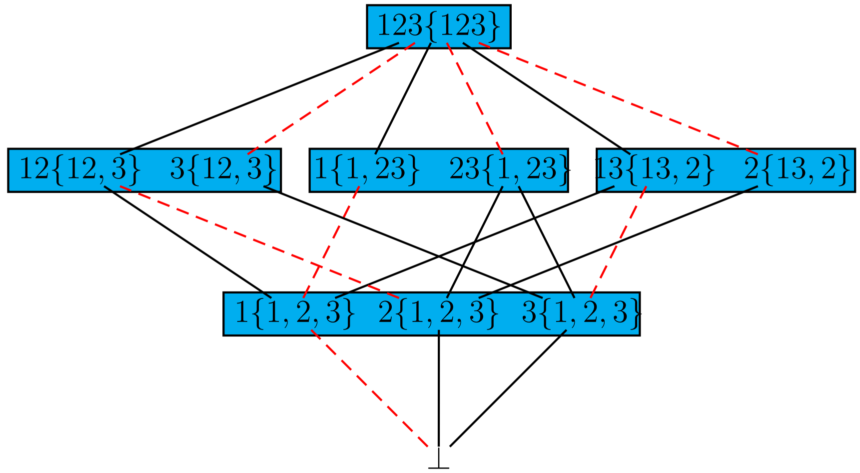


Figure 4: Hasse diagram of $(\mathfrak{C}(3)_\perp, \leq)$. Elements with the same partition are framed in blue. Elements in nullity condition of player 1 are linked in dashed red line

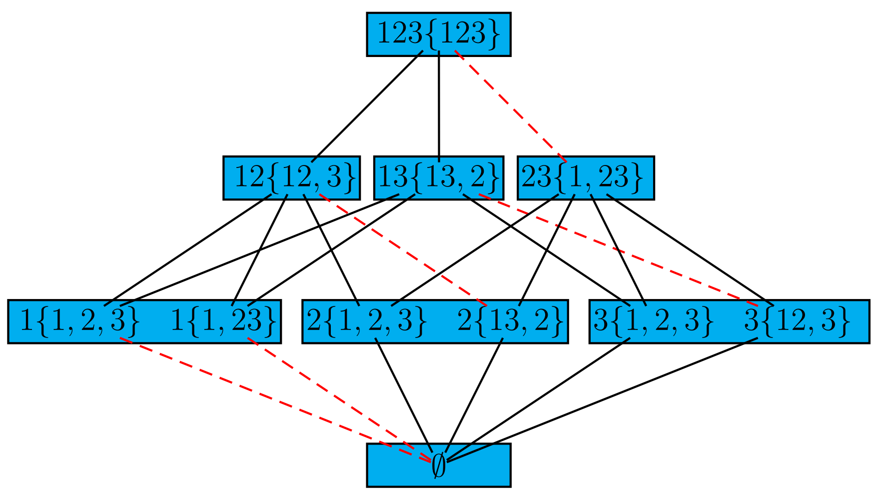


Figure 5: Hasse diagram of the structure of Wettstein for $n = 3$. Elements with same coalition (duplicates) are framed in blue. Elements in nullity condition of player 1 are linked in dashed red line

[4] F. Bloch. Sequential formation of coalitions in games with externalities and fixed payoff divisions. *Games and Economic Behavior*, 14:90–123, 1996.

[5] E. M. Bolger. A set of axioms for a value for partition funcion games. *Int. J. of Game Theory*, 18:37–44, 1989.

[6] G. de Clippel and R. Serrano. Marginal contributions and externalities in the value. Economics Working Papers we057339, Universidad Carlos III, Departamento de Economia, 2005.

[7] E. Diamantoudi and L. Xue. Coalitions, agreements and efficiency. *J. of Economic Theory*, 136:105–125, 2007.

[8] K. H. P. Do and H. Norde. The shapley value for partition function form games. Technical Report DP 2002-04, CenTER, Tilburg University, 2002.

[9] U. Faigle and W. Kern. The Shapley value for cooperative games under precedence constraints. *Int. J. of Game Theory*, 21:249–266, 1992.

[10] Y. Funaki and T. Yamato. The core of an economy with a common pool resource: A partition function approach. *International Journal of Game Theory*, 28:157–171, 1999.

[11] I. Macho-Stadler, D. Pérez-Castillo, and D. Wettstein. Sharing the surplus: An extension of the Shapley value for environments with externalities. *J. of Economic Theory*, 135:339–356, 2007.

[12] R. B. Myerson. Values of games in partition function form. *International Journal of Game Theory*, 6:23–31, 1977.

[13] D. Ray and R. Vohra. Equilibrium binding agreement. *J. of Economic Theory*, 73:30–78, 1997.

[14] R. M. Thrall and W. F. Lucas. n-person games in partition function form. *Naval Research Logistic Quarterly*, 10:281–298, 1963.

A Lattices, partitions and embedded coalitions

(see essentially Aigner [1]) A lattice $(L, \leq)$ is a set L endowed with a partial order (reflexive, antisymmetric and transitive), such that for any two elements $x, y \in L$, there is a least upperbound (or supremum), denoted by $x \vee y$, and a greatest lower bound (or infimum), denoted by $x \wedge y$. We give some definitions on lattices.

Definition 7 *Let $(L, \leq)$ be a lattice.*

(i) For $x, y \in L$, x covers *y (denoted by $x \succ y$) if $x \geq z \geq y$ implies $z = x$ or $z = y$.*

(ii) Atoms *in L are elements which covers $\perp$. A lattice is* atomistic *if all join-irreducible elements (i.e., elements which cover only one element) are atoms.*

(iii) A lattice is distributive *if* sup, inf *obey distributivity.*

(iv) $(L, \leq)$ *is* lower semimodular *(resp.* upper semimodular*) if for all* $x, y \in L$, $x \vee y \succ x$ *and* $x \vee y \succ y$ *imply* $x \succ x \wedge y$ *and* $y \succ x \wedge y$ *(resp.* $x \succ x \wedge y$ *and* $y \succ x \wedge y$ *imply* $x \vee y \succ x$ *and* $x \vee y \succ y$*).*

(v) height function h: $h(x) =$ *length of a longest chain from* $\perp$ *to* x*.*

(vi) A lattice is geometric *if it is atomistic and for every atom* a, $h(a \vee x) = h(x) + 1$ *(equivalently if it is atomistic and upper semimodular). (equivalently if* $x \succ y \neq \perp$ *then there exists an atom* $z \not\leq y$ *s.t.* $x = y \vee z$*.*

The following proposition summarizes known results on $(\Pi(n), \leq)$.

Proposition 7 *(i) The height of the lattice is* $n - 1$*, the height function is* $h(\pi) = n - |\pi|$*. Hence, all partitions of same number of blocks are of the same height (or are on the same level), and the lattice is ranked (any maximal chain from* $\pi^{\perp}$ *to* π *has same length).*

(ii) Atoms are of the form $\pi^{\perp}_{\{i,j\}}$, $\{i, j\} \subseteq N$ *(denoted* $\pi^{\perp}_{ij}$ *for short).*

(iii) Each π *covers* $\sum_{S \in \pi} 2^{|S|-1} - |\pi|$ *partitions. Each* k*-partition is covered by* $\binom{k}{2}$ *partitions.*

(iv) $(\Pi(n), \leq)$ *is geometric (and hence atomistic), indecomposable (i.e., it cannot be written as a product* $[\perp, a] \times [\perp, a']$ *with* $a, a' \neq \perp, \top$*), but not distributive.*

(v) Every lattice is a sublattice of a partition lattice.

(vi) The number of partitions of k *blocks is* $S_{n,k}$ *(Stirling number of the second kind), with*

$$S_{n,k} := \frac{1}{k!} \sum_{i=0}^{n} (-1)^{k-i} \binom{k}{i} i^n, \quad n \geq 0, k \leq n.$$

We have $S_{0,0} = 1$, $S_{n,0} = 0$ *for* $n > 0$*, and*

$$S_{n+1,k} = S_{n,k-1} + kS_{n,k}$$
$$S_{n+1,k} = \sum_{j=1}^{n} \binom{n}{j} S_{j,k-1}.$$

$S_{n,k}$	$k = 1$	2	3	4	5	6	7	8
$n = 1$	1							
2	1	1						
3	1	3	1					
4	1	7	6	1				
5	1	15	25	10	1			
6	1	31	90	65	15	1		
7	1	63	301	350	140	21	1	
8	1	127	966	1701	1050	266	28	1

(vii) The total number of partitions in $\Pi(n)$ *is the Bell number* B_n.

$$B_n = \sum_{k=1}^{n} S_{n,k}.$$

$B_0 = 1$, *and*

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
B_n	*1*	*2*	*5*	*15*	*52*	*203*	*877*	*4140*

(viii) Let $\pi := \{S_1, \ldots, S_k\}$ *a* k*-partition. Then*

$$[\pi, \pi^\top] \equiv \Pi(k)$$

$$[\pi^\perp, \pi] \equiv \prod_{i=1}^{k} \Pi(s_i)$$

$$[\pi, \pi'] \equiv \prod_{i=1}^{|\pi'|} \Pi(m_i) \text{ for some } m_i\text{'s with } \sum_{i=1}^{|\pi'|} m_i = k.$$

(ix) The number of partitions of type $1^{k_1} 2^{k_2} \cdots n^{k_n}$ *is*

$$\frac{n!}{(1!)^{k_1} (2!)^{k_2} \cdots (n!)^{k_n} k_1! k_2! \cdots k_n!}.$$

To our knowledge, the two following properties were not yet shown, and they are fundamental in our work.

Proposition 8 *The number of maximal chains of* $\Pi(n)$ *is*

$$|\mathcal{C}(\Pi(n))| = \frac{n}{2^{n-1}}((n-1)!)^2 = \frac{n!(n-1)!}{2^{n-1}}.$$

Proof: Starting from $\pi^\perp$, we have $n-1$ steps (height of the lattice) for reaching $\pi^\top$. At first step, we have $\frac{n(n-1)}{2}$ possibilities to go from $\pi^\perp$ to second level (atoms) (say, $\pi^\perp_{ij}$). From second to 3d level we have $\frac{(n-1)(n-2)}{2}$, etc., so in total, this gives $\frac{n}{2^{n-1}}((n-1)!)^2$ maximal chains. ■

n	1	2	3	4	5	6	7	8
$\lvert\mathcal{C}(\Pi(n))\rvert$	1	1	3	18	180	2 700	56 700	1 587 600

Proposition 9 *Let* $\pi := \{S_1, \ldots, S_k\}$ *be a* k*-partition. Then*

(i) The number of maximal chains from $\pi^\perp$ *to* π *is*

$$|\mathcal{C}([\pi^\perp, \pi])| = \frac{(n-k)!}{2^{n-k}} s_1! s_2! \cdots s_k!.$$

(ii) The number of maximal chains from π to $\pi^\top$ is

$$|\mathcal{C}([\pi^\perp, 1^{k_1}2^{k_2}\cdots n^{k_n}])| = |\mathcal{C}(\Pi(k))|.$$

Proof: (i) We use the fact that for any k-partition $\pi = \{S_1, \dots, S_k\}$, $[\pi^\perp, \pi] = \prod_{i=1}^k \Pi(s_i)$. We have the general following fact: if $L = L_1 \times \cdots \times L_k$, then to obtain all maximal chains in L, we select a k-uple of maximal chains $C_1, \dots, C_k$ in $L_1, \dots, L_k$ respectively, say of length $c_1, \dots, c_k$. Then the number of chains induced by $C_1, \dots, C_k$ in L is equal to the number of chains in the lattice $C_1 \times \cdots \times C_k$, isomorphic to $c_1 \times \cdots \times c_k$. This is known to be (cf., e.g., Faigle and Kern [9], established for multichoice games)

$$\frac{(\sum_{i=1}^k c_i)!}{\prod_{i=1}^k (c_i!)}.$$

Applied to our case, this gives

$$|\mathcal{C}([\pi^\perp, \pi])| = \prod_{i=1}^k |\mathcal{C}(\Pi(s_i)| \frac{\big(\sum_{i=1}^k (s_i - 1)\big)!}{\prod_{i=1}(s_i - 1)!}$$

which after simplification gives the desired result.

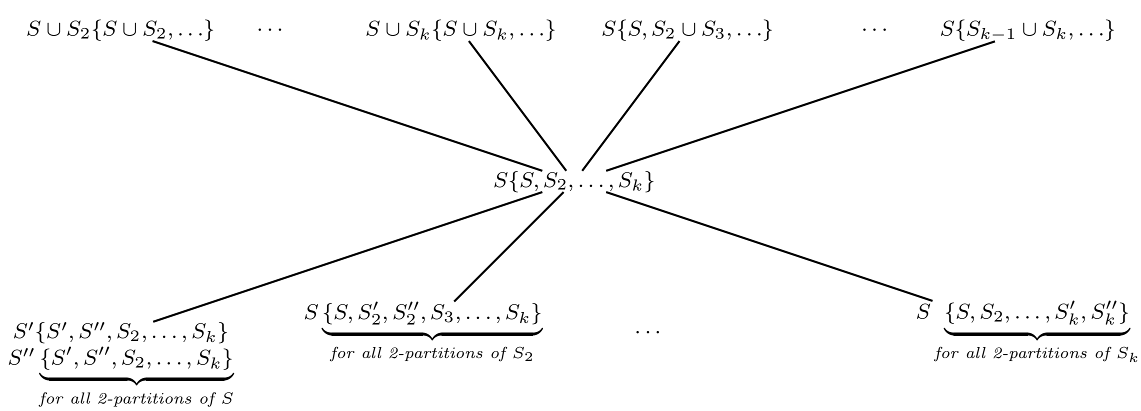
(ii) Immediate from $[\pi, \pi^\perp] \equiv \Pi(k)$. ■

Example: there are 270 maximal chains from $\{1, 2, 3, 4, 5, 6, 7, 8\}$ to $\{123, 456, 78\}$.

Proposition 10 *The following holds.*

(i) $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ is a lattice, whose top and bottom elements are $(N, \pi^\top)$ and $\perp$.

(ii) Each element (S, π) where $\pi := \{S, S_2, \dots, S_k\}$ is a partition in k blocks is covered by $\binom{k}{2}$ elements, and covers $\sum_{T\in\pi} 2^{t-1} - |\pi| + 2^{s-1} - 1$ elements. Specifically:



(iii) Every maximal chain from $\perp$ to an element (S, π) has the same length, which is $n - k + 1$, if π is a partition in k blocks. Hence, the height of the lattice is n.

(iv) The total number of elements is $\sum_{k=1}^n kS_{n,k} + 1$, where $S_{n,k}$ is the Stirling number of second kind.

n	1	2	3	4	5	6	7	8
$\lvert\mathfrak{C}(n)_\perp\rvert$	2	4	11	38	152	675	3264	17008

(v) The number of maximal chains from $\perp$ to $(N, \{N\})$ is $\frac{(n!)^2}{2^{n-1}}$, which is also the number of maximal longest chains in $\mathfrak{C}(N)$.

n	1	2	3	4	5	6	7	8
$\lvert\mathcal{C}(\mathfrak{C}(n)_\perp)\rvert$	1	2	9	72	900	16 200	396 900	12 700 800

(vi) Let (S,π) be an embedded coalition, with $\pi := \{S, S_2 \ldots, S_k\}$, and $|S| = s$. The number of maximal chains from $\perp$ to (S,π) is

$$|\mathcal{C}([\perp,(S,\pi)])| = \frac{s(n-k)!}{2^{n-k}} s!s_2! \cdots s_k!.$$

For example, there are 9 chains from $\perp$ to $123\{123, 4\}$.

(vii) Let (S,π) be an embedded coalition, with $\pi := \{S, S_2 \ldots, S_k\}$. The number of maximal chains from (S,π) to $N\{N\}$ is

$$|\mathcal{C}([(S,\pi), N\{N\}])| = \frac{1}{k}|\mathcal{C}(\mathfrak{C}(k)_\perp)| = \frac{k!(k-1)!}{2^{k-1}}.$$

Proof: (of Proposition 10) (i) We prove that $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ is a lattice, with infimum and supremum given by

$$(S,\pi) \vee (S',\pi') = (T \cup T', \rho) \text{ (see below)}$$
$$(S,\pi) \wedge (S',\pi') = (S \cap S', \pi \wedge \pi') \text{ if } S \cap S' \neq \emptyset, \text{ and } \perp \text{ otherwise.}$$

Consider $(S,\pi), (S',\pi') \in \mathfrak{C}(n)_\perp$. Then (K,ρ) is an upper bound of both elements iff $K \supseteq S \cup S'$ and $\rho \geq \pi \vee \pi'$ (clearly exists). If $S \cup S' \in \pi \vee \pi'$ then $(S \cup S', \pi \vee \pi')$ is the least upper bound. If not, since $S \in \pi$ and $S' \in \pi'$ and by definition of $\pi \vee \pi'$, there exist blocks T, T' of $\pi \vee \pi'$ such that $T \supseteq S$ and $T' \supseteq S'$. Then $(T \cup T', \rho)$ where ρ is the partition obtained by merging T and T' in $\pi \vee \pi'$ is the least upper bound of $(S,\pi), (S',\pi')$.

Next, $(S \cap S', \pi \wedge \pi')$ would be the infimum if $S \cap S'$ is a block of $\pi \wedge \pi'$. If $S \cap S' \neq \emptyset$, then this is the case. If not, then $\perp$ is the only lower bound. This proves that $(\mathfrak{C}(N)_\perp, \sqsubseteq)$ is a lattice.

(ii) Clear.

(iii) From (ii), (S,π) covers (S',π') implies that if π is a k-partition, then π' is a $(k+1)$-partition. Hence, a maximal chain from (S,π) to the bottom element has length $n-k+1$, which proves the Jordan-Dedekind chain condition. Now, the height function is $h(S,\pi) = n-k+1$.

(iv) Clear from the results on $\Pi(N)$.

(v) Consider the element $(i, \pi^\perp)$ in $\mathfrak{C}(N)_\perp$, $i \in N$. Then $[(i, \{\pi^\perp\}), (N, \pi^\top)]$ is a sublattice isomorphic to $\Pi(N)$, since by (ii) the number of elements covering $(i, \pi^\perp)$ is the same as the number of elements covering $\pi^\perp$ in $\Pi(n)$, and that this property remain true for all elements above $(i, \pi^\perp)$. Since by Prop. 8, $\mathcal{C}(\Pi(N)) = \frac{n((n-1)!)^2}{2^{n-1}}$, and there are n mutually incomparable elements $(i, \pi^\perp)$ in $\mathfrak{C}(n)_\perp$, the result follows.

(vi) The proof follows the same technique as for Prop. 9 (i). Remarking that for every embedded coalition $(S_1, \{S_1, \dots, S_k\})$

$$[\perp, (S_1, \{S_1, \dots, S_k\})] = \big(\mathfrak{C}(S_1) \times \Pi(S_2) \times \cdots \times \Pi(S_k)\big) \cup \{\perp\},$$

and noting that deleting the bottom element does not change the number of maximal (longest) chains, we can write immediately

$$|\mathcal{C}([\perp, (S, \pi)])| = \prod_{i=1}^{k} |\mathcal{C}(\Pi(s_i))||\mathcal{C}(\mathfrak{C}(s))| \frac{\left(\sum_{i=1}^k s_i - 1\right)!}{\prod_{i=1}^k (s_i - 1)!}.$$

The result follows by using Prop. 8 and (v).

(vii) Clear since $[(S, \pi), N\{N\}]$ is isomorphic to $[i\{i, i_2, \dots, i_k\}, K\{K\}]$. ■

Further properties are given below.

Proposition 11 *The following holds for $(\mathfrak{C}(N)_\perp, \sqsubseteq)$.*

(i) Its join-irreducible elements are $(i, \pi^\perp)$, $i \in N$ (atoms), and $(i, \pi^\perp_{jk})$, $i, j, k \in N$, $i \notin \{j, k\}$. Its meet-irreducible elements are (S, π) where π is any 2-partition (co-atoms).

(ii) The lattice is not distributive (and even neither upper nor lower locally distributive), not atomic (hence not geometric) but upper semimodular.

Proof: (i) Clear from Prop. 10 (ii).

(ii) The lattice is not (upper or lower locally) distributive since it contains diamonds. For example, with $n = 3$, the following 5 elements form a diamond:

$$(1, \{1, 2, 3\}), (12, \{12, 3\}), (1, \{1, 23\}), (13, \{13, 2\}), (123, \{123\}).$$

For atomicity see (iv). Let us prove it is upper semimodular. Since the lattice is ranked, configuration $x \succ x \wedge y$ and $y \succ x \wedge y$ implies that both x and y are one level above $x \wedge y$. Hence using (iii), if $x \wedge y := (S, \{S, S_2, \dots, S_k\})$, then x has either the form $(S, \{S, S_i \cup S_j, \dots\})$ or $(S \cup S_i, \{S \cup S_i, \dots\})$, and similarly $y = (S, \{S, S_k \cup S_l, \dots\})$ or $(S \cup S_j, \{S \cup S_j, \dots\})$. To compute $x \vee y$, we have three cases:

- $x = (S, \{S, S_i \cup S_j, \dots\})$ and $y = (S, \{S, S_k \cup S_l, \dots\})$: then $x \vee y = (S, \{S, S_i \cup S_j, S_k \cup S_l, \dots\})$ if $k, l \neq i, j$. If, e.g., $k = i$, $x \vee y = (S, \{S, S_i \cup S_j \cup S_l, \dots\})$.
- $x = (S \cup S_i, \{S \cup S_i, \dots\})$ and $y = (S \cup S_j, \{S \cup S_j, \dots\})$: then $x \vee y = (S \cup S_i \cup S_j, \{S \cup S_i \cup S_j, \dots\})$.
- $x = (S, \{S, S_i \cup S_j, \dots\})$ and $y = (S \cup S_k, \{S \cup S_k, \dots\})$: then $x \vee y = (S \cup S_k, \{S \cup S_k, S_i \cup S_j, \dots\})$ if $k \neq i, j$. If $k = i$, $x \vee y = (S \cup S_i \cup S_j, \{S \cup S_i \cup S_j, \dots\})$.

In all cases, we get a $(k-1)$-partition, so upper modularity holds. ■