# On a class of vertices of the core

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- Our results extend the families introduced so far for classical TU-games, and generalize the framework to games with restricted cooperation
- Still not all vertices are known in the general case.

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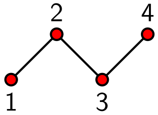


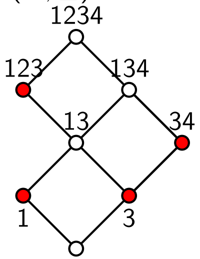

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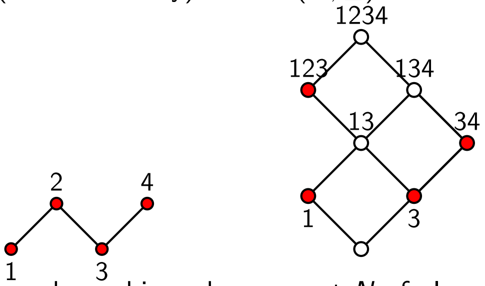


Conclusion: an order or hierarchy on a set $N$ of players produces a set of feasible coalitions $\mathcal{F}$ which is a distributive lattice (Faigle and Kern 1992: *games with precedence constraints*)

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- A *game on $\mathcal{F}$* is a function $v : \mathcal{F} \to \mathbb{R}$ s.t. $v(\emptyset) = 0$. Notation: $(N, \preceq, v)$.
- A game is *supermodular (or convex)* if for all $S, T \in \mathcal{F}$,

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T)$$

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- The extremal rays of the core are of the form $1_{\{i\}} - 1_{\{j\}}$ with $i \prec\cdot\, j$

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Theorem

*(Fujishige and Tomizawa 1983, Derks and Gilles 1995) The game $(N, \preceq, v)$ is supermodular if and only if every marginal vector $m^{\pi,v}$ with $\pi \in \Pi(\mathcal{F})$ is a vertex of $C(N, \preceq, v)$.*

# How many vertices?

Adapting an argument of Derks and Kuipers (2002) for classical games, we can show:

**Theorem**

*Let $\mathcal{F} = \mathcal{O}(N, \preceq)$ be given, and let $\kappa(\mathcal{F})$ be the number of linear extensions of $(N, \preceq)$. The core of any game $v$ on $\mathcal{F}$ has at most $\kappa(\mathcal{F})$ vertices.*

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This bound is attained for strictly supermodular games. For classical games, $\kappa(2^N) = n!$.

# Reduced games

Definition of the *reduced game* $v_{S,x}$ on set system $\mathcal{F}(S) = \{T \cap S \mid T \in \mathcal{F}\}$ w.r.t. $x \in \mathbb{R}^N$:

$$v_{S,x}(T) = \begin{cases} v(N) - x(N \setminus S), & \text{if } T = S \\ 0, & \text{if } T = \emptyset \\ \max_{R \subseteq N \setminus S,\, T \cup R \in \mathcal{F}} \{v(T \cup R) - x(R)\}, & \text{otherwise} \end{cases}$$

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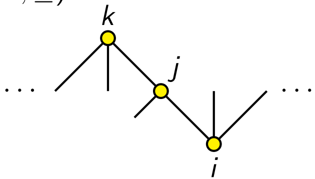
The core satisfies the *RGP (reduced game property)* and the *RCP (reconfirmation property)*:

- RGP: for every $x \in C(N, \preceq, v)$ and $\emptyset \neq S \subseteq N$, $x_S \in C(S, \preceq, v_{S,x_{N \setminus S}})$
- RCP: for every $x \in C(N, \preceq, v)$ and $\emptyset \neq S \subseteq N$, $y_S \in C(S, \preceq, v_{S,x_{N \setminus S}})$ implies $(x_{N \setminus S}, y_S) \in C(N, \preceq, v)$.

# The basic idea of min-max vertices

- Consider a hierarchy $(N, \preceq)$



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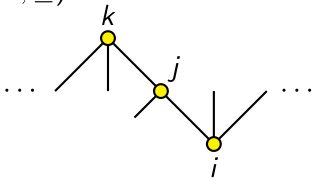


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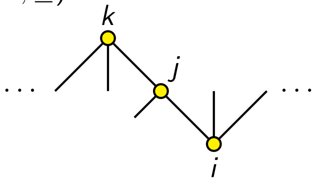


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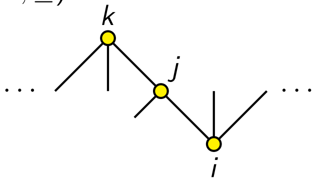


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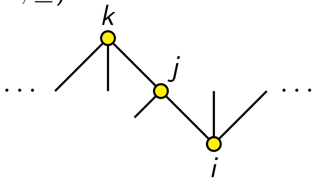


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- Supposing that some core element satisfies $x_i = v(\{i\})$, by RCP it suffices to find $x_{N\setminus i} \in C(N \setminus i, \preceq, v_{N\setminus i, x_i})$ (nonempty by RGP) to ensure that $(x_i, x_{N\setminus i})$ is a core element (same for $k$)

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Basic algorithm:

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The algorithm will end up with a core element if at each step there exists a core element with coordinate attaining the lower or upper bound. Hence, the key point of this procedure will be to find valid bounds for core elements.

# Min-max vertices (1/4)

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**Lemma**

*Let $(N, \preceq, v)$ be a game with precedence constraints and $i \in N$.*

1. *Let $n \geqslant 2$. Then $x_i \in \mathbb{R}^{\{i\}}$ is core extendable if and only if*
   - 1.1 $x_i \geqslant v(\{i\})$ *if $i$ is a minimal element of $(N, \preceq)$,*
   - 1.2 $x_i \leqslant v(N) - v(N \setminus \{i\})$ *if $i$ is a maximal element of $(N, \preceq)$, and*
   - 1.3 $(N \setminus \{i\}, \preceq, v_{N\setminus\{i\},x_i})$ *is balanced.*
2. *Assume that $(N, \preceq, v)$ is balanced. The set $\{x_i : x \in C(N, \preceq, v)\}$ is convex and bounded*
   - 2.1 *from below if and only if $i$ is a minimal element of $(N, \preceq)$;*
   - 2.2 *from above if and only if $i$ is a maximal element of $(N, \preceq)$.*

# Min-max vertices (2/4)

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- A *decision vector* is any vector in $\{-1, 1\}^N$. Given an admissible order $\pi$ and a decision vector $d$, $(\pi, d)$ is a *consistent pair* if the following conditions are satisfied for $i = 1, \dots, n$:

$$\begin{aligned} d_i = -1 &\implies \pi(i) \text{ is minimal in the poset } (A_i^\pi, \preceq);\\ d_i = 1 &\implies \pi(i) \text{ is maximal in the poset } (A_i^\pi, \preceq). \end{aligned}$$

# Min-max vertices (3/4)

Assume $(N, \preceq, v)$ is balanced. For any consistent pair $(\pi, d)$, recursively define the vector $x = x^{\pi,d,v} \in \mathbb{R}^N$ as follows:

$$x_{\pi(i)} = \max\left\{z_{\pi(i)} d_i \ : \ z \in C\left(A_i^\pi, \preceq, v_{A_i^\pi, x_{B_{i-1}^\pi}}\right)\right\} \text{ for all } i = 1, \ldots, n.$$

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**Theorem**

*Let $(N, \preceq, v)$ be a balanced game, $\pi$ be an admissible order of $N$, and $d$ a decision vector. If $(\pi, d)$ is consistent, then the vector $x^{\pi,d,v}$ is well-defined, and it is a vertex of $C(N, \preceq, v)$.*

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- $x^{\pi,d,v}$ with $d = (1, 1, \ldots, 1)$ and $\mathcal{O}(N, \preceq) = 2^N$ was introduced by Tijs (2005) by the above formula under the name of *leximal*.

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- **Remark:** Corresponds to the "intuitive bounds".
- Not always a core element! But $y^{\pi,d,v}$ is a core element iff $y^{\pi,d,v} = x^{\pi,d,v}$, i.e., it is a min-max vertex.

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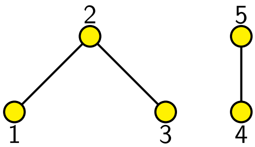


**Example:** A hierarchy with 5 players. Consider $\pi = 13524$ and $d = (-1, -1, 1, 1, -1)$. Then $\pi^d = 13425$, and the maximal chain $B_0^{\pi^d}, \ldots, B_n^{\pi^d}$ is $\emptyset, 1, 13, 134, 1234, N$.

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We have obtained:

**Theorem**

*Let $(N, \preceq, v)$ be a game with $(N, \preceq)$ a connected hierarchy. Then for any consistent pair $(\pi, d)$ where $\pi$ is a simple order, the induced vector $y^{\pi,d,v}$ is a min-max vertex.*

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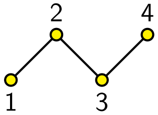
**Lemma**

*Any poset $(N, \preceq)$ has a total order that is simple and admissible.*

# Example



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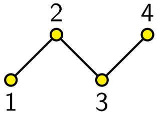


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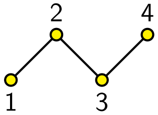


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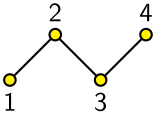


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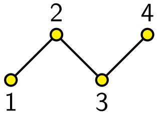


- Linear extensions: 1324, 1342, 3124, 3142, 3412
- Admissible orders: all
- Simple orders: 1234, 1243, 1423, 1432, 4321, 4312, 4123, 4132
- If $v$ is strictly supermodular, with $\pi = 1234$ and $d = (-1, 1, -1, 1)$, we find:

$$
\begin{aligned}
x_1 &= v(1) \\
x_2 &= v_{234,x}(234) - v_{234,x}(34) = v(N) - v(1) - \max(v(34), v(134) - v(1)) \\
&= v(N) - v(134) \\
x_3 &= v_{34,x}(3) = \max(v(3), v(13) - x_1, v(123) - x_1 - x_2) \\
&= \max(v(3), v(13) - v(1), v(123) - v(N) + v(134) - v(1)) \\
&= v(13) - v(1).
\end{aligned}
$$

We get $x = m^{1342,v}$. Order 1243 yields the same vertex.

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**Theorem**

*If $y$ is a vertex of $C(\mathcal{R}, v_{\mathcal{R}})$ then every min-max vertex of $C_y(N, \preceq, v)$ is a vertex of $C(N, \preceq, v)$.*

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  3. $(A^{\pi}_{k+2}, \preceq)$ is connected (where $A^{\pi}_{n+1} = \emptyset$ and $\emptyset$ is assumed to be connected).

# Equivalent consistent pairs

### Theorem

*Let $(N, \preceq)$ be a poset and $(\pi, d)$ and $(\pi', d')$ be consistent pairs. Then the following statements are equivalent:*

1. *The pairs $(\pi, d)$ and $(\pi', d')$ are equivalent.*
2. *There is a sequence $(\pi^1, d^1), \ldots, (\pi^t, d^t)$ of consistent pairs such that $(\pi^1, d^1) = (\pi, d), (\pi^t, d^t) = (\pi', d')$, and for any $\ell \in \{1, \ldots, t-1\}$, either $(\pi^\ell, d^\ell)$ and $(\pi^{\ell+1}, d^{\ell+1})$ are neighbors or they only differ by an irrelevant switch.*

# Limits of the min-max approach

**Example**

- Let $N = \{1, \ldots, 5\}$, $\mathcal{S} = \{\{1,2,3\}, \{2,3\}, \{2,4\}, \{3,4\}, N\}$, and let $(N, \preceq)$ be a poset that such that $\mathcal{S} \subseteq \mathcal{O}(N, \preceq)$.

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- $x = (0,0,0,0,0)$ is a core element. Since $x(S) = v(S)$ for all $S \in \mathcal{S}$, $x$ is a vertex of the core.

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- Consider the following core elements:

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- Hence $x$ can never be attained by coordinatewise minimization/maximization over the core.

**Theorem**

*For any balanced game $(N, \preceq, v)$, every vertex of the core is a min-max vertex if and only if $n \leqslant 4$.*