# Cutting matroids like cakes

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# Outline

Allocation of goods

Matroids

Our problem(s)

Divide-and-Choose : 2 agents

Divide-Ask-and-Choose : 3 agents

Wrap-up

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# Allocation of a divisible resource

...also known as cake cutting

- $n$ agents $\{1, \ldots, n\}$
- a divisible resource (a cake)
- A solution is a partition of the resource $A_1 \cup \ldots \cup A_n$ such that agent $i$ receives $A_i$
- Agent $i$'s utility for $A_i$ is denoted by $u_i(A_i)$

Normalization assumption: every agent has a utility of 1 for the entire resource

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# Two desirable properties

Proportionality : $u_i(A_i) \geq 1/n$ for all $i \in [n]$

Envy-freeness : $u_i(A_i) \geq u_i(A_j)$ for all $i, j \in [n]$

It is possible to find an envy-free (and thus proportional) solution

2 agents : `Divide-and-Choose`

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Bibliographic notes

- 1940's : proportional protocol for any number of agents
- late 1950's : envy-free protocol for 3 agents
- 1995 : envy-free protocol for any number of agents

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# Allocation of **in**divisible goods

- $n$ agents $\{1, \ldots, n\}$
- $m$ indivisible items $\{1, \ldots, m\}$
- $u_i(j)$, nonnegative utility of agent $i$ for item $j$
- A solution/allocation is a partition of the set of items $A_1 \cup \ldots \cup A_n$ so that agent $i$ receives $A_i$
- Agent $i$'s utility for $A_i$ is denoted by $u_i(A_i)$ and defined as $\sum_{j \in A_i} u_i(j)$

Normalization assumption: $\sum_{j=1}^{m} u_i(j) = 1$ for all $i \in [n]$

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# Proportionality

Proportionality : $u_i(A_i) \geq 1/n$ for all $i \in [n]$

A proportional allocation is not guaranteed to exist

| | item 1 | item 2 |
|---|---|---|
| agent 1 | 0.3 | 0.7 |
| agent 2 | 0.4 | 0.6 |

The agent who does not receive item 2 has total utility $< \frac{1}{2}$

# Hill's substitute

Ted Hill [The Annals of Probability, 1987] tried to characterize the value $t_n$ such that, in any case, there exists an allocation $A$ satisfying

$$u_i(A_i) \geq t_n \text{ for all } i \in [n]$$

The result relies on a parameter

$$\alpha := \max_{i \in [n], j \in [m]} u_i(j)$$

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| | item 1 | item 2 |
|---|---|---|
| agent 1 | 0.2 | 0.8 |
| agent 2 | 0.1 | 0.9 |

$\alpha = 0.9$

There exists an allocation $A$ such that $u_i(A_i) \geq 0.2$ for $i = 1, 2$

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The result relies on a parameter

$$\alpha := \max_{i \in [n], j \in [m]} u_i(j)$$

| | item 1 | item 2 | item 3 |
|---|---|---|---|
| agent 1 | 0.4 | 0.3 | 0.3 |
| agent 2 | 0.4 | 0.5 | 0.1 |

$\alpha = 0.5$

There exists an allocation $A$ such that $u_i(A_i) \geq 0.5$ for $i = 1, 2$

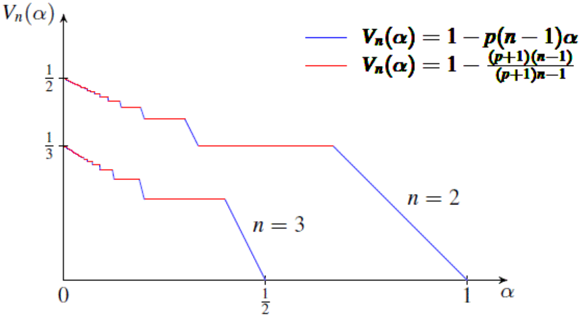
# Hill's substitute

Hill's characterization is a non increasing function $V_n : [0,1] \to [0, n^{-1}]$

$$V_n(x) = \begin{cases} 1 - p(n-1)x, & x \in I(n,p) \\ 1 - \frac{(p+1)(n-1)}{(p+1)n-1}, & x \in NI(n,p) \end{cases}$$

where for any integer $p \geq 1$,
$I(n,p) = \left[\frac{p+1}{p((p+1)n-1)}, \frac{1}{pn-1}\right]$ and
$NI(n,p) = \left]\frac{1}{(p+1)n-1}, \frac{p+1}{p((p+1)n-1)}\right[$

# Hill's substitute




If there are $n$ agents and $\alpha := \max_{i\in[n], j\in[m]} u_i(j)$ then there must exist an allocation $A$ such that $u_i(A_i) \geq V_n(\alpha)$ for all $i \in [n]$

# An algorithmic approach

Markakis & Psomas [WINE 2011] proposed a centralized algorithm called ALLOCATE, polynomial in $(n, m)$, that builds an allocation $A$ such that $u_i(A_i) \geq V_n(\alpha)$

Moreover, the allocation is such that $u_i(A_i) \geq V_n(\alpha_i) \geq V_n(\alpha)$ where

$$\alpha_i = \max_{j \in [m]} u_i(j)$$

and

$$\alpha := \max_{i \in [n], j \in [m]} u_i(j) = \max_{i \in [n]} \alpha_i$$

Note that the utility of the least happy agent remains the same

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# Centralized algorithms versus protocols

Centralized algorithms

- A third party collects the utilities of the agents for the different portions and computes an allocation
- An agent is often reluctant to disclose his utilities (is the third party trustworthy?)
- Issue with the communication of the utilities for every possible portion

Protocols

- There is no third party to compute the allocation; each agent takes part of the construction of the final solution
- The communication problem is avoided

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# Matroids

A matroid $\mathcal{M} = (X, \mathcal{F})$ consists of a finite set of elements $X$ and a collection $\mathcal{F}$ of subsets of $X$ such that:

(i) $\emptyset \in \mathcal{F}$,

(ii) if $F_2 \subseteq F_1$ and $F_1 \in \mathcal{F}$ then $F_2 \in \mathcal{F}$,

(iii) for every couple $F_1, F_2 \in \mathcal{F}$ such that $|F_1| < |F_2|$, $\exists\, e \in F_2 \setminus F_1$ such that $F_1 \cup \{e\} \in \mathcal{F}$.

Every $F \subseteq X$ s.t. $F \in \mathcal{F}$ is said independent

Every $C \subseteq X$ s.t. $C \notin \mathcal{F}$ is said dependent

A base $B$ is an independent of maximal size

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# Example 1 : Forests

A forest in a graph $G$ is a subset of its edges *without* any cycle

We can define a matroid $\mathcal{M} = (X, \mathcal{F})$ over a graph $G$

- $X$ is the edges of $G$
- $\mathcal{F}$ is the set of all forests of $G$

A base is a spanning tree

# Example 2 : Allocation of indivisible goods

$n$ agents, $m$ indivisible items

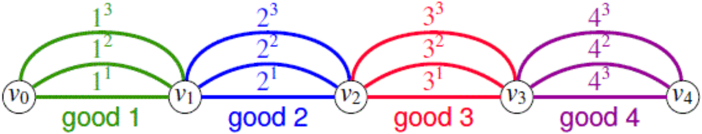
For every item $j$, build the set $E_j = \{j^1, j^2, \ldots, j^n\}$

We can define a matroid $\mathcal{M} = (X, \mathcal{F})$ as follows

- $X = \bigcup_{j=1}^{m} E_j$
- $\mathcal{F} = \{F \subseteq X : |F \cap E_j| \leq 1, j \in [m]\}$

# Example 2 : Allocation of indivisible goods

3 agents, 4 goods...




An allocation is a spanning tree in this graph

Taking edge $j^i$ means allocating item $j$ to agent $i$

# Example 3 : Seminar

- $m$ speakers $\{1, \ldots, m\}$
- $\ell$ days $\{d_1, \ldots, d_\ell\}$

A speaker $j$ is available for a subset of $\{d_1, \ldots, d_\ell\}$

At most one speaker per day

Find a subset of speakers that we can assign to $\{d_1, \ldots, d_\ell\}$

| | $d_1$ | $d_2$ | $d_3$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | 0 | 1 | 0 |
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- $X = \{1, \ldots, m\}$
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# Matroids' background

Introduced by H. Whitney (1935), a matroid is a structure that generalizes the notion of linear independence in vector spaces

Many applications in combinatorial optimization:

- spanning trees
- assignment problem

Polynomial algorithms

- greedy: independent of maximum weight in a matroid
- maximum weight independent in the intersection of 2 matroids defined on the same set of elements

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# The setting

Input

- a matroid $\mathcal{M} = (X, \mathcal{F})$
- $n$ agents
- a utility function $u_i : X \to \mathbb{R}^+$ for every agent $i$

Additive utility: Agent $i$ has utility $u_i(F) := \sum_{x \in F} u_i(x)$ for every set $F \in \mathcal{F}$

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# The setting

### Remark

Since the utility for every element is nonnegative, one can restrict ourselves to the **bases** of $\mathcal{M}$

### Normalization to 1

For every agent $i$ and $F \in \mathcal{F}$ we have $0 \leq u_i(F) \leq 1$

$u_i(B_i^*) = 1$ where $B_i^* \in \mathcal{F}$ is the base that agent $i$ likes the most

No loss of generality (rescaling)

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# Example : Seminar

- $n$ agents $\{a_1, \ldots, a_n\}$
- $m$ speakers $\{1, \ldots, m\}$
- $\ell$ days $\{d_1, \ldots, d_\ell\}$

A speaker is available for a subset of $\{d_1, \ldots, d_\ell\}$, at most one speaker per day, find a subset of speakers that we can assign to $\{d_1, \ldots, d_\ell\}$

| $a_1$ | $a_2$ |
|---|---|
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| 0.4 | 0.3 |
| 0.1 | 0.2 |
| 0.3 | 0.4 |
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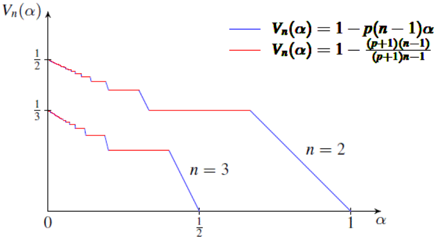
# Example : Seminar

| maximal feasible sets | $u_1$ | $u_2$ | $\min\{u_1, u_2\}$ |
|---|---|---|---|
| $\{1, 2, 3\}$ | 0.8 | 0.6 | 0.6 |
| $\{1, 2, 5\}$ | 1 | 0.8 | 0.8 |
| $\{1, 3, 4\}$ | 0.7 | 0.7 | 0.7 |
| $\{1, 3, 5\}$ | 0.7 | 0.7 | 0.7 |
| $\{1, 4, 5\}$ | 0.9 | 0.9 | 0.9 |
| $\{2, 3, 5\}$ | 0.8 | 0.9 | 0.8 |
| $\{3, 4, 5\}$ | 0.7 | 1 | 0.7 |

# Centralized algorithm

One can extend ALLOCATE, the centralized algorithm by Markakis & Psomas, to matroids

For every number of agents $n$, and every matroid $\mathcal{M}$, there exists a centralized algorithm which outputs a base $B$ of $\mathcal{M}$ such that $u_i(B) \geq V_n(\alpha_i)$ for all agent $i$.

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# Divide-and-Choose on a matroid (2 agents)

The input is a matroid $(X, \mathcal{F})$

1. Agent 1 computes a base and cuts it in two parts $S_1 \cup T_1$
2. Agent 2 finds $S_2$ and $T_2$ s.t. $S_1 \cup S_2$ and $T_1 \cup T_2$ are bases
3. Agent 2 chooses the base that she prefers between $S_1 \cup S_2$ and $T_1 \cup T_2$

# The viewpoint of Agent 1

## Lemma

Given a base $B$ of a matroid partitioned in $S \cup T$ and another base $B^*$ with maximal utility, one can always partition $B^*$ in $S^* \cup T^*$ such that $\min\{u(S^*), u(T^*)\} \geq \min\{u(S), u(T)\}$

Hence, in the DaC mechanism, the first agent should compute a base which maximizes her utility (greedy algorithm) and partition it

Partitioning such that the lightest part is maximized is an **NP**-hard problem (SUBSET-SUM); but one can use ALLOCATE which is polynomial

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3. Agent 2 chooses the base that she prefers between $S_1 \cup S_2$ and $T_1 \cup T_2$

# The viewpoint of Agent 2

## Theorem [Brylawski '73, Greene '73, Woodall '74]

Given two bases $B_1$ and $B_2$ of a matroid $\mathcal{M}$, and a partition $B_1 = X_1 \cup Y_1$, there is a partition $B_2 = X_2 \cup Y_2$ such that $X_1 \cup Y_2$ and $X_2 \cup Y_1$ are two bases of $\mathcal{M}$

Hence, in the DaC mechanism, the second agent can compute a base $B_2^*$ which maximizes her utility and find the partition $S_2 \cup T_2$ of it such that $S_1 \cup S_2$ and $T_1 \cup T_2$ are both bases

- the first step is polynomial (greedy algorithm)
- the second step is also polynomial (matroid intersection)

# The viewpoint of Agent 2

## Theorem [Brylawski '73, Greene '73, Woodall '74]

Given two bases $B_1$ and $B_2$ of a matroid $\mathcal{M}$, and a partition $B_1 = X_1 \cup Y_1$, there is a partition $B_2 = X_2 \cup Y_2$ such that $X_1 \cup Y_2$ and $X_2 \cup Y_1$ are two bases of $\mathcal{M}$

Hence, in the `DaC` mechanism, the second agent can compute a base $B_2^*$ which maximizes her utility and find the partition $S_2 \cup T_2$ of it such that $S_1 \cup S_2$ and $T_1 \cup T_2$ are both bases

- the first step is polynomial (greedy algorithm)
- the second step is also polynomial (matroid intersection)

By doing so the second agent is guaranteed to have a utility of $\max\{u_2(S_1 \cup S_2), u_2(T_1 \cup T_2)\} \geq \max\{u_2(S_2), u_2(T_2)\} \geq u(B_2^*)/2 = 1/2 > V_2(\alpha_2)$

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# Summary

With the matroid extension of Divide-and-Choose, Agent 1 and 2 can guarantee to themselves $V_2(\alpha_1)$ and $0.5 > V_2(\alpha_2)$, respectively.

| 1 | 2 |
|---|---|
| 0.3 | 0.1 |
| 0.4 | 0.3 |
| 0.1 | 0.2 |
| 0.3 | 0.4 |
| 0.3 | 0.4 |

| | $d_1$ | $d_2$ | $d_3$ |
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| 2 | 0 | 1 | 0 |
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Agent 1 proposes $\{\{1,5\},\{2\}\}$. Agent 2 can partition his base $\{3,4,5\}$ into $\{\{4\},\{3,5\}\}$ such that $\{1,4,5\}$ and $\{2,3,5\}$ are two bases. Agent 2 likes them equally; suppose $\{1,4,5\}$ is returned. The two agents have utility 0.9

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# Outline

Allocation of goods

Matroids

Our problem(s)

Divide-and-Choose : 2 agents

Divide-Ask-and-Choose : 3 agents

Wrap-up

# A protocol for 3 agents on a matroid

1. Agent 1 computes a base and cuts it in 3 parts $A_1 \cup A_2 \cup A_3$
2. Agent 2 chooses one part, say $A_i$, and asks Agent 3 if he agrees to "give" this part to Agent 1
3. If Agent 3 agrees then $A_i$ is in the final solution and Agent 2 and 3 apply Divide-and-Choose on the contraction of $\mathcal{M}$ by $A_i$
4. Otherwise, Agents 1 and 2 apply Divide-and-Choose on the contraction of $\mathcal{M}$ by $A_i$; $R$ denotes the resulting independent set
5. Agent 3 completes $R$ into a base of $\mathcal{M}$

### Theorem

Using Divide-Ask-and-Choose, Agents 1, 2 and 3 can guarantee to themselves $V_3(\alpha_1)$, $\frac{2}{3}V_2(\alpha_2) \geq V_3(\alpha_2)$ and $1/3 \geq V_3(\alpha_3)$, respectively.

This result partially relies on the following lemma

### Multiple exchange property (Greene & Magnanti 1975)

Let $A$ and $B$ be bases of a matroid $\mathcal{M}$ such that $A$ is partitioned into $\{A_1, \ldots, A_n\}$. Then there exists a partition of $B$ into $\{B_1, \ldots, B_n\}$ such that $(A - A_i) \cup B_i$ is a base for all $i \in [n]$.

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The fair allocation of (in)divisible goods is a challenging problem with a lot of applications

We work with matroids, a structure that extends the allocation of indivisible goods

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- centralized (any number of agents)
- decentralized (limited number of agents, $n \leq 8$)

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THANK YOU FOR YOUR ATTENTION