Location Games on Networks

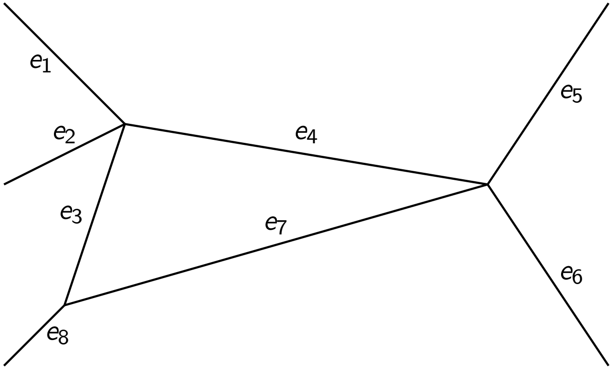
Gaëtan FOURNIER

IAST, TSE

Games and Optimization, November 2016

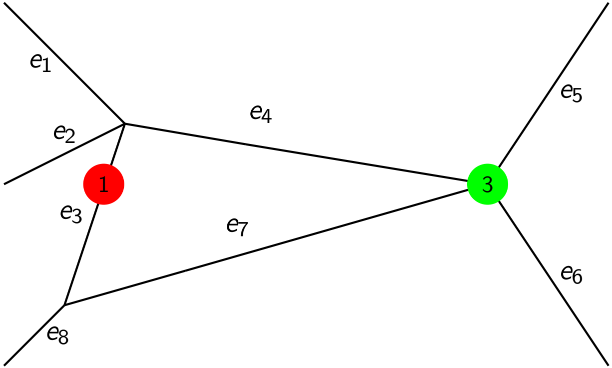
Hypothesis on buyers

1. Non-strategic continuum of buyers, distributed on a network generated by a metric graph.
2. They buy a given quantity of a good whose price is fixed: they shop to the closest location.



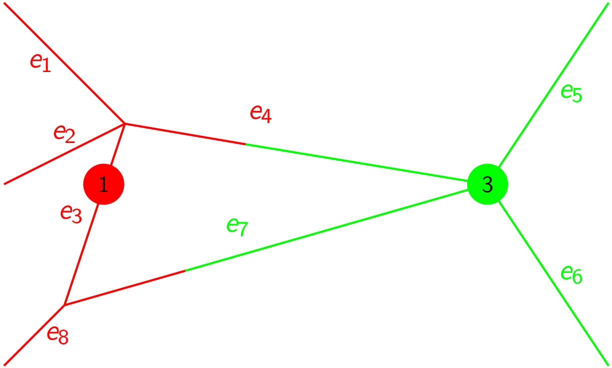
Hypothesis on sellers (= players)

1. A fixed number of strategic sellers simultaneously choose their locations.
2. They want to sell as much as possible.



Hypothesis on sellers (= players)

1. A fixed number of strategic sellers simultaneously choose their locations.
2. They want to sell as much as possible.



The unit interval [Eaton-Lipsey,1975]

1. For $n = 2$, there exists a pure Nash equilibrium.
2. For $n = 3$, there is no pure Nash equilibrium.
3. For $n \geq 4$,there exists a pure Nash equilibrium.

The star $S_k(r)$

1. For $n \leq k$, there exists a pure Nash equilibrium.
2. For $n \in [k + 1, 3k - 2]$, there is no pure Nash equilibrium.
3. For $n \geq 3k - 1$, there exists a pure Nash equilibrium.

Results with uniform density (Fournier-Scarsini[2015])

- Existence of pure Nash equilibrium for any graph when the number of player is large enough.

Theorem: Fournier-Scarsini [2015]

On any finite graph, Hotelling games admit a pure Nash equilibrium, provided the number of players is larger than

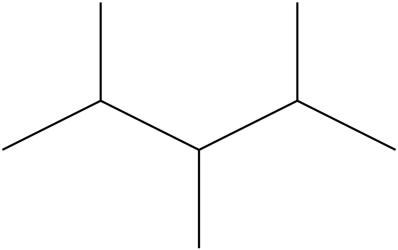
$$N := 3\,\text{card}(E) + \sum_{e \in E} \left\lceil \frac{5\lambda(e)}{\lambda^\star} \right\rceil$$

where $\lambda^\star = \min_E \lambda$ (the length of the shortest edge).

Extensive analysis (also with small number of players): not easy.

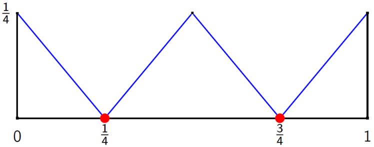
Counterexample [D. Palvolgyi]

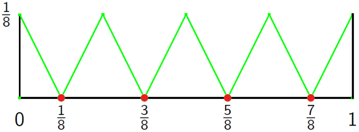
1. For $n = 2$, there exists a pure Nash equilibrium.
2. For $n \in \{3, 4\}$, there is no pure Nash equilibrium.
3. For $n \in \{5, 6\}$, there exists a pure Nash equilibrium.
4. For $n \in \{7, \ldots, 16\}$, there is no pure Nash equilibrium.
5. For $n = 17$, there exists a pure Nash equilibrium.



- Existence of pure Nash equilibrium for any graph when the number of player is large enough.
- Efficiency of these equilibria: not for the players (sellers) but for the consumers.

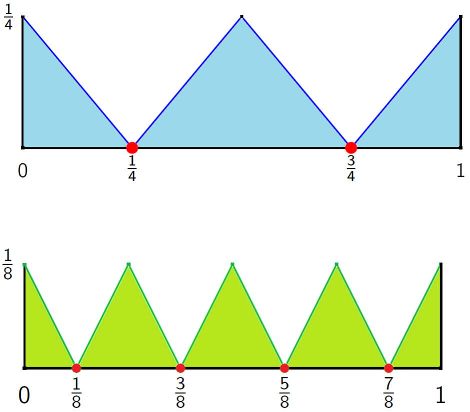
Efficiency of the equilibria





Traveling distances of consumers, in equilibrium and in social optimum.

Equilibrium social cost: ?
Optimum social cost: ?



Social costs in equilibrium and in social optimum.

Equilibrium social cost: $\frac{1}{8}$
Optimum social cost: $\frac{1}{16}$

- For $\boldsymbol{x} \in S^n$, the **social cost** $\sigma(\boldsymbol{x})$ is given by:

$$\sigma(\boldsymbol{x}) := \int_S \min_{i \in \{1,\dots,n\}} d(x_i, y) dy$$

- **The price of anarchy** is given by:

$$\text{PoA}(n) := \frac{\max_{\boldsymbol{x} \in \mathcal{E}_n(\mathcal{H})} \sigma(\boldsymbol{x})}{\min_{\boldsymbol{x} \in S^n} \sigma(\boldsymbol{x})},$$

- **The price of stability** is given by:

$$\text{PoA}(n) := \frac{\min_{\boldsymbol{x} \in \mathcal{E}_n(\mathcal{H})} \sigma(\boldsymbol{x})}{\min_{\boldsymbol{x} \in S^n} \sigma(\boldsymbol{x})},$$

where $\mathcal{E}_n(\mathcal{H})$ is the set of equilibrium with n players.

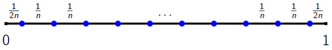


Figure: Social optimum $\bar{x}$ with n players.

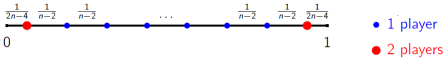


Figure: Best equilibrium $\check{x}$ with n players.

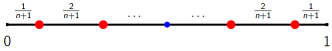


Figure: Worst equilibrium $\hat{x}$ with n players (n odd)

Figure: Worst equilibrium $\hat{x}$ with n players (n even).

On the unit interval, we have:

$$\text{PoA}(n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 2\left(\dfrac{n}{n+1}\right) & \text{if } n > 3 \text{ is odd.} \end{cases}$$

For $n \geq 4$

$$\text{PoS}(n) = \frac{n}{n-2}$$

- Efficiency of these equilibria: not for the players (sellers) but for the consumers.

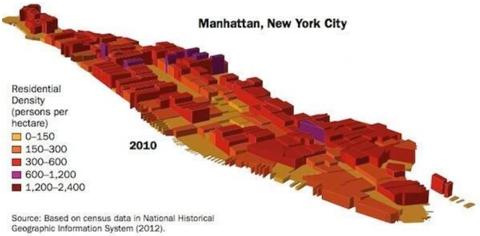
Theorem

Suppose that the game $\mathcal{H}(n, G)$ has an equilibrium. Then

$$\mathrm{PoA}(n) \underset{n \to +\infty}{\longrightarrow} 2$$

$$\mathrm{PoS}(n) \underset{n \to +\infty}{\longrightarrow} 1$$

General distributions of consumers [Fournier 2015]



For a subset $A \subset G$, the quantity of consumers located in A is:

$$\int_A g(x) d\mathcal{L}(x)$$

where $\mathcal{L}$ is the Lebesgue measure, and $g > 0$.

Counterexamples with general distribution of consumers

Counterexamples

(1) For any $\epsilon > 0$, the function $g : [0, 1] \to \mathbb{R}^+$:

$$g := x \mapsto 1 + \epsilon x$$

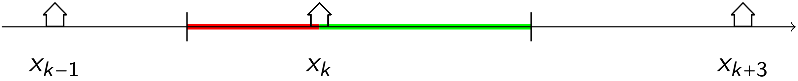
is arbitrary close to **1** but the game $\mathcal{H}(n, [0, 1], g)$ doesn't admit any Nash equilibrium in pure strategies for $n > 2$.

(2) In fact, when the number of player is small, the class of distributions such that the corresponding Hotelling games admit an exact pure Nash equilibrium is small.

- Suppose that $\boldsymbol{x} = (x_1, \ldots, x_n)$ is an equilibrium $(x_1 \leq \cdots \leq x_n)$ in the game $\mathcal{H}(n, [0, 1], f)$. We first claim that all players are coupled, i.e. that $x_1 = x_2 < x_3 = x_4 < \cdots < x_{n-1} = x_n$.

→ Suppose that 3 players (or more) share the same location

$$x_k = x_{k+1} = x_{k+2}$$



$$p_k(x_1, \ldots, x_n) = \frac{1}{3} \int_{{\color{red}\blacksquare} \cup {\color{green}\blacksquare}} (1 + \epsilon x) dx$$

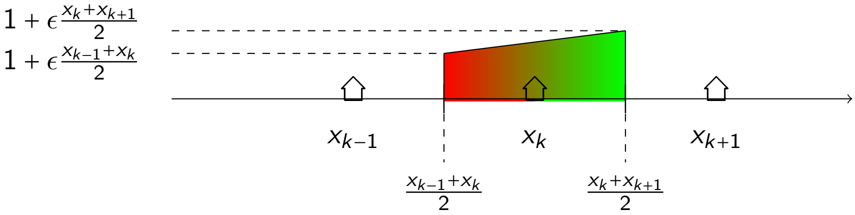
Either:

$$\int_{\color{red}\blacksquare} (1 + \epsilon x) dx > p_k(x_1, \ldots, x_n)$$

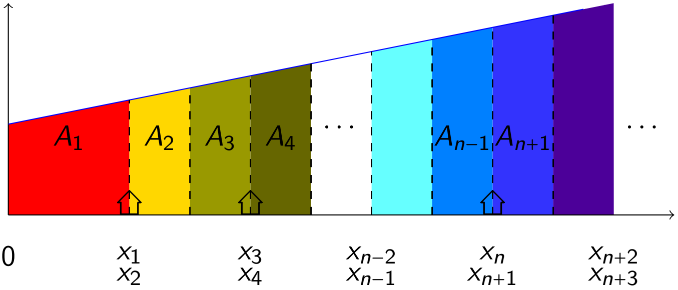
or:

$$\int_{\color{green}\blacksquare} (1 + \epsilon x) dx > p_k(x_1, \ldots, x_n)$$

→ Suppose now that there exists a location $x_k \in [0, 1]$ with a single player k. His payoff is equal to a right trapezoid's area:



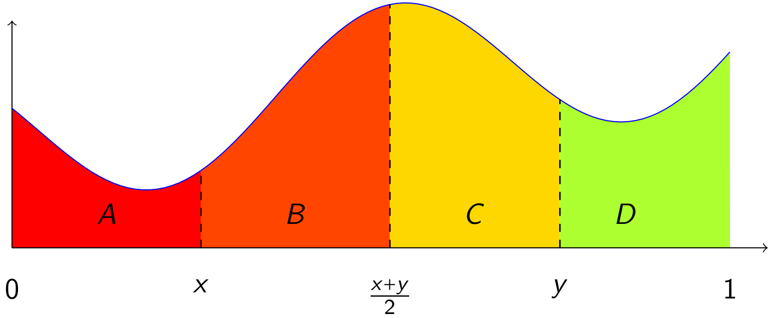
- All players are coupled.



- $p_1(\boldsymbol{x}) = p_2(\boldsymbol{x}) = \frac{A_1+A_2}{2} = A_1 = A_2$
  - □ $A_1 > A_2 \Rightarrow$ player 1 has a profitable deviation: $x_1 - \delta$.
  - □ $A_2 > A_1 \Rightarrow$ player 1 has a profitable deviation: $x_1 + \delta$.
- Same: $p_3(\boldsymbol{x}) = p_4(\boldsymbol{x}) = A_3 = A_4$
- $A_2 = A_3$ ↯
  - □ $A_2 < A_3 \Rightarrow$ player 2 has a profitable deviation: $x_3 - \delta$.
  - □ $A_2 > A_3 \Rightarrow$ player 3 has a profitable deviation: $x_2 + \delta$.

Counterexample 2

There exists a pure Nash equilibrium on the unit interval with 4 players and with density g if and only if g satisfies $Q_{\frac{1}{2}} = \frac{Q_{\frac{1}{4}} + Q_{\frac{3}{4}}}{2}$



$$A = B = C = D \Rightarrow x = Q_{\frac{1}{4}}, \quad y = Q_{\frac{3}{4}}, \quad \frac{x+y}{2} = Q_{\frac{1}{2}}$$

Definition

A profile of actions $\boldsymbol{x} := (x_1, \dots, x_n)$ is a pure additive ϵ-equilibrium of $\mathcal{H}(n, G, g)$ if and only if for all $i \in \{1, \dots, n\}$ and all $y \in G$, we have:

$$p_i(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) - p_i(\boldsymbol{x}) \leq \epsilon$$

Definition

A profile of actions $\boldsymbol{x} := (x_1, \dots, x_n)$ is a pure multiplicative ϵ-equilibrium of $\mathcal{H}(n, G, g)$ if and only if for all $i \in \{1, \dots, n\}$ and for all $y \in G$ we have:

$$p_i(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \leq (1 + \epsilon) p_i(\boldsymbol{x})$$

Asymptotic existence of ϵ-equilibrium.

Suppose that:

1. g is K-Lipschitz
2. There exist m and M such that for all x, $0 < m \leq g(x) \leq M$

Then:

$$\forall \epsilon > 0,\ \exists N(\epsilon) \in \mathbb{N},\ \forall n \geq N(\epsilon),$$

there exists an $\epsilon-$ pure equilibrium in the game with n players and density distribution g.

$$N(\epsilon) \sim \frac{1}{\epsilon}$$

Sketch of the proof

1/ We approximate the density function g by a step function $\hat{g}(\epsilon_1)$, where ϵ_1 is a parameter playing a role in the length of the steps. Because g is K-Lipschitz, the step function $\hat{g}(\epsilon_1)$ is such that $\|g - \hat{g}(\epsilon_1)\|_\infty \leq \epsilon$ when ϵ_1 is small enough.

2/ We prove that there exists an (exact) equilibrium in pure strategies in the game $\mathcal{H}(n, G, \hat{g}(\epsilon_1))$, when the number of players n is larger than a lower-bound $N(\epsilon_1)$. This lower bound increases when ϵ_1 goes to zero.

3/ If ϵ_1 is small enough, the equilibrium constructed in the previous step is a multiplicative ϵ-equilibrium in the game $\mathcal{H}(n, S, g)$.
We obtain therefore a lower bound $N(\epsilon)$ on the number of players n that guarantees the existence of a pure multiplicative ϵ-equilibrium in $\mathcal{H}(n, G, g)$.

Probabilistic Hotelling game [Fournier 2016]

Motivation: find a continuous version of the Hotelling game.

1. n players chose a location in $[0,1]$.
2. The probability that consumer $t \in [0,1]$ shops to location x_k is equal to:

$$\frac{f(|x_k - t|)}{\sum_{i=1}^{n} f(|x_i - t|)}$$

   for a given positive and decreasing function f.
3. The payoff of a player k is equal to:

$$\pi_k(x_1, \ldots, x_n) = \int_0^1 \frac{f(|x_k - t|)}{\sum_{i=1}^{n} f(|x_i - t|)} g(t) dt$$

Theorem

Suppose that f is C^2, symmetric, strictly positive, decreasing and concave. Then there exists a symmetric equilibrium in pure strategies. This equilibrium is $(x, \ldots, x)$, where x satisfies:

$$\int_0^1 \frac{f'(x-t)g(t)}{f(x-t)} dt = 0 \tag{1}$$

i.e:

$$\left[\frac{f'}{f} * g\right](x) = 0 \tag{2}$$

Remark: x doesn't depend on the number of players.

- Simple cases: if $g = \mathbf{1}$, then $x = \frac{1}{2}$
- In general: either compute $\frac{f'}{f} * g$, or because $\pi_k(x)$ is strictly concave in x_k, the best response dynamics is well defined.

$f[x] = (1 - x^2)$ and $g[x] = x$

0.2	0.4	0.6	0.8
0.661548	0.665754	0.670023	0.674283
0.657028	0.657122	0.657218	0.657313
0.657887	0.657889	0.657891	0.657893
0.657842	0.657842	0.657842	0.657842
0.657845	0.657845	0.657845	0.657845

$f[x] = (1 - x^2)$ and $g[x] = x^2$

0.2	0.4	0.6	0.8
0.748681	0.751422	0.754195	0.756968
0.741933	0.741976	0.742018	0.742061
0.742493	0.742493	0.742494	0.742494
0.742471	0.742471	0.742471	0.742471
0.742472	0.742472	0.742472	0.742472

$f[x] = Cos[x]$ and $g[x] = x$

0.2	0.4	0.6	0.8
0.665875	0.667724	0.669664	0.671602
0.663837	0.663855	0.663874	0.663893
0.664008	0.664009	0.664009	0.664009
0.664004	0.664004	0.664004	0.664004
0.664005	0.664005	0.664004	0.664005

$f[x] = Cos[x]$ and $g[x] = x^2$

0.2	0.4	0.6	0.8
0.750645	0.751845	0.753123	0.754417
0.747649	0.747657	0.747665	0.747674
0.747758	0.747758	0.747758	0.747758
0.747756	0.747756	0.747756	0.747756
0.747756	0.747756	0.747756	0.747756

Open questions:

(1) Can we approximate the standard Hotelling game with a sequence of continuous games? We can't with concave function f.
(2) Impact of price competition?
(3) Voting models
(4) What if consumers do not always buy?

Theorem

Suppose that consumers are distributed on the real line according to $\mathcal{N}(0,\sigma)$ and that they buy iif $v - p - f(d) \geq 0 \Leftrightarrow d \leq \delta$. This game always has at least one equilibrium in pure strategies:

(1) $\frac{\delta}{\sigma} < \sqrt{\frac{\ln 2}{2}}$,

$$NE = \{(t_A, t_A + 2\delta) \mid t_A \in [\alpha,\beta] \in [-2\delta, 0]\}$$

(2) $\sqrt{\frac{\ln 2}{2}} < \frac{\delta}{\sigma} < \sqrt{2\ln 2}$ then

$$NE = \{(-t_A, t_A) \mid t^A = \sigma\sqrt{2\ln 2} - \delta \in [0,\delta]\}$$

(3) $\sqrt{2\ln 2} < \frac{\delta}{\sigma}$ then

$$NE = \{(0,0)\}$$

Thank you

Stochastic dominance / Majorization

For a vector $\boldsymbol{z} = (z_1, \ldots, z_n)$, we denote $z_{[1]} \geq \cdots \geq z_{[n]}$ its decreasing rearrangement.

Definition

Let $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ be such

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

if, for all $k \in \{1, \ldots, n\}$

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}.$$

then we say that $\boldsymbol{x}$ is *majorized* by $\boldsymbol{y}$ ($\boldsymbol{x} \prec \boldsymbol{y}$).

Definition

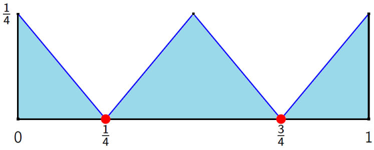
A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said **Schur-convex** if $\mathbf{x} \prec \mathbf{y}$ implies $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$.

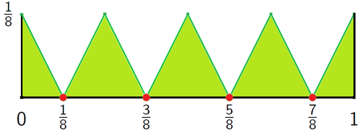
Proposition

If $\psi : \mathbb{R} \to \mathbb{R}$ is a convex function,

$$\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \psi(x_i),$$

then ϕ is Schur-convex.





Suppose that f is C^2, strictly concave, and bounded below by $\epsilon > 0$.

Then the function $x_k \mapsto \pi_k(x_k, x^{-k})$ is concave and (jointly) continuous.

It follows from the **Nash-Glicksberg theorem** that the game admits an equilibrium in pure strategies.

Moreover, due to the symmetry of the problem, there exists at least one symmetric pure Nash equilibrium.