Location Games on Networks

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Games and Optimization, November 2016
Hypothesis on buyers

1. Non-strategic continuum of buyers, distributed on a network generated by a metric graph.
2. They buy a given quantity of a good whose price is fixed: they shop to the closest location.
Hypothesis on sellers (= players)

1. A fixed number of strategic sellers simultaneously choose their locations.
2. They want to sell as much as possible.
Hypothesis on sellers (= players)

1. A fixed number of strategic sellers simultaneously choose their locations.
2. They want to sell as much as possible.
The unit interval [Eaton-Lipsey,1975]

1. For $n = 2$, there exists a pure Nash equilibrium.
2. For $n = 3$, there is no pure Nash equilibrium.
3. For $n \geq 4$, there exists a pure Nash equilibrium.

The star $S_k(r)$

1. For $n \leq k$, there exists a pure Nash equilibrium.
2. For $n \in [k + 1, 3k - 2]$, there is no pure Nash equilibrium.
3. For $n \geq 3k - 1$, there exists a pure Nash equilibrium.
Results with uniform density (Fournier-Scarsini[2015])

- Existence of pure Nash equilibrium for any graph when the number of player is large enough.

**Theorem: Fournier-Scarsini [2015]**

On any finite graph, Hotelling games admit a pure Nash equilibrium, provided the number of players is larger than

\[
N := 3 \text{card}(E) + \sum_{e \in E} \left\lceil \frac{5 \lambda(e)}{\lambda^*} \right\rceil
\]

where \( \lambda^* = \min_{E} \lambda \) (the length of the shortest edge).
Extensive analysis (also with small number of players): not easy.

Counterexample [D. Palvolgyi]

1. For $n = 2$, there exists a pure Nash equilibrium.
2. For $n \in \{3, 4\}$, there is no pure Nash equilibrium.
3. For $n \in \{5, 6\}$, there exists a pure Nash equilibrium.
4. For $n \in \{7, \ldots, 16\}$, there is no pure Nash equilibrium.
5. For $n = 17$, there exists a pure Nash equilibrium.
Existence of pure Nash equilibrium for any graph when the number of player is large enough.

Efficiency of these equilibria: not for the players (sellers) but for the consumers.
Efficiency of the equilibria

Traveling distances of consumers, in equilibrium and in social optimum.

Equilibrium social cost: ?
Optimum social cost: ?
Social costs in equilibrium and in social optimum.

Equilibrium social cost: $\frac{1}{8}$
Optimum social cost: $\frac{1}{16}$
For $x \in S^n$, the **social cost** $\sigma(x)$ is given by:

$$\sigma(x) := \int_s \min_{i \in \{1, \ldots, n\}} d(x_i, y) dy$$

**The price of anarchy** is given by:

$$PoA(n) := \frac{\max_{x \in \mathcal{E}_n(\mathcal{H})} \sigma(x)}{\min_{x \in S^n} \sigma(x)},$$

**The price of stability** is given by:

$$PoA(n) := \frac{\min_{x \in \mathcal{E}_n(\mathcal{H})} \sigma(x)}{\min_{x \in S^n} \sigma(x)},$$

where $\mathcal{E}_n(\mathcal{H})$ is the set of equilibrium with $n$ players.
On the unit interval, we have:

\[ \text{PoA}(n) = \begin{cases} 
2 & \text{if } n \text{ is even}, \\
2 \left( \frac{n}{n+1} \right) & \text{if } n > 3 \text{ is odd.} 
\end{cases} \]

For \( n \geq 4 \)

\[ \text{PoS}(n) = \frac{n}{n-2} \]
Efficiency of these equilibria: not for the players (sellers) but for the consumers.

Theorem

Suppose that the game $\mathcal{H}(n, G)$ has an equilibrium. Then

$$\text{PoA}(n) \xrightarrow[n \to +\infty]{} 2$$

$$\text{PoS}(n) \xrightarrow[n \to +\infty]{} 1$$
For a subset $A \subset G$, the quantity of consumers located in $A$ is:

$$\int_A g(x) \, d\mathcal{L}(x)$$

where $\mathcal{L}$ is the Lebesgue measure, and $g > 0$. 

Source: Based on census data in National Historical Geographic Information System (2012).
Counterexamples with general distribution of consumers

Counterexamples

(1) For any $\epsilon > 0$, the function $g : [0, 1] \rightarrow \mathbb{R}^+$:

$$g := x \mapsto 1 + \epsilon x$$

is arbitrary close to 1 but the game $\mathcal{H}(n, [0, 1], g)$ doesn’t admit any Nash equilibrium in pure strategies for $n > 2$.

(2) In fact, when the number of player is small, the class of distributions such that the corresponding Hotelling games admit an exact pure Nash equilibrium is small.

Suppose that $x = (x_1, \ldots, x_n)$ is an equilibrium ($x_1 \leq \cdots \leq x_n$) in the game $\mathcal{H}(n, [0, 1], f)$. We first claim that all players are coupled, i.e. that $x_1 = x_2 < x_3 = x_4 < \cdots < x_{n-1} = x_n$. 
Suppose that 3 players (or more) share the same location

\[ X_k = X_{k+1} = X_{k+2} \]

Either:

\[ \int (1 + \epsilon x) \, dx > p_k(x_1, \ldots, x_n) \]

or:

\[ \int (1 + \epsilon x) \, dx > p_k(x_1, \ldots, x_n) \]

where

\[ p_k(x_1, \ldots, x_n) = \frac{1}{3} \int \bigcup \mathbb{R} (1 + \epsilon x) \, dx \]
Suppose now that there exists a location $x_k \in [0, 1]$ with a single player $k$. His payoff is equal to a right trapezoid’s area:

$$1 + \epsilon \frac{x_k + x_{k+1}}{2}$$

$$1 + \epsilon \frac{x_{k-1} + x_k}{2}$$
All players are coupled.

\[ p_1(x) = p_2(x) = \frac{A_1 + A_2}{2} = A_1 = A_2 \]

\[ A_1 > A_2 \Rightarrow \text{player 1 has a profitable deviation: } x_1 - \delta. \]

\[ A_2 > A_1 \Rightarrow \text{player 1 has a profitable deviation: } x_1 + \delta. \]

Same: \[ p_3(x) = p_4(x) = A_3 = A_4 \]

\[ A_2 = A_3 \Rightarrow \]

\[ A_2 < A_3 \Rightarrow \text{player 2 has a profitable deviation: } x_3 - \delta. \]

\[ A_2 > A_3 \Rightarrow \text{player 3 has a profitable deviation: } x_2 + \delta. \]
Counterexample 2

There exists a pure Nash equilibrium on the unit interval with 4 players and with density \( g \) if and only if \( g \) satisfies

\[
Q_{\frac{1}{2}} = \frac{Q_{\frac{1}{4}} + Q_{\frac{3}{4}}}{2}
\]

A = B = C = D \( \Rightarrow \) \( x = Q_{\frac{1}{4}}, \ y = Q_{\frac{3}{4}}, \ \frac{x + y}{2} = Q_{\frac{1}{2}} \)
### Definition

A profile of actions \( x := (x_1, \ldots, x_n) \) is a pure additive \( \epsilon \)-equilibrium of \( \mathcal{H}(n, G, g) \) if and only if for all \( i \in \{1, \ldots, n\} \) and all \( y \in G \), we have:

\[
p_i(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) - p_i(x) \leq \epsilon
\]

### Definition

A profile of actions \( x := (x_1, \ldots, x_n) \) is a pure multiplicative \( \epsilon \)-equilibrium of \( \mathcal{H}(n, G, g) \) if and only if for all \( i \in \{1, \ldots, n\} \) and for all \( y \in G \) we have:

\[
p_i(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \leq (1 + \epsilon) p_i(x)
\]
Asymptotic existence of $\epsilon$-equilibrium.

Suppose that:

1. $g$ is $K$-Lipschitz
2. There exist $m$ and $M$ such that for all $x$, $0 < m \leq g(x) \leq M$

Then:

$$\forall \epsilon > 0, \\exists N(\epsilon) \in \mathbb{N}, \ \forall n \geq N(\epsilon),$$

there exists an $\epsilon$-pure equilibrium in the game with $n$ players and density distribution $g$.

$$N(\epsilon) \sim \frac{1}{\epsilon}$$
Sketch of the proof

1/ We approximate the density function $g$ by a step function $\hat{g}(\epsilon_1)$, where $\epsilon_1$ is a parameter playing a role in the length of the steps. Because $g$ is $K$-Lipschitz, the step function $\hat{g}(\epsilon_1)$ is such that $\|g - \hat{g}(\epsilon_1)\|_{\infty} \leq \epsilon$ when $\epsilon_1$ is small enough.

2/ We prove that there exists an (exact) equilibrium in pure strategies in the game $\mathcal{H}(n, G, \hat{g}(\epsilon_1))$, when the number of players $n$ is larger than a lower-bound $N(\epsilon_1)$. This lower bound increases when $\epsilon_1$ goes to zero.
3/ If $\epsilon_1$ is small enough, the equilibrium constructed in the previous step is a multiplicative $\epsilon$-equilibrium in the game $\mathcal{H}(n, S, g)$. We obtain therefore a lower bound $N(\epsilon)$ on the number of players $n$ that guarantees the existence of a pure multiplicative $\epsilon$-equilibrium in $\mathcal{H}(n, G, g)$. 
Motivation: find a continuous version of the Hotelling game.

1. $n$ players chose a location in $[0, 1]$.
2. The probability that consumer $t \in [0, 1]$ shops to location $x_k$ is equal to:

$$\frac{f(|x_k - t|)}{\sum_{i=1}^{n} f(|x_i - t|)}$$

for a given positive and decreasing function $f$.
3. The payoff of a player $k$ is equal to:

$$\pi_k(x_1, \ldots, x_n) = \int_0^1 \frac{f(|x_k - t|)}{\sum_{i=1}^{n} f(|x_i - t|)} g(t) dt$$
Theorem

Suppose that $f$ is $C^2$, symmetric, strictly positive, decreasing and concave. Then there exists a symmetric equilibrium in pure strategies. This equilibrium is $(x, \ldots, x)$, where $x$ satisfies:

$$\int_0^1 \frac{f'(x-t)g(t)}{f(x-t)} dt = 0 \quad (1)$$

i.e:

$$\left[ \frac{f'}{f} \ast g \right](x) = 0 \quad (2)$$

**Remark:** $x$ doesn’t depend on the number of players.
Simple cases: if \( g = 1 \), then \( x = \frac{1}{2} \)

In general: either compute \( \frac{f'}{f} * g \), or because \( \pi_k(x) \) is strictly concave in \( x_k \), the best response dynamics is well defined.
Open questions:

(1) Can we approximate the standard Hotelling game with a sequence of continuous games? We can’t with concave function $f$.
(2) Impact of price competition?
(3) Voting models
(4) What if consumers do not always buy?
Theorem

Suppose that consumers are distributed on the real line according to $\mathcal{N}(0, \sigma)$ and that they buy iff $v - p - f(d) \geq 0 \iff d \leq \delta$. This game always has at least one equilibrium in pure strategies:

1. $\frac{\delta}{\sigma} < \sqrt{\frac{\ln 2}{2}}$,

   $$NE = \left\{ (t_A, t_A + 2\delta) \mid t_A \in [\alpha, \beta] \in [-2\delta, 0] \right\}$$

2. $\sqrt{\frac{\ln 2}{2}} < \frac{\delta}{\sigma} < \sqrt{2 \ln 2}$ then

   $$NE = \left\{ (-t_A, t_A) \mid t^A = \sigma \sqrt{2 \ln 2} - \delta \in [0, \delta] \right\}$$

3. $\sqrt{2 \ln 2} < \frac{\delta}{\sigma}$ then

   $$NE = \left\{ (0, 0) \right\}$$
Thank you
Stochastic dominance / Majorization

For a vector \( z = (z_1, \ldots, z_n) \), we denote \( z_{[1]} \geq \cdots \geq z_{[n]} \) its decreasing rearrangement.

**Definition**

Let \( x, y \in [0, 1]^n \) be such

\[
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i
\]

if, for all \( k \in \{1, \ldots, n\} \)

\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i].
\]

then we say that \( x \) is *majorized* by \( y \) (\( x < y \)).
Definition

A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said Schur-convex if $x < y$ implies $\phi(x) \leq \phi(y)$.

Proposition

If $\psi : \mathbb{R} \to \mathbb{R}$ is a convex function,

$$\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \psi(x_i),$$

then $\phi$ is Schur-convex.
Suppose that $f$ is $C^2$, strictly concave, and bounded below by $\epsilon > 0$.

Then the function $x_k \mapsto \pi_k(x_k, x^{-k})$ is concave and (jointly) continuous.

It follows from the **Nash-Glicksberg theorem** that the game admits an equilibrium in pure strategies.

Moreover, due to the symmetry of the problem, there exists at least one symmetric pure Nash equilibrium.