A game theoretic generalized additive model on networks: theory and applications

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Outline

- Our model: *Generalized Additive Games* (GAGs)
- Some theoretical results
- GAGs and Argumentation Theory
- GAGs and Biomedicine

Cooperative games and networks

Cooperative TU-game

A couple (N, v), where N is the set of players and $v : 2^N \to \mathbb{R}$ is the utility function, s.t. $v(\emptyset) = 0$.

- **Coalition formation**: which coalitions of players are going to form?
- **Solution** of a game: $\varphi : \mathcal{G}_N \to \mathbb{R}^n$ s.t. $\varphi_i(v)$ is the amount given to player i in game v.
    - Core, nucleolus
    - Power indices: Shapley value, Banzaf value, semivalues

Network

$\Gamma = (V, E)$ where V is the set of vertices (nodes) and E is the set of edges (links) between couple of nodes.

Graph games

- A cooperative game (N, v):
  - N is the set of nodes in a network
  - the value of each coalition depends on the properties of an underlying network
- **Goal**: extract some information from the network (how to share costs/benefits, which are the influential nodes, community detection and so on)
- Examples from the literature: *airport games, peer games, maintenance cost games, minimum cost spanning tree games, argumentation games*

Generalized Additive Games

Motivation

In many models, the worth of each coalition can be computed from the contributions of single players via an additive operator describing how the individual abilities interact within groups.

Examples:

- airport games (Littlechild and Owen 1973; Littlechild and Thompson 1977)
- maintenance games (Koster 1999)
- peer games (Branzei et al. 2002)
- minimum cost spanning tree games (Moretti et al. 2002)
- connectivity games (Amer et al. 2003, Lindelauf et al. 2013)
- argumentation games (Bonzon et al. 2014)

In some cases, the worth of a coalition $S \subseteq N$ is strongly related to the sum of the individual values over another subset $T \subseteq N$, not necessarily included in S.

Example: *glove game*

- $N = L \cup R$, where the players in L that own a left-hand glove, and those in R with a right-hand glove.
- the worth of a coalition of players $S \subseteq N$ is defined as the number of pairs of gloves owned by the coalition S:

$$v(S) = min\{|S \cap L|, |S \cap R|\}$$

We can represent this game by assigning value 1 to each player and by describing the worth of each coalition S as the sum of single players' values over the smallest subset among $S \cap L$ and $S \cap R$.

A *Generalized Additive Situation* (GAS) is a triple $\langle N, v, \mathcal{M} \rangle$, where:

- N is the set of players;
- $v : N \to \mathbb{R}$;
- $\mathcal{M} : 2^N \to 2^N$ is a coalitional map.

Definition

Given a GAS $\langle N, v, \mathcal{M} \rangle$, the associated *Generalized Additive Game* (GAG) is the TU-game $(N, v^{\mathcal{M}})$ such that $v(\emptyset) = 0$ and for $S \neq \emptyset$:

$$v^{\mathcal{M}}(S) = \sum_{i \in \mathcal{M}(S)} v(i).$$

Example: *simple games*

Let w be a simple game. Then w is described by the GAG associated to $\langle N, v, \mathcal{M} \rangle$ with $v(i) = 1$ for all i and

$$\mathcal{M}(S) = \begin{cases} \{i\} \subseteq S & \text{if } S \in W \\ \emptyset & \text{otherwise} \end{cases}$$

where W is the set of the winning coalitions in w.

An interesting subclass: basic GAGs

Let $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$, where $\mathcal{C}_i = \{F_i^1, \ldots, F_i^m, E_i\}$ is a collection of subsets of N such that $F_i^j \neq \emptyset$ and $F_i^j \cap E_i = \emptyset$ for all $i \in N$ and for all $j = 1, \cdots, m$.

Definition

We denote by $\langle N, v, \mathcal{C} \rangle$ the *basic* GAS associated with the coalitional map $\mathcal{M}$ defined as:

$$\mathcal{M}(S) = \{i \in N : S \cap F_i^1 \neq \emptyset, \ldots, S \cap F_i^m \neq \emptyset, S \cap E_i = \emptyset\}$$

and by $\langle N, v^{\mathcal{C}} \rangle$ the associated *basic* GAG.

- $F_i^1, \ldots, F_i^m$: sets of *friends* of player i;
- E_i: set of *enemies* of player i.

Examples of basic GAGs

We show that several classes of games can be described as basic GAGs:

- airport games ($F_i = N_i$, $E_i = N_{i+1} \cup \ldots \cup N_k$, $v(i) = \frac{c_i}{|N_i|}$)
- argumentation games ($F_i = \{i\}$, $E_i =$ set of attackers , $v(i) =$ worth of argument i)
- some classes of operation research games:
  - peer games ($F_i^1, \cdots F_i^m$ are singleton sets and are superiors of i, $E_i = \emptyset$)
  - (dichotomous) mountain situations ($F_i = \{i\}$, $E_i =$ set of best connections of i, $v(i)$ can be 0 or c)
  - maintenance cost games (F_i, $E_i = \emptyset$)

Maintenance problem

A mantainance problem is a couple (T, t), where:

- T=$(N \cup \{0\}, E)$ is a tree;
- 0 is the root of the tree having only one adjacent edge;
- $t : E \to \mathbb{R}^+$ is a nonnegative cost function on the edges of the tree.

Each vertex $i \in N$ is connected to the root 0 by a unique path P_i; we shall denote by e_i the edge in P_i that is incident to i.

A *precedence relation* $\preceq$ is defined by: $j \preceq i$ if and only if j is on the path P_i.

A *trunk* $R \subseteq N \cup \{0\}$ is a set of vertices which is closed under the relation $\preceq$, i.e. if $i \in R$ and $j \preceq i$, then $j \in R$. The cost of a trunk R is then defined as

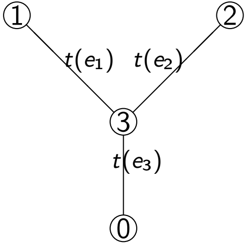
$$C(R) = \sum_{i \in R \setminus \{0\}} t(e_i).$$

Maintenance cost game

The associated *maintenance cost game* (N, c) is defined by

$$c(S) = \min\{C(R) : S \subseteq R \text{ and } R \text{ is a trunk}\}.$$

Example:



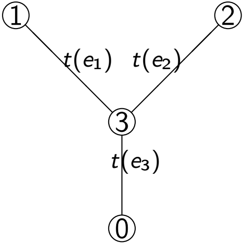
$P_1 = (1, 3, 0)$; $P_2 = (2, 3, 0)$;
$P_3 = (3, 0)$.

$c(\{1\}) = c(\{1, 3\}) = C(\{1, 3\}) = t(e_1) + t(e_3)$;
$c(\{2\}) = c(\{2, 3\}) = C(\{1, 3\}) = t(e_2) + t(e_3)$;
$c(\{3\}) = C(\{3\}) = t(e_3)$;
$c(\{1, 2\}) = c(\{1, 2, 3\}) = C(\{1, 3\}) = t(e_1) + t(e_2) + t(e_3)$

Let $F(i) = \{j \in N | i \preceq j\}$ be the set of *followers* of player i (note that $i \in F(i)$ for each $i \in N$).

Maintenance cost GAS

- $v(i) = t(e_i)$
- $\mathcal{C}_i = \{F_i, E_i\}$ such that $F_i = F(i)$ and $E_i = \emptyset$ for every $i \in N$



$F_1 = \{1\}$; $F_2 = \{2\}$; $F_3 = \{1, 2, 3\}$.

Some results

Results

- We provide a general framework for describing several classes of games studied in the literature that are ascribable to this notion of additivity over individual values
- We study the properties of GAGs and provide a characterization of basic GAGs without enemies
- We provide some results on classical solution concepts for basic GAGs:
  - sufficient conditions for the non-emptiness of the core;
  - concise formula for semivalues with one or two set of friends;
  - formula for semivalues with multiple disjoint sets of friends.

When a GAG can be represented as a basic GAG without enemies?

Proposition

A GAS $\langle N, v, \mathcal{M} \rangle$ be a GAS can be represented as a basic GAS $\langle N, v, \mathcal{C} \rangle$ via collections $\mathcal{C}_i = \{F_i^1, \ldots, F_i^{m_i}, E_i = \emptyset\}$, for each $i \in N$, if and only if $\mathcal{M}$ is monotonic.

Example

Consider a GAS $\langle N, v, \mathcal{M}\rangle$ with $N = \{1, 2, 3, 4\}$ and $\mathcal{M}$ monotonic such that $\mathcal{M}(\{1, 2, 3\}) = \{3\}$, $\mathcal{M}(\{3, 4\}) = \{2, 3\}$, $\mathcal{M}(\{1, 3, 4\}) = \{2, 3, 4\}$, $\mathcal{M}(\{1, 2, 3, 4\}) = \{2, 3, 4\}$, and $\mathcal{M}(S) = \emptyset$ or all other coalitions.

The sets of minimal coalitions are as follows: $\mathcal{S}^{\mathcal{M},1} = \emptyset$, $\mathcal{S}^{\mathcal{M},2} = \{\{3, 4\}\}$, $\mathcal{S}^{\mathcal{M},3} = \{\{1, 2, 3\}, \{3, 4\}\}$, $\mathcal{S}^{\mathcal{M},4} = \{\{2, 3, 4\}\}$.

Such a map can be represented as a basic GAG with no enemies and with the collection of friends:
$F_1^1 = \emptyset$, $\{F_2^1, F_2^2\} = \{\{3\}, \{4\}\}$,
$\{F_3^1, \ldots, F_3^4\} = \{\{1, 4\}, \{2, 4\}, \{3\}, \{3, 4\}\}$,
$\{F_4^1, \ldots, F_4^3\} = \{\{2\}, \{3\}, \{4\}\}$.

Decomposition of basic GAGs

The basic GAG $v^{\mathcal{C}}$ associated with a basic GAS can be decomposed in the following sense:

$$v^{\mathcal{C}} = \sum_{i=1}^{n} v^{\mathcal{C}_i},$$

where for $i = 1, \ldots, n$:

$$v^{\mathcal{C}_i}(S) = \begin{cases} v(i) & \text{if } S \cap E_i = \emptyset, S \cap F_i^k \neq \emptyset, k = 1, \ldots, m \\ 0 & \text{otherwise.} \end{cases}$$

Non-emptiness of the core

Proposition

Let $\langle N, v, \mathcal{M}\rangle$ be a GAS such that $v \in \mathbb{R}_+^N$ and

- *for every i there is j such that $F_i^j = \{i\}$*
- *$E_i = \emptyset$ for all i.*

Then, the core of the associated (reward) GAG $(N, v^{\mathcal{M}})$ is non-empty.

Proposition

Let $\langle N, v, \mathcal{C} \rangle$ *be a basic GAS with* $\mathcal{C}_i = \{F_i^1, \dots, F_i^m, E_i = \emptyset\}$ *and* $v(i) \geq 0$ *for each* $i \in N$*. Suppose there exists a coalition* $S \subseteq N$*,* $S \neq \emptyset$*, satisfying the following two conditions:*
i) $S \subseteq I_i$ *for each* $i \in \mathcal{I}$*;*
ii) for each $i \in N \setminus \mathcal{I}$*, there exists* $k \in \{1, \dots, m\}$ *s.t.* $F_i^k = S$*,*
where $\mathcal{I}$ *is the set of players that have at least one singleton among their sets of friends.*
Define the equal split allocation among players in S *as the vector* y *such that*

$$y = e^S \frac{v^{\mathcal{C}}(N)}{s},$$

where s *is the cardinality of* S *and where* $e^S \in \{0,1\}^N$ *is such that* $e_k^S = 1$*, if* $k \in S$ *and* $e_k^S = 0$*, otherwise. The allocation* y *is in the core of the corresponding GAG* $v^{\mathcal{C}}$ *if and only if*

$$v^{\mathcal{C}}(N) \geq s \sum_{i \in N \setminus \mathcal{I}} v(i).$$

Formulas for the semivalues

Proposition

Let us consider a basic GAS $\langle N, v, \{\mathcal{C}_i = \{F_i, E_i\}\}_{i \in N} \rangle$*. Then the Shapley and Banzhaf values for the game* $v^{\mathcal{C}_i}$ *are given, respectively, by:*

$$\sigma_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if} \quad j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{f+e} & \text{if} \quad j \in F_i \\ -v(i)\frac{f}{e(f+e)} & \text{if} \quad j \in E_i \end{cases}$$

and

$$\beta_j(v^{\mathcal{C}_i}) = \begin{cases} 0 & \text{if} \quad j \in N \setminus (F_i \cup E_i) \\ \frac{v(i)}{2^{f+e-1}} & \text{if} \quad j \in F_i \\ -v(i)\frac{2^f - 1}{2^{f+e-1}} & \text{if} \quad j \in E_i. \end{cases}$$

Theorem

Consider a GAS situation $\langle N, v, \{F_i^1, \ldots, F_i^m, E_i\}_{i \in N}\rangle$ *with* $F_i^j \cap F_i^k = \emptyset$ *for all* $i \in N$ *and* $j, k = 1, \cdots, m$, $j \neq k$. *For all* $j \in N \setminus (F \cup E_i)$, *we have that* $\pi_j^{\mathbf{p}}(v^{\mathcal{C}_i}) = 0$.

Take $j \in F_i^b$, *with* $b \in \{1, \ldots, m\}$. *Then* $\pi_j^{\mathbf{p}}(v^{\mathcal{C}_i})$ *is equal to the following expression:*

$$v(i) \sum_{(k_i^{\mathbf{1}}, \ldots, k_i^{b-\mathbf{1}}, 0, k_i^{b+\mathbf{1}}, \ldots, k_i^m) \in \Gamma} \sum_{l=0}^{n-e-f} \binom{f_i^{\mathbf{1}}}{k_i^{\mathbf{1}}} \times \ldots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l} p_{h+l}$$

where $e = |E_i|$ *and* $h = \sum_{j=1}^m k^j$. *Now, take* $j \in E_i$. *Then*

$$\pi_j^{\mathbf{p}}(v^{\mathcal{C}_i}) = -v(i) \sum_{(k_i^{\mathbf{1}}, \ldots, k_i^m) \in \Gamma} \sum_{l=0}^{n-t-f} \binom{f_i^{\mathbf{1}}}{k_i^{\mathbf{1}}} \times \ldots \times \binom{f_i^m}{k_i^m} \times \binom{n-e-f}{l} p_{h+l}.$$

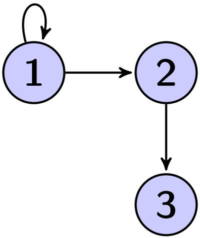
Applications

- **Argumentation Theory**: Basic GAGs and Shapley value to provide a game-theoretical interpretation of a conflict index in an argumentation framework.
- **Biomedicine**: Basic GAGs and Shapley value to measure the relevance of genes in a biological network.

GAGs and Argumentation Theory

An **argumentation graph A** $= \langle A, R \rangle$ is a directed graph:

- nodes are arguments
- edges represent attacks between pairs of arguments



Semantics and acceptability of arguments

Semantics are criteria to determine which arguments are acceptable in an argumentation framework (Dung et al.1995, Caminada et al. 2008).

- Game theory has been recently used to define gradual semantics
- *Acceptability* is not the only attribute studied from a gradual perspective

Motivation

Ranking arguments according to their *controversiality* by using a game-theoretical framework.

Disagreement and controversiality of arguments

Disagreement measure in an argumentation graph through an axiomatic approach (Amgoud 2015).

Results

- We show that the properties can be reformulated for single arguments.
- We define a *conflict index* that satisfies those properties.
- We show that the conflict index may be re-interpreted in terms of a classical solution for coalitional games.

Our approach

- We define a cooperative game on an argumentation graph, where the value of a coalition $S \subseteq A$ expresses the total disagreement within the coalition:

$$v(S) = \frac{max - D(S)}{max - min},$$

where $D(S) = \sum_{i,j \in S} d_{i,j}$, $max = n^2(n+1)$, and $min = n^2$.

The conflict in a coalition of arguments inversely depends on the distance among arguments.

Our approach

- We show that the so-defined game is representable in terms of basic GAGs.
- We show that the Shapley value of such game coincides with the *conflict index* that measures the controversiality of arguments:

$$K_i(\mathbf{A}) = \frac{1}{(max - min)}(\frac{max}{n} - \varphi_i),$$

where $\varphi_i = \frac{1}{2} \sum_{j \in A \setminus i} d_{i,j} + \frac{1}{2} \sum_{j \in A \setminus i} d_{j,i} + d_{i,i}$.

Game theoretical interpretation of the measure of controversiality: the average marginal contribution of each argument to the disagreement induced by all possible coalitions of arguments.

GAGs and Biomedicine

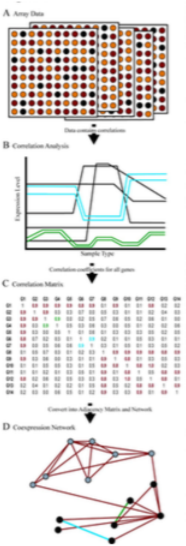
Motivation

Problem: assessing the relevance of genes in a biological network.

- To understand the interaction mechanisms between genes, proteins and other molecules within a cell, described by *gene regulatory networks* or *gene regulatory pathways*.
- To understand the function of genes in determining a certain biological condition of interest, such as the onset of a disease.

Gene co-expression networks

- **Gene network**: the nodes are the genes under analysis and the edges represent the *interactions* among genes.
- **Co-expression network**: a network is constructed from microarray gene expression data, where the level of interaction is expressed by the Pearson correlation between the gene expression profiles.



Centrality in gene co-expression networks

- **Centrality analysis**: centrality measures represent an important tool for the interpretation of the interaction of genes in a co-expression network.
- Recently, several measures based on **cooperative games** have been successfully applied to different kinds of biological networks.

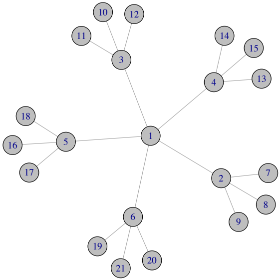
We propose a game theoretical relevance measure based on the model of basic GAGs.

Our approach

Objective: to measure the potential of a gene in preserving the regulatory activity in a biological network and acting as intermediary between hub nodes and leaf nodes.

Target genes:

- through their connections, they are able to influence the expression of the highest number of genes in the network
- when removed (or inhibited), they break down the regulatory activity of the network, by leaving many nodes isolated or in small connected components



Our approach

- We introduce a basic GAG with a biological interpretation, where the valuable players for a coalition are those in its neighbourhood in the gene network $\langle N, E \rangle$:

$$v_E^k(S) = \sum_{j \in S \cup N_S(E)} k_j$$

where $k \in \mathbb{R}^N$ is a vector that specifies the a priori importance of each gene.

v_E^k is the basic GAG associated to the GAS $\langle N, v, \{\mathcal{C}_i\}_{i \in N} \rangle$, where $v(i) = k_i$ for every $i \in N$ and $\mathcal{C}_i = \{F_i = N_i(E), E_i = \emptyset\}$ $\forall i \in N$.

Our approach

Our model generalizes the game introduced in Suri et al. (2008) for determining the "top-k nodes" in a co-authorship network.

- The parameter vector k allows for an a priori ranking of the genes according to their importance, while in the previous model by Moretti et al. (2010) only key-genes were considered.
- Moreover, it allows to combine the game-theoretical approach with other tools from network analysis to assess the relevance of genes.

A relevance index for genes

We propose the Shapley value of such a game as a new relevance index for genes in gene co-expression networks.

Results

- The index is characterized by a set of axioms with biological intepretation
- The index is easily computable and has a straightforward interpretation:

$$\rho_i(v_E^k) = \sum_{j \in (N_i(E) \cup \{i\})} \frac{k_j}{d_j(E)+1}$$

An application to real data

- Network (2242 genes and 25878 links) generated from a gene expression dataset of patients affected by lung cancer.
- Analysis with three different choices of the parameter k:
  - first analysis: $(k_i = 1 \forall i \in N)$
  - second analysis: 23 known key-genes ($k_i = 1$ if i is a key-gene, $k_i = 0$ otherwise)
  - third analysis: k_i is the number of clusters to whom gene i belongs, found by ClusterONE (algorithm that captures overlapping clusters of genes in a network)

Results

A preliminary analysis on the real dataset and the comparison with classical centrality measures show some interesting results.

Conclusions & future work

- We provide a general framework for describing several classes of games studied in the literature
- We study the properties of GAGs and provide a characterization of basic GAGs without enemies
- We provide some results on classical solution concepts for basic GAGs
- We present two approaches using basic GAGs to two very different fields: Argumentation Theory and Biomedicine

Future work

To study the coalition formation process (hedonic games based on basic GAGs).

- "C.G., Lucchetti R., Moretti S. *Generalized Additive Games*, To appear in: International Journal of Game Theory".
- "C.G., Fossati F., Moretti S., *A conflict index for arguments in an argumentation graph*, Submitted to the 2^{nd} European Conference on Argumentation (ECA2017)".
- "C.G., Algaba E., Moretti S., Nepomuceno J.A., *A game theoretic neighborhood-based relevance index to evaluate nodes in gene co-expression networks*, Submitted to: Computer Methods and Programs in Biomedicine".
- "C.G., Ferrari M.M., *On the Position Value for Special Classes of Networks*, Recent Advances in Game Theory and Applications, Springer International Publishing, 29-47 (2016)".