The Aggregate-Monotonic Core.

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Abstract

The main objective of the paper is to study the locus of all core selection and aggregate monotonic point solutions: the aggregate-monotonic core. Furthermore it is shown that any allocation in this set induces a special core selection and aggregate monotonic point solution. Further, the aggregate monotonic core is characterized by means of aggregate monotonicity for set solutions. To finish, we determine the class of games for which the core and the aggregate-monotonic core coincide.

1 Introduction.

The core of a cooperative TU-game, Gillies (1953, 1959), is one of the most important and intuitive solution concepts. Roughly speaking, the core is the set of feasible outcomes that can not be improve upon by any coalition of players. From Bondareva (1963) and Shapley (1967), we know algebraic conditions of the characteristic function that guarantee its non-emptiness. Since the core of a cooperative game may be empty, its generalizations and modifications have been taken into account from the very beginning.

If one is interested in solutions, as we shall be, the core selection property seems natural to be requested. It says that if a game has a non-empty core, then the proposed solution has to belong to it. From the main two generic single point solutions, the (pre)nucleolus (Schmeidler, 1969) satisfies this property, while the Shapley value (Shapley, 1953) does not.

Another important and not too demanding property for a solution concept is aggregate monotonicity, introduced by Megiddo (1974). Roughly speaking it says that everybody will be weakly better off if efficiency grows.

The Shapley value satisfies this property, in fact it satisfies a stronger version: coalitional monotonicity, which is aggregate monotonicity extended to all coalitional values. The nucleolus does not satisfy aggregate monotonicity, not even on the class of convex games (Hokari, 2000).

From the above results there seems to be some kind of incompatibility between monotonicity properties and core selection for solutions concepts. In fact, such an incompatibility was shown by Young (1985) and Housman and Clark (1998). It is proved that there is no point solution concept in the whole class of cooperative TU-games satisfying core selection and coalitional monotonicity properties for $n \geq 4$.

This paper is devoted to analyze the behavior of point solution concepts when we combine aggregate monotonicity and core selection properties. Neither the nucleolus nor the Shapley value satisfy both properties together. Nevertheless, aggregate monotonicity and core selection are compatible in the class of cooperative TU$-$games independently of the number of agents involved. This is important, and can be checked by looking at the per-capita prenucleolus (Grotte, 1970), a variant of the classical prenucleolus, defined by means of the per-capita excesses instead of the classical excesses. This point solution concept satisfies core selection and aggregate monotonicity (see for example Moulin, 1988 or Young et al., 1982). From its existence and also from the importance of the requirements, it seems natural to analyze the locus of all aggregate monotonic point solutions which are core selections.

In fact, this two properties are independent but also mutually conditioned, in the sense that there may be core elements not selected by point solutions satisfying both properties. Intuitively, if one faces an arbitrary game, increasing or decreasing only its efficiency level, will get a special efficiency level, from which any level of efficiency gives rise to a balanced game. Any point solution having the core selection property must pick out a core element at this particular level of efficiency. Then, and imposing aggregate monotonicity, one can push this core element up or down to get the original efficiency level. Point solutions candidates to satisfy core selection and aggregate monotonicity are those attainable from the core of the minimum balancedness level of efficiency game imposing aggregate monotonicity.

The set formed for all these points is always well defined for any cooperative game, and consists of a subset of the core of the original game whenever it is balanced. Such a set has been already used only for balanced games and from a dynamic point of view in Dementieva (2004). In particular they take care of time consistency of solutions for dynamic cooperative games. We call this set: the aggregate-monotonic core and we study it in section 2. We prove that this is the locus of all aggregate monotonic point solutions which are core selections. Furthermore, this set is characterized by means of aggregate monotonicity for set solutions. In section 3, we determine those games for which the core and the aggregate-monotonic core coincide; that is, those games for which aggregate monotonicity does not make any influence with core selection. Finally, in section 4 we conclude with some final remarks concerning special classes of games and point solutions.

2 The aggregate-monotonic core.

A cooperative TU-game (a game) is a pair (N, v) (v for short) where $N = \{1, ..., n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ the characteristic function, with $v(\varnothing) = 0$; $v(S)$ is the worth of coalition S. By G^N we denote the space of all TU-games with players set N. One of the main purposes of the theory of cooperative games is to study solutions or allocations of the total amount players can achieve together. Given a game v, a preimputation is a vector $x \in \mathbb{R}^n$ distributing the worth of the grand coalition, i.e. $\sum_{i \in N} x_i = v(N)$. The preimputation set is denoted by $I^*(v)$. Formally, a point solution concept (a point solution for short) is a function $\alpha : G^N \to \mathbb{R}^n$, such that $\alpha(v) \in I^*(v)$ for any $v \in G^N$.

Along this section we are interested in combining two properties of point solutions. The first is core selection.

The core, $C(v)$, of a game (N, v) (Gillies, 1959) consists of the payoff vectors satisfying coalitional rationality and efficiency, formally,

$$C(v) = \{x \in \mathbb{R}^n : x(S) \geq v(S) \text{ for all } S \subset N \text{ and } x(N) = v(N)\},$$

where $x(S) = \sum_{i \in S} x_i$, by convention $x(\varnothing) = 0$ and by $\subset$ we denote strict set inclusion, whereas $\subseteq$ denotes weak set inclusion. An allocation in the core divides the benefits of the grand coalition in such a way that no player or group of players has incentives to split off from the grand coalition. According to a well known theorem (Bondareva (1963) and Shapley (1967)) the core of a game is non-empty if and only if the game is balanced. A collection $C = \{S_1, ..., S_r\}$ of non empty subsets of N is said to be balanced if there exist positive constants $\gamma_1, ..., \gamma_r \in \mathbb{R}_{++}$, the balancing coefficients of C, such that $\sum_{j : i \in S_j} \gamma_j = 1$ for all $i \in N$. We denote by $\mathcal{C}$ the set of all balanced collections. A game is said to be balanced if the following inequality holds:

$$\sum_{j=1}^{r} \gamma_j v(S_j) \leq v(N),$$

for all $C = \{S_1, ..., S_r\} \in \mathcal{C}$. By B^N we denote the set of all balanced games.

A point solution is said to satisfy the core selection property (**CS**) if whenever the game is balanced, $v \in B^N$, then $\alpha(v) \in C(v)$.

The second property is aggregate monotonicity (**AM**). A point solution is said to satisfy aggregate monotonicity (Megiddo, 1974) if for any two games, $v, v' \in G^N$, with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, it holds that $\alpha(v) \leq \alpha(v')$, where $\leq$ in $\mathbb{R}^n$ is the pairwise order, i.e. $x \leq y$ if $x_i \leq y_i$, for all $i \in N$, while $x < y$ if $x_i < y_i$, for all $i \in N$. Aggregate monotonicity states that if only the value of the grand coalition grows, no player can suffer from it.

Given an arbitrary game v we focus on the set of efficient allocations which a point solution should pick up to hold aggregate monotonicity and core selection. For this aim we first define the *root* game associated to v.

Definition 1 *The root game* (N, v_R) *of a given game* (N, v) *is defined by* $v_R(S) = v(S)$ *for all* $S \subset N$ *and* $v_R(N) = \min_{x \in \mathbb{R}^n} \{x(N) : x(S) \geq v(S) \text{ for all } S \subset N\}$.

Note that the *root* game coincides with the original one in all coalitional values except the grand coalition. Instead, we take the minimum level of efficiency in order to get balancedness. Indeed

$$v_R(N) = \max_{\substack{C=\{S_1,\ldots,S_r\}\in\mathcal{C} \\ C\neq N}} \left\{ \sum_{j=1}^{r} \gamma_j v(S_j) \right\},$$

and the maximum is always attained in a minimal balanced collection, i.e. a balanced collection which balancing coefficients are unique (Owen, 1995).

The *root* game (N, v_R) is uniquely determined and can be alternatively described as $v_R = v + \varepsilon_R \cdot u_N$. Here, $\varepsilon_R = \min \left\{ \varepsilon \in \mathbb{R} : v + \varepsilon \cdot u_N \in B^N \right\}$, and for any $S \subseteq N$, u_S denotes the well known unanimity basis of the linear space G^N, i.e. $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. Note also that a game v can be rewritten in terms of its *root* game, in fact,

$$v = v_R + (v(N) - v_R(N)) \cdot u_N,$$

where the coefficient $(v(N) - v_R(N))$ does not need to be positive. Indeed if $v(N) \geq v_R(N)$ then $C(v) \neq \varnothing$, while if $v(N) < v_R(N)$ then $C(v) = \varnothing$.

A game (N, v) is said to be a root game if it coincides with its root game (N, v_R).

Next we define the central concept of the paper which is the aggregate-monotonic core.

Definition 2 *The aggregate-monotonic core of* (N, v)*,* $\mathcal{AC}(v)$*, is defined by*

$$\mathcal{AC}(v) = C(v_R) + (v(N) - v_R(N)) \cdot \Delta_n,$$

where Δ_n *denotes the unit-simplex, i.e.* $\Delta_n = \left\{ x \in \mathbb{R}^n_+ : x_1 + \cdots + x_n = 1 \right\}$.

Let us point out that the aggregate-monotonic core is well defined since v_R is always a balanced game. Note also that to get the aggregate-monotonic core of a game v, we follow two sequential steps. First, we select an element in the core of the *root* game v_R. Afterwards, it is enough to add a non-negative (non-positive) vector to return to the original efficiency level in v. Indeed, a cooperative phenomenon can be faced as a problem of allocating the worth of the grand coalition at cooperation birth, that is in the root game, and afterwards bring this allocation up or down to the final efficiency level in a reasonable (monotonic) way.

One can easily check following this procedure that an allocation in the aggregate-monotonic core of v is an allocation in the core of v, in case the core is non empty, i.e. $\mathcal{A}C(v) \subseteq C(v)$ whether $v \in B^N$. Indeed, from the definition it follows easily that if $v(N) \geq v_R(N)$ then $\mathcal{A}C(v) = \bigcup_{y \in C(v_R)} \{x \in I^*(v) : x \geq y\}$, while if $v(N) \leq v_R(N)$ then $\mathcal{A}C(v) = \bigcup_{y \in C(v_R)} \{x \in I^*(v) : x \leq y\}$. The relevance of the aggregate-monotonic is given in next proposition.

Proposition 3 *Let $\alpha : G^N \to \mathbb{R}^n$ be an arbitrary point solution satisfying the core selection and aggregate monotonicity properties. Then $\alpha(v) \in \mathcal{A}C(v)$ for any $v \in G^N$.*

Proof. Let v be a game, and let v_R be its *root* game. First, if $v(N) = v_R(N)$ then $v = v_R$ and clearly $\mathcal{A}C(v) = C(v)$, hence by **CS**, $\alpha(v) \in \mathcal{A}C(v)$. Second, if $v(N) > v_R(N)$ by **CS**, $\alpha(v_R) \in C(v_R)$, moreover by **AM**, $\alpha(v) \geq \alpha(v_R)$; since $\alpha(v) \in I^*(v)$ it follows that $\alpha(v) \in \mathcal{A}C(v)$. Finally, if $v(N) < v_R(N)$ by **CS**, $\alpha(v_R) \in C(v_R)$, and by **AM**, $\alpha(v) \leq \alpha(v_R)$; since $\alpha(v) \in I^*(v)$ it follows that $\alpha(v) \in \mathcal{A}C(v)$ and we are finished. ■

Apart from the fact that an AM and CS point solution always pick out an allocation in the aggregate-monotonic core, any allocation in the aggregate-monotonic core induces an AM and CS point solution defined on a restricted domain. By $[v_R]$ we denote the family of games that can be obtained from the root game v_R only varying its efficiency level, i.e. $[v_R] = \left\{v \in G^N : v = v_R + \varepsilon \cdot u_N \text{ with } \varepsilon \in \mathbb{R}\right\}$.

To do this end, if we are working on a specific subclass $\mathcal{G} \in G^N$ a point solution on $\mathcal{G}$ is a function $\alpha : \mathcal{G} \to \mathbb{R}^n$ with $\alpha(v) \in I^*(v)$ for any $v \in \mathcal{G}$. We say that α satisfies core selection (**CS**) on $\mathcal{G}$ if for all $v \in \mathcal{G}$ whenever $v \in B^N$, then $\alpha(v) \in C(v)$. Moreover, we say that α satisfies aggregate monotonicity (**AM**) on $\mathcal{G}$ if for any two games, $v, v' \in \mathcal{G}$, with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, then $\alpha(v) \leq \alpha(v')$. Next proposition links an allocation in the aggregate-monotonic core with an **AM** and **CS** point solution defined on the domain $[v_R]$.

Proposition 4 *Let (N, v) be an arbitrary game and $x \in AC(v)$. There exists a point solution $\alpha : \mathcal{G} = [v_R] \to \mathbb{R}^n$ satisfying core selection and aggregate monotonicity on $\mathcal{G} = [v_R]$, furthermore $\alpha(v) = x$*

Proof. We distinguish two cases.

Case I) $v(N) = v_R(N)$

We define the point solution $\alpha(w) = x + (w(N) - v_R(N)) \cdot \frac{e_N}{n}$ for all $w \in [v_R]$.

To show core selection notice that $\alpha(w) \in AC(w) \subseteq C(w)$ for any balanced game $w \in [v_R]$, which follows from $x \in C(v_R)$ and $\frac{e_N}{n} \in \Delta_n$. Aggregate monotonicity follows easily observing that for any $w, w' \in [v_R]$ with $w(N) < w'(N)$, $\alpha(w') - \alpha(w) = (w'(N) - w(N)) \frac{e_N}{n} \geq 0$. Finally, it is obvious that $\alpha(v) = x$.

Case II) $v(N) \neq v_R(N)$

Due to $x \in AC(v)$, there exists $y \in C(v_R)$ and $h \in \Delta_n$ such that $x = y + (v(N) - v_R(N)) \cdot h$. Notice that x and y determines a line in $\mathbb{R}^n$, we will show that the point solution following this line is AM and CS on $[v_R]$.

This point solution on the domain $[v_R]$ is defined by $\alpha(w) = \lambda x + (1-\lambda)y$ for all $w \in [v_R]$, with $\lambda = \frac{w(N)-v_R(N)}{v(N)-v_R(N)}$.

To show core selection notice that $\alpha(w) = \lambda x + (1-\lambda)y = y + \lambda(x-y)$, with $y \in C(v_R)$ and $\lambda(x-y) = (w(N) - v_R(N)) \frac{(x-y)}{v(N)-v_R(N)}$. Note also that $\lambda(x-y) \in (w(N) - v_R(N)) \cdot \Delta_n$, which follows observing that if $v(N) > v_R(N)$ then $x > y$ while if $v(N) < v_R(N)$ then $x < y$. Hence $\alpha(w) \in AC(w) \subseteq C(w)$ whenever w is balanced.

Aggregate monotonicity follows from the fact that for any $w, w' \in [v_R]$ with $w(N) < w'(N)$, $\alpha(w') - \alpha(w) = (w'(N) - w(N)) \frac{(x-y)}{v(N)-v_R(N)} \geq 0$. Last inequality holds by the same arguments discussed to show core selection.

Finally, it obviously holds that $\alpha(v) = x$. ■

As an interesting property of the above proposition, we can always extend (not in a unique way) a core selection point solutions defined on the class of the root games, i.e. $\mathcal{G} = \{v \in G^N : v \text{ is a root game}\}$, to a core selection and aggregate monotonic point solution on the whole class of games G^N.

In order to compute the aggregate-monotonic core of a game note that

$$\mathcal{A}C(v) = convex \{x + (v(N) - v_R(N)) \cdot e_i\}_{\substack{x \in Ext(C(v_R)), \\ i \in N}} \qquad (1)$$

where *convex* means convex hull and *Ext* means extreme points of the corresponding convex set. This expression says that the set of extreme points of the $\mathcal{A}C(v)$ is a subset of the allocations we get adding the extreme points of the set $(v(N) - v_R(N)) \cdot \Delta_n$ to the extreme core elements of the *root* game. The above expression can be alternatively written as $\mathcal{A}C(v) = convex \left\{ \underset{x \in Ext(C(v_R))}{\cup} \Delta_x \right\}$ where $\Delta_x = convex \{x + (v(N) - v_R(N)) \cdot e_i\}_{i \in N}$ for all $x \in \mathbb{R}^N$. This gives an intuitive view of the aggregate-monotonic core as the convex hull of a finite union of simplexes (triangles for $n = 3$), any of them from a different extreme point in the core of v_R. Note also that this last representation of the aggregate-monotonic core is equivalent to $\mathcal{A}C(v) = \underset{x \in (C(v_R))}{\cup} \Delta_x$.

The following example shows that the aggregate-monotonic core of a game may be a proper subset of the core, even for a convex game (Shapley, 1971). A game $v \in G^N$ is said to be convex if it satisfies $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$ for all $i \in N$ and all $S \subset T \subseteq N \backslash \{i\}$. In Shapley (1971) it is shown that convex games are balanced, and the extreme points of the core consist on all marginal worth vectors. Given a game $v \in G^N$ and an ordering on the players set, $\theta = (i_1, ..., i_n)$ then the vector $m^\theta(v) \in \mathbb{R}^n$ defined by $m^\theta_{i_1}(v) = v(i_1)$ and $m^\theta_{i_k}(v) = v(i_1, ..., i_k) - v(i_1, ..., i_{k-1})$ for all $k = 2, ..., n$, is the marginal worth vector of v with respect to θ. The value $m^\theta_{i_k}(v)$ is the marginal contribution of player i_k with respect to θ.

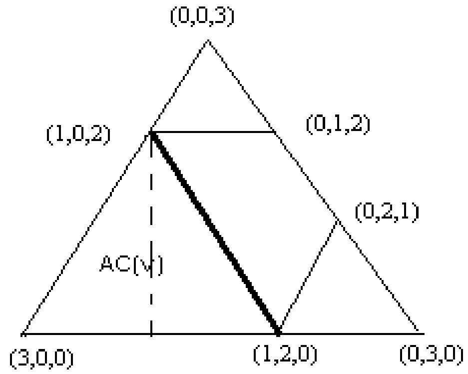

Figure 1: The aggregate-monotonic core of a three player game

Example 5 *Let (N, v) be the three players game: $v(i) = 0$ for all $i \in N$, $v(12) = v(13) = 1$, $v(23) = 0$ and $v(123) = 3$.*

It is easy to see that $v_R(N) = 1$ due to $\{\{12\}, \{3\}\}$ forms a minimal balanced collection with largest value $v(12) + v(3) = 1$ (there are others). The core of the root game reduces to one point, indeed $C(v_R) = \{(1, 0, 0)\}$. To compute the extreme points of the $\mathcal{AC}(v)$ it is enough to add the extreme points of the set $(3-1) \cdot \Delta_3$ to the unique core element of the root game. Consequently, $\mathcal{AC}(v) = convex\{(3, 0, 0), (1, 2, 0), (1, 0, 2)\}$.

The original game is convex, from which it follows that the core of the game is the convex hull of all its marginal worth vectors. Therefore, $C(v) = convex\{(3, 0, 0), (1, 2, 0), (1, 0, 2), (0, 1, 2), (0, 2, 1)\}$. The aggregate-monotonic core and the core of this three player game are depicted in figure 1.

Though, the aggregate-monotonic core can be seen as a set solution concept (a set solution for short); that is, a correspondence $\Gamma : G^N \rightarrow \mathbb{R}^n$, such that $\Gamma(v) \subseteq I^*(v)$ for any $v \in G^N$. A set solution is said to satisfy core selection if $\Gamma(v) \subseteq C(v)$ for any $v \in B^N$. And according to Megiddo (1974) a set solution satisfies what he calls monotonicity if for any two games, $v, v' \in G^N$, with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, it holds that for all $x \in \Gamma(v)$ there exists a $y \in \Gamma(v')$ such that $y \geq x$. Notice that this is indeed a kind of aggregate monotonicity property. Meggido shows that the bargaining set (Aumann and Maschler, 1964) and the Kernel (Davis and Maschler, 1965) do not satisfy this property. On the other hand, one can be easily check that the core of a game satisfies this property, and trivially also core selection.

This aggregate monotonicity property for set solutions prevents players of being damaged when the worth of the grand coalition grows, although it admits payoffs of some players grow as the worth of the grand coalition decreases. Example 5 illustrates this difference. It is easy to see that for any $x \in C(v)$

not belonging to the aggregate-monotonic core, i.e. $x \notin AC(v)$, there is no $y \in C(v_R)$ such that $y \leq x$. To avoid that some players take profit when the value of the grand coalition decreases we introduce *upper* and *lower aggregate-monotonicity* for set solutions.

Definition 6 *A set solution is said to satisfy upper aggregate monotonicity if for any two games, $v, v' \in G^N$, with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, it holds that for all $x \in \Gamma(v)$ there exists a $y \in \Gamma(v')$ such that $y \geq x$.*

Definition 7 *A set solution is said to satisfy lower aggregate monotonicity if for any two games, $v, v' \in G^N$, with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, it holds that for all $y \in \Gamma(v')$ there exists a $x \in \Gamma(v)$ such that $x \leq y$.*

Notice that the core is an upper aggregate monotonic set solution, although it does not satisfy lower aggregate monotonicity. For a complete comparative study of the core, see Ichiishi (1990) Note also that aggregate monotonic point solutions satisfy both requirements. Moreover, the aggregate-monotonic core left characterized by upper and lower aggregate monotonicity for set solutions. In fact:

Theorem 8 *The aggregate-monotonic core is the 'largest' (with respect to inclusion) core selection, upper and lower aggregate monotonic set solution.*

Proof. First note that the aggregate-monotonic core satisfies core selection since $\mathcal{A}C(v) \subseteq C(v)$ whether $v \in B^N$. To show that it satisfies upper and lower aggregate monotonicity, let $v, v' \in G^N$ be such that $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$. Obviously, v and v' have the same associated *root* game v_R. It is easy to check that if $v(N) \leq v'(N) \leq v_R(N)$ or $v_R(N) \leq v(N) \leq v'(N)$ then the following equalities hold,

$$\begin{aligned} \mathcal{A}C(v') &= C(v_R) + (v'(N) - v_R(N)) \cdot \Delta_n \\ &= C(v_R) + (v(N) - v_R(N)) \cdot \Delta_n + (v'(N) - v(N)) \cdot \Delta_n \\ &= \mathcal{A}C(v) + (v'(N) - v(N)) \cdot \Delta_n. \end{aligned}$$

Consequently, if these are the cases upper and lower aggregate monotonicity are satisfied. There is still to treat the case when $v(N) \leq v_R(N) \leq v'(N)$. We first check upper aggregate monotonicity, let $x \in \mathcal{A}C(v)$, there exists $z \in C(v_R)$ such that $z \geq x$, and there exists $y \in \mathcal{A}C(v')$ such that $y \geq z$, and consequently $y \geq x$. Analogously, to check lower aggregate monotonicity, let $y \in \mathcal{A}C(v')$, there exists $z \in C(v_R)$ such that $z \leq y$, and there exists $x \in \mathcal{A}C(v)$ such that $x \leq z$, and consequently $x \geq y$.

To finish it is enough to show that if Γ is a core selection, upper and lower aggregate monotonic set solution then $\Gamma(v) \subseteq \mathcal{A}C(v)$ for all $v \in G^N$. Let v be an arbitrary game and v_R its *root* game. First suppose that $v(N) = v_R(N)$ then $v = v_R$ and by core selection $\Gamma(v) \subseteq C(v) = \mathcal{A}C(v)$.

Secondly, if $v(N) > v_R(N)$, by core selection $\Gamma(v_R) \subseteq C(v_R)$. Now, let $x \in \Gamma(v)$, by lower aggregate monotonicity there exists a $y \in \Gamma(v_R)$ such that $y \leq x$, therefore $x \in \mathcal{A}C(v)$.

Finally, if $v(N) < v_R(N)$, again by core selection $\Gamma(v_R) \subseteq C(v_R)$. Now, let $x \in \Gamma(v)$, by upper aggregate monotonicity there exists a $y \in \Gamma(v_R)$ such that $y \geq x$, therefore $x \in \mathcal{A}C(v)$.

This finishes the proof. ■

3 The core and the aggregate-monotonic core.

In this section we study those games for which the aggregate-monotonic core coincides with the core. Note that for any *root* game v_R it holds that $\mathcal{A}C(v_R) = C(v_R)$, nevertheless example 5 shows that this coincidence is not necessarily always true for an arbitrary game v. In the following example we show that both sets might coincide.

Example 9 *Let (N, v) be the three player game defined by $v = u_{12} + 2 \cdot u_{123}$. It is easy to see that $v_R(N) = 1$ and consequently $v_R = u_{12}$. Note also that the game v is convex since it is a positive linear combination of unanimity games. Moreover, $C(v) = C(u_{12} + 2 \cdot u_{123}) = C(u_{12}) + 2 \cdot C(u_{123}) = C(u_{12}) + 2 \cdot \Delta_3 = \mathcal{A}C(v)$.*

In this section, the concept of large core plays an important role. A game (N, v) is said to have a large core (Sharkey, 1982) if for every vector $y \in \mathbb{R}^n$ with $y(S) \geq v(S)$ for all $S \subseteq N$ there exists a core element $x \in C(v)$ with $x \leq y$.

A vector y satisfying the conditions $y(S) \geq v(S)$ for all $S \subset N$ (note that we take all $S \neq N$) will be called an *upper vector* of the game (N, v). The set of all upper vectors is denoted by $U(v)$, i.e. $U(v) = \{y \in \mathbb{R}^n : y(S) \geq v(S) \text{ for all } S \subset N\}$. The following theorem due to van Gellekom et al. (1999) connects largeness of the core with the extreme points of $U(v)$.

Theorem 10 *(van Gellekom et al.,1999) Let (N, v) be a balanced game. Then (N, v) has a large core if and only if $z(N) \leq v(N)$ for all extreme points z of $U(v)$.*

The above theorem together with the next interesting theorem from Ichiishi (1990) are the tools to prove the main result of this section. Ichiishi (1990) introduces the *extended exact envelope* of a balanced game v as the function $\bar{v} : \mathbb{R}^n_+ \to \mathbb{R}$ defined by $\bar{v}(p) = \min_{x \in C(v)} p \cdot x$ where $p \cdot x$ denotes the Euclidean inner product of p and x, $\sum_{i \in N} p_i x_i$. And the theorem states the following:

Theorem 11 *(Ichiishi, 1990)Let v, w be balanced games, and let $\bar{v}$ and $\bar{w}$ be their extended exact envelopes respectively. Then, the following two conditions are equivalent:*

(1) For every $y \in C(w)$ there exists $x \in C(v)$ such that $x \leq y$.

(2) $\bar{v}(p) \leq \bar{w}(p)$ for every $p \in \mathbb{R}^n_+$.

Next we show that the coincidence of the core and the aggregate-monotonic core of an arbitrary game depends completely on the largeness of the core of its *root* game.

Theorem 12 *Let (N, v) be a balanced game and let (N, v_R) be its root game with $v \neq v_R$. Then $\mathcal{A}C(v) = C(v)$ if and only if (N, v_R) has a large core*

Proof. Let us prove first the if part; since $v \neq v_R$ and v is balanced then $v(N) > v_R(N)$. Take an arbitrary $y \in C(v)$, clearly $y(S) \geq v(S) \geq v_R(S)$ for all $S \subseteq N$; hence y is an *upper vector* of (N, v_R). Since (N, v_R) has a large core there exists $x \in C(v_R)$ such that $x \leq y$. Therefore $y \in \mathcal{A}C(v)$. Moreover, $\mathcal{A}C(v) \subseteq C(v)$ and consequently $\mathcal{A}C(v) = C(v)$.

Proving the only if part will require more arguments. Let us suppose that the *root* game does not have a large core, we will show the fact that $\mathcal{A}C(v) \subset C(v)$. Again, since $v \neq v_R$ and v is balanced we have $v(N) > v_R(N)$. Due to v_R has not a large core and by theorem 10 there exists an extreme point y^* of $U(v_R)$ with $y^*(N) > v_R(N)$.

By the fact that y^* is an extreme point of $U(v_R)$ there exists a set of coalitions $\mathcal{S} = \{S_1, \ldots, S_n\}$ such that the vectors $e_{S_1}, \ldots, e_{S_n}$ form a basis of $\mathbb{R}^n$ and $y^*(S_j) = v_R(S_j)$ for all $j = 1, \ldots, n$. From this, observe that for all $i \in N$ there is a $S \in \mathcal{S}$ with $i \in S$, and consequently there can not be any $x \in C(v_R)$ such that $x \leq y^*$. Suppose the contrary, let $x \in C(v_R)$ be such that $x \leq y^*$. By x belonging to $C(v_R)$ it follows that $x(S) \geq v_R(S) = y^*(S)$ for all $S \in \mathcal{S}$. Hence $x(S) \geq y^*(S)$ for all $S \in \mathcal{S}$ and due to $x \leq y^*$ and for all $i \in N$ there is a $S \in \mathcal{S}$ with $i \in S$ it follows that $x_i = y_i^*$ for all $i \in N$. As a consequence, $x(N) = y^*(N)$, and by $x \in C(v_R)$ we have that $x(N) = v_R(N)$, which contradicts the fact that $y^*(N) > v_R(N)$.

Define now the game (N, v_{y^*}) by $v_{y^*} = v_R + (y^*(N) - v_R(N)) \cdot u_N$ where coalitional worths do not vary from v_R, but the worth of the grand coalition increases up to $y^*(N)$, i.e. $v_{y^*}(N) = y^*(N)$. Clearly $y^* \in C(v_{y^*})$ since $y^* \in U(v_R)$ and $v_R(S) = v_{y^*}(S)$ for all $S \subset N$. Note also that $v_{y^*}(S) = v(S)$ for all $S \subset N$.

In the following, we structure the only if part of the proof in three different cases:

Case I) $v_{y^*}(N) = v(N)$. In this case $v_{y^*} = v$ and from the above discussion clearly $y^* \in C(v_{y^*})$ although $y^* \notin \mathcal{A}C(v_{y^*})$ due to the fact that there is no $x \in C(v_R)$ such that $x \leq y^*$. And this finishes case I.

Case II) $v(N) < v_{y^*}(N)$. We know that $y^* \in C(v_{y^*})$ and there is no $x \in C(v_R)$ such that $x \leq y^*$. Hence, by theorem 11 there exists $p \in \mathbb{R}^n_+$ such that $\bar{v}_R(p) > \bar{v}_{y^*}(p)$; that is, $p \cdot \hat{x} = \min\limits_{x \in C(v_R)} p \cdot x > \min\limits_{y \in C(v_{y^*})} p \cdot y = p \cdot \hat{y}$ where $\hat{x} \in C(v_R)$ and $\hat{y}.\in C(v_{y^*})$.

Define the vectors set $z_\lambda = \lambda \hat{x} + (1 - \lambda)\, \hat{y}$ for all $\lambda \in (0, 1)$ and let $\hat{\lambda}$ be such that $z_{\hat{\lambda}}(N) = v(N)$, indeed $0 < \hat{\lambda} = \frac{\hat{y}(N) - v(N)}{\hat{y}(N) - \hat{x}(N)} = \frac{v_{y^*}(N) - v(N)}{v_{y^*}(N) - v_R(N)} < 1$ since $v_R(N) < v(N) < v_{y^*}(N)$. Clearly, $p \cdot z_{\hat{\lambda}} = \hat{\lambda}\,(p \cdot \hat{x}) + \left(1 - \hat{\lambda}\right)(p \cdot \hat{y}) < \hat{\lambda}\,(p \cdot \hat{x}) + \left(1 - \hat{\lambda}\right)(p \cdot \hat{x}) = p \cdot \hat{x}$ due to $p \cdot \hat{x} > p \cdot \hat{y}$. Furthermore, $z_{\hat{\lambda}}(S) =$

$\hat{\lambda}\hat{x}(S) + \left(1 - \hat{\lambda}\right)\hat{y}(S) \geq v(S)$ for all $S \subset N$ since $\hat{x} \in C(v_R)$, $\hat{y} \in C(v_{y^*})$ and $v_R(S) = v_{y^*}(S) = v(S)$, hence $z_{\hat{\lambda}} \in C(v)$.

Finally, to finish with case II note that $\bar{v}_R(p) = p \cdot \hat{x} > p \cdot z_{\hat{\lambda}} \geq \min_{z \in C(v)} p \cdot z = \bar{v}(p)$. Again, by theorem 11 there exist $z \in C(v)$ such that there is no $x \in C(v_R)$ with $x \leq z$, from which we conclude that $z \notin \mathcal{A}C(v)$.

Case III) $v(N) > v_{y^*}(N)$. We will construct from y^* which is an extreme point of $U(v_R)$ a vector $\bar{y}^*$ such that $\bar{y}^* \in C(v)$ and $\bar{y}^* \notin \mathcal{A}C(v)$. To do it, let $i \in N$ be such that

$$|\{S \in \mathcal{S} : i \in S\}| \leq |\{S \in \mathcal{S} : j \in S\}| \text{ for all } j \in N \tag{2}$$

Notice that this player always exists, since we take one of the players belonging to the least minimum number of coalitions from the set $\mathcal{S}$.

Now, define $\bar{y}^* = y^* + (v(N) - v_{y^*}(N)) \cdot e_i$; note that $\bar{y}^*_j = y^*_j$ for all $j \neq i$, so we give all the extra surplus of the grand coalition to player i. It follows easily that $\bar{y}^* \in C(v)$ since $\bar{y}^*(N) = v(N)$ and $\bar{y}^*(S) \geq y^*(S) \geq v(S)$ for all $S \subset N$ due to $y^* \in C(v_{y^*})$ and $v_{y^*}(S) = v(S)$ for all $S \subset N$.

By R we denote the set of players different from i for which there is a coalition in $\mathcal{S}$ that does not include player i, i.e. $R = \underset{S \in \mathcal{S}: i \notin S}{\cup} S$. Notice that $R \neq \varnothing$. Suppose the contrary, if $R \neq \varnothing$, then $i \in S$ for all $S \in \mathcal{S}$, and consequently $|\{S \in \mathcal{S} : i \in S\}| = n$. Since by definition $|\{S \in \mathcal{S} : i \in S\}| \leq |\{S \in \mathcal{S} : j \in S\}|$ for all $j \in N$ it follows that $|\{S \in \mathcal{S} : j \in S\}| = n$ for all $j \in N$. Hence $S = N$ for all $S \in \mathcal{S}$ which contradicts the fact that the corresponding set of characteristic vectors form a basis of $\mathbb{R}^n$.

Then, for any $j \in R$ there exists a coalition $S \in \mathcal{S}$ with $j \in S$ and $i \notin S$ that holds $\bar{y}^*(S) = y^*(S) = v_{y^*}(S) = v(S)$. Observe that if there exists a $x \in C(v_R)$ with $x \leq \bar{y}^*$ then by x belonging to $C(v_R)$ it follows that $x(S) \geq v_R(S) = v(S) = \bar{y}^*(S)$. Hence $x(S) \geq \bar{y}^*(S)$ and due to $x \leq \bar{y}^*$ it follows that $x_j = \bar{y}^*_j = y^*_j$. Consequently allocation x holds that $x(R) = \bar{y}^*(R) = y^*(R)$.

Now, let $j \in N \backslash R$, (note that since $i \notin R$ then $N \backslash R \neq \varnothing$) it follows that any coalition $S \in \mathcal{S}$ such that $j \in S$ does also include player i and consequently $|\{S \in \mathcal{S} : i \in S\}| \geq |\{S \in \mathcal{S} : j \in S\}|$. However by hypothesis (2) the equality holds, which means that for any $S \in \mathcal{S}$ such that $i \in S$ then $j \in S$, and consequently all players in $N \backslash R$ go together in any coalition $S \in \mathcal{S}$. Suppose there exists $x \in C(v_R)$ such that $x \leq \bar{y}^*$ and let $S \in \mathcal{S}$ be such that $i \in S$ and consequently any player $j \in N \backslash R$ belongs to S, too. Computing the amount players in S can get according to x we get

$$\begin{aligned} x(S) &= x(S \cap R) + x(S \cap (N \backslash R)) \\ &= x(S \cap R) + x(N \backslash R) \\ &\geq v_R(S) = v_{y^*}(S) \\ &= y^*(S) \\ &= y^*(S \cap R) + y^*(N \backslash R). \end{aligned}$$

Here, the inequality follows from $x \in C(v_R)$ and fourth equality from $S \in \mathcal{S}$. From this and the fact that $x(S \cap R) = y^*(S \cap R)$ due to $x_j = y^*_j$ for all $j \in R$ it follows from $x(S \cap R) + x(N \backslash R) \geq y^*(S \cap R) + y^*(N \backslash R)$ that $x(N \backslash R) \geq y^*(N \backslash R)$.

Combining the amount $x \in C(v_R)$ with $x \leq \bar{y}^*$ can assign to players in R and in $N \backslash R$ we get that $x(N) = x(R) + x(N \backslash R) \geq y^*(R) + y^*(N \backslash R) = y^*(N) > v_R(N)$. Hence there is no $x \in C(v_R)$ with $x \leq \bar{y}^*$ which implies $\bar{y}^* \notin \mathcal{A}C(v)$, being $\bar{y}^* \in C(v)$. With this we finish case III and close the proof. ■

The above theorem has some implications on the coincidence of the core and the aggregate-monotonic core for the family of balanced games that can be derived from a particular *root* game by increasing the efficiency level, i.e. the family $[v_R]_B = \{v \in G^N : v = v_R + \varepsilon \cdot u_N \text{ with } \varepsilon > 0\}$. The next two corollaries state that either the core and the aggregate-monotonic core coincide or not for all games in such a family.

Corollary 13 *Let v_R be a root game. Then the following two conditions are equivalent:*
(1) $\mathcal{A}C(v) = C(v)$ for all $v \in [v_R]_B$.
(2) (N, v_R) has a large core.

Corollary 14 *Let v_R be a root game. Then the following two conditions are equivalent:*
(1) $\mathcal{A}C(v) \subset C(v)$ for all $v \in [v_R]_B$.
(2) (N, v_R) has not a large core.

Proof. Both corollaries are straightforward from theorem 12. ■

But theorem 12 does also says something about games with large core. In fact, it follows easily from the proof of the theorem that whenever an arbitrary balanced game v has not large core then there exists $y \in \mathbb{R}^n$, $y(S) \geq v(S)$ for all $S \subseteq N$ such that there is no $x \in C(v)$ with $x \leq y$ at any level of efficiency larger than $v(N)$. Furthermore, the theorem has a nice implication on the additivity of the cores of two games with different efficiency levels, indeed,

Corollary 15 *Let (N, v) and (N, w) be two balanced games such that $w(S) = v(S)$ for all $S \subset N$ and $w(N) < v(N)$. Then $C(v) = C(w) + (v(N) - w(N)) \cdot \Delta_n$ if and only if (N, w) has large core.*

Proof. The proof follows directly with the same arguments as in the proof of theorem 12, but now with game w playing the role of game v_R. ■

4 Concluding Remarks.

Throughout this work a new set solution concept has been introduced. Its interest lays not only in its properties, it is the locus in the core of core selection and aggregate monotonic point solution, but also in the fact that induces a new way of looking at the cooperative phenomenon. Indeed, a cooperative problem can be faced as a problem of allocating the worth of the grand coalition at cooperation birth, that is in the root game, and afterwards bring this allocation up or down until the final efficiency level in a reasonable (monotonic) way.

In fact, any stable allocation in the root game, extended monotonically to an allocation in the aggregate-monotonic core will have strong arguments to be proposed as a reasonable one.

With respect to special classes of games. As example 5 shows, convexity of a game does not necessarily imply the coincidence of the core and the aggregate-monotonic core. In fact, convexity of a game does not imply convexity of its *root* game which is a sufficient condition for largeness of the core (Sharkey, 1982). However, the *root* game associated to a convex game is almost convex, i.e. $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$ for all $i \in N$ and all $S \subset T \subset N \setminus \{i\}$. As a consequence, and since extreme points of the core of an almost convex games are known we can derive extreme points of the aggregate-monotonic core of a convex games by using (**??**). Núñez and Rafels (1998) introduce the reduced marginal worth vectors and show that those are the extreme points of the core of an almost convex game.

Assignment games were introduced by Shapley and Shubik (1971) as a model for a two-sided market with transferable utility. For this well known class of balanced games, the value of the grand coalition can be seen as the sum of the worths of a set of coalitions forming a partition of the players set. Consequently, an assignment game is a *root* game itself and the core coincides with the aggregate-monotonic core. But, we can still say something else. From Solymosi and Raghavan (2001) we know that an assignment game has a large core if and only if its corresponding matrix A has a dominant and doubly dominant diagonal. It is then obvious that given an assignment game the core and the aggregate-monotonic core coincide for the family of balanced games that can be derived increasing only the efficiency level if and only if its associated matrix A has a dominant and doubly dominant diagonal. Convexity and subconvexity are sufficient conditions for largeness of the core (Sharkey, 1982), also exactness (Schmeidler, 1972) of a symmetric game is a necessary and sufficient condition for largeness of the core (Biswas et al., 1999). Therefore, a *root* game holding any of these conditions also behaves in such a way.

Many interesting questions arise with respect to this new set. In particular we are thinking in properties of the core that the aggregate-monotonic core may preserve. How does the kernel behaves with respect to the aggregate-monotonic core?

Many other questions arise with respect to point solutions, which are those satisfying CS and AM? Are these two properties compatible with the dummy player property, symmetry, or individual rationality? Hokari (2000) shows that the nucleolus is not an aggregate-monotonic point solution in the domain of convex games, neither the prenucleolus is. The Shapley value is not core selection. But the per-capita prenucleolus is an aggregate monotonic and core selection point solution (see for instance Moulin 1988 or Young et al. 1982). However, as quoted in Young et al. (1982) this point solution may fail individual rationality and dummy player property (a fact first noted by Reinhard Selten). Instead, we can take the per-capita nucleolus which imposes individual rationality and does have the dummy player property. Nevertheless and this is still open we do not know if it looses aggregate monotonicity.

If we look at other known point solutions. we realize that non of them satisfies CS and AM. The Tau value (Tijs, 1981) neither possesses core selection property nor aggregate monotonicity, even in the class of convex games (see Driesen (1985) for $n = 6$ and Hokari and van Gellekom (2002) for $n = 7$). Separable cost remaining benefits solution (see James and Lee, 1971) is neither aggregate monotonic (see Young et al. (1982) and Young (1985)) nor core selection (Young et al., 1982). To finish, although the Dutta-Ray solution (Dutta and Ray, 1989) does hold aggregate monotonicity it fails core selection.

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