A UNIFIED APPROACH OF VALUES FOR GAMES ON UNION STABLE SYSTEMS

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- ✓ ***Union stable systems***
- ✓ ***Restricted and conference games***
- ✓ ***Myerson and position values***
- ✓ ***Characterizations***
- ✓ ***Harsanyi power solutions***
- ✓ ***Characterizations***

Myerson (1977)

Graphs and Cooperation in Games

$(N, v), \quad G = (N, E)$

- Rectricted game by a communication graph

$$(N, v^G), \qquad v^G : 2^N \longrightarrow \mathbb{R}$$

$$v^G(S) = \sum_{S_i \in S/G} v(S_i)$$

- Myerson value

$$\mu(N, v, E) = \Phi(N, v^G)$$

COMMUNICATION SITUATION

(N, v, E)

A set system $(N, \mathcal{F})$ is $\cup$*-stable* if

A1. $\varnothing \in \mathcal{F}$

A2. For $S, T \in \mathcal{F}$ with $S \cap T \neq \varnothing$, we have $S \cup T \in \mathcal{F}$

Union stable cooperation structures

$(N, v, \mathcal{F})$

Comunication situations

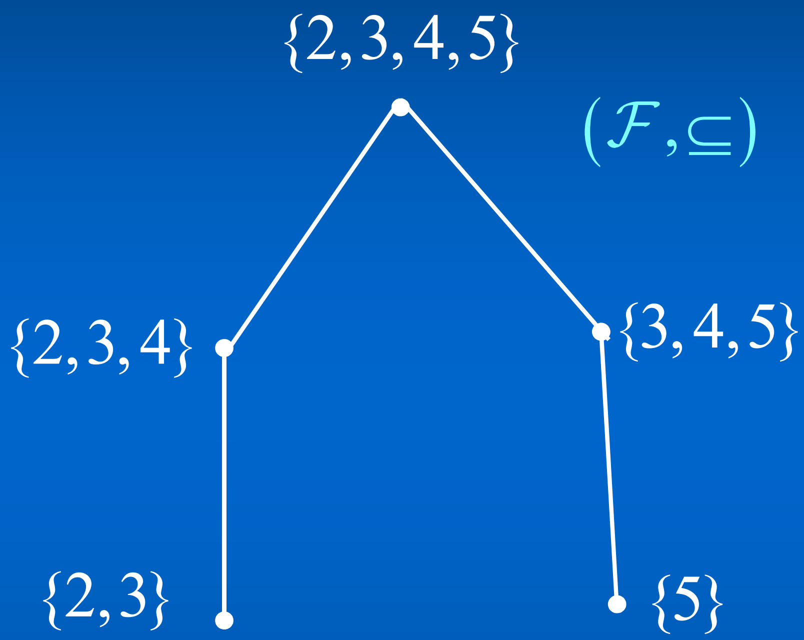
(N, v, E)

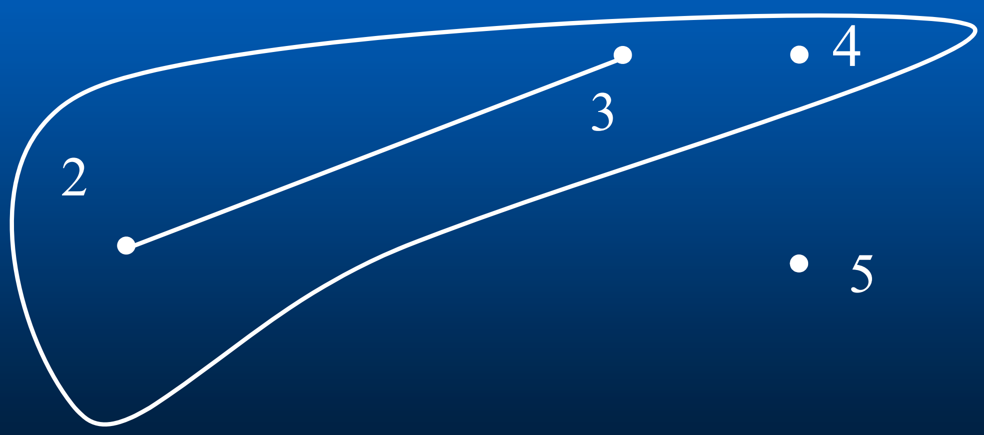
- **Myerson (1977),**
- **Owen (1986)**
- **Borm, Owen and Tijs (1992)**
- **Van den Nouweland, Borm, Tijs (1992)**

Example

$N = \{1,2,3,4,5\}$

$\mathcal{F} = \{\ \{5\}, \{2,3\}, \{2,3,4\}, \{3,4,5\}, \{2,3,4,5\}\ \}$





Example

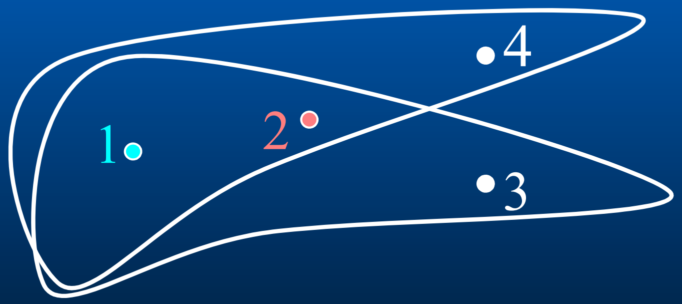
$N = \{1, 2, 3, 4\}$

$\mathcal{F} = \{ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\} \}$

1 / Seller

2 / Stage agent

3, 4 / Buyers



$\mathcal{G} \subseteq \mathcal{F}$

$$\mathcal{G}^{(0)} = \mathcal{G}$$

$$\vdots$$

$$\mathcal{G}^{(n)} = \left\{ S \cup T : S, T \in \mathcal{G}^{(n-1)}, S \cap T \neq \emptyset \right\}$$

$$\mathcal{G}^{(0)} \subseteq \cdots \subseteq \mathcal{G}^{(n)} \subseteq \cdots \subseteq \mathcal{F}$$

If k is the smallest integer such that $\mathcal{G}^{(k+1)} = \mathcal{G}^{(k)}$

$$SG(\mathcal{G}) = \mathcal{G}^{(k)}$$

$SG(\mathcal{G})$ is a union stable system

Example

$$N = \{1,2,3,4\}$$

$$\mathcal{F} = \{\,\{1\},\{3\},\{1,2\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\},\{2,3,4\},N\,\}$$

$$\mathcal{G} = \{\,\{1,2\},\{2,3\},\{2,4\}\,\} \subseteq \mathcal{F}$$

$$\mathcal{G}^{(1)} = \{\,\{1,2\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\},\{2,3,4\}\,\}$$

$$\mathcal{G}^{(2)} = \{\,\{1,2\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\},\{2,3,4\},N\,\}$$

$$\mathcal{G}^{(2)} = \mathcal{G}^{(3)}$$

$$SG\left(\mathcal{G}\right) = \mathcal{G}^{(2)} = \{\,\{1,2\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\},\{2,3,4\},N\,\}$$

$(N,\mathcal{F})$ union stable system

$$\varphi : 2^{\mathcal{F}} \to 2^{\mathcal{F}}, \qquad \varphi(\mathcal{G}) = \overline{\mathcal{G}} = SG(\mathcal{G})$$

$\varphi : 2^{\mathcal{F}} \to 2^{\mathcal{F}}$ is a closure operator

(a) $\forall \mathcal{G} \in 2^{\mathcal{F}} \Rightarrow \mathcal{G} \subseteq \varphi(\mathcal{G})$

(b) $\mathcal{G} \subseteq \mathcal{R} \subseteq \mathcal{F} \Rightarrow \varphi(\mathcal{G}) \subseteq \varphi(\mathcal{R})$

(c) $\forall \mathcal{G} \in 2^{\mathcal{F}}, \varphi(\varphi(\mathcal{G})) = \varphi(\mathcal{G})$

$(\mathcal{F}, -)$ is a closure space

$\mathcal{G} \in 2^{\mathcal{F}}$ is a closed set $\Leftrightarrow (N, \mathcal{G},)$ is a union stable system

Basis of $\mathcal{F}$

$$B(\mathcal{F}) = \mathcal{F} \setminus D(\mathcal{F})$$

$$D(\mathcal{F}) = \left\{ F \in \mathcal{F} : F = A \cup B,\ \ A, B \neq F,\ \ A, B \in \mathcal{F},\ \ A \cap B \neq \emptyset \right\}$$

$B \in B(\mathcal{F})$ support

Example

$$N = \{1, 2, 3, 4\}$$

$$\mathcal{F} = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \}$$

$$B(\mathcal{F}) = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{3,4\}, \{2,3,4\} \}$$

Union stable cooperation structures

$(N, v, \mathcal{F})$

Comunication situations

(N, v, E)

- **Myerson (1977),**
- **Owen (1986)**
- **Borm, Owen and Tijs (1992)**
- **Van den Nouweland, Borm, Tijs (1992)**

(N, v, E) $\cup$-*stable* cooperation structure

$$\mathcal{F} = \left\{ S \subseteq N : \left(S, E(S)\right) \text{ connected subgraph of } (N, E) \right\}$$

$$\mathcal{B}(\mathcal{F}) = \left\{ \{i, j\} : \{i, j\} \in E \right\} \cup \left\{ \{i\} : i \in N \right\}$$

Union stable cooperation structures

$(N, v, \mathcal{F})$

Comunication situations

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Example

$$N = \{1,2,3,4\}$$

$$\mathcal{F} = \{\,\{1,2\},\{1,3\},\{3,4\},\{1,2,3\},\{1,3,4\},\{2,3,4\},N\}$$

$$\mathcal{B}(\mathcal{F}) = \{\,\{1,2\},\{1,3\},\{3,4\},\{2,3,4\}\,\}$$

$$\{3,4\} \subseteq \{2,3,4\}$$

$B(\mathcal{F})$ basis of $\mathcal{F}$

Proposition

$B(\mathcal{F})$ is the minimal subset of $\mathcal{F}$ such that $SG(B(\mathcal{F})) = \mathcal{F}$

Proposition

$$B(\mathcal{F}) = ex(\mathcal{F})$$

$ex(\mathcal{F}) = \{F \in \mathcal{F} : \mathcal{F} \setminus F \text{ is union stable}\}$

$(N,\mathcal{F}),\quad S \subseteq N$

$T \subseteq S \quad \mathcal{F}$-component of S:

$T \in \mathcal{F}$ and there exists no $T' \in \mathcal{F},\ T \subset T' \subseteq S$

Proposition

$(N,\mathcal{F})$ U-stable $\Leftrightarrow$ $S \subseteq N,\ C_{\mathcal{F}}(S)$ partition of $S' \subseteq S$

$(N,\mathcal{F})$ union stable system, $S \subseteq N$

Proposition

$$\mathcal{F}_S = \{F \in \mathcal{F} : F \subseteq S\} \subseteq \mathcal{F}$$

(a) $(N, \mathcal{F}_S)$ is a union stable system

(b) $C_{\mathcal{F}}(S) = C_{\mathcal{F}_S}(N)$

(c) $\mathcal{B}(\mathcal{F}_S) = \{B \in \mathcal{B}(\mathcal{F}) : B \subseteq S\} \subseteq \mathcal{B}(\mathcal{F})$

$USI^N \subseteq US^N$ $(N, \mathcal{F})$ satisfying:

(1) For all $S, T \in \mathcal{F}$ with $|S \cap T| \geq 2$, we have $S \cap T \in \mathcal{F}$

(2) all non-unitary feasible coalition can be expressed in a unique way as union of non-unitary supports

Example

$$N = \{1,2,3,4\}$$

$$\mathcal{F} = \{\{1,2\},\{1,2,3\},\{2,3,4\},N\}$$

$$\mathcal{B} = \mathcal{C} = \{\{1,2\},\{1,2,3\},\{2,3,4\}\}$$

$$(N, v, \mathcal{F}) \notin USI^N$$

$$\{1,2,3\} \cap \{2,3,4\} = \{2,3\} \notin \mathcal{F}$$

$$N = \{1,2\} \cup \{2,3,4\} = \{1,2,3\} \cup \{2,3,4\}$$

$USI^N \subseteq US^N$ $(N, \mathcal{F})$ satisfying:

(1) For all $S, T \in \mathcal{F}$ with $|S \cap T| \geq 2$, we have $S \cap T \in \mathcal{F}$

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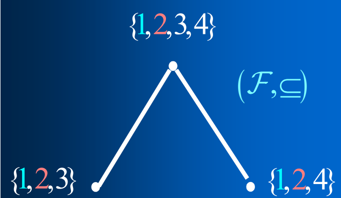
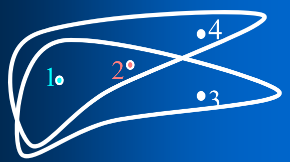
1/ Seller

2/ Stage agent

3,4/ Buyers





Example

$$N = \{1, 2, 3, 4\}$$

$$\mathcal{F} = \{ \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\} \}$$

$$(N, v, \mathcal{F}) \notin USI^N$$

$$\{1,2,3\} \cap \{1,2,4\} = \{1,2\} \notin \mathcal{F}$$

RESTRICTED GAME

$$v^{\mathcal{F}} : 2^N \longrightarrow \mathbb{R},$$

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$$

CONFERENCE GAME

$(N, v, \mathcal{F}) \rightarrow (\mathcal{C}, v^{\mathcal{C}})$ conference game

$\mathcal{A} \subseteq \mathcal{C} \rightarrow (N, v, \overline{\mathcal{A}})$ $\cup$-stable cooperation structure

$\mathcal{C} = \{C \in \mathcal{C} : |C| \geq 2\}$

$$v^{\mathcal{C}} : 2^{\mathcal{C}} \longrightarrow \mathbb{R}, \quad v^{\mathcal{C}}(\mathcal{A}) = v^{\overline{\mathcal{A}}}(N)$$

RESTRICTED GAME

$$v^{\mathcal{F}} : 2^N \longrightarrow \mathbb{R},$$

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$$

MYERSON VALUE

$$\mu(N, v, \mathcal{F}) = \Phi(N, v^{\mathcal{F}})$$

CONFERENCE GAME

$(N, v, \mathcal{F}) \rightarrow (\mathcal{C}, v^{\mathcal{C}})$ conference game

$\mathcal{A} \subseteq \mathcal{C} \rightarrow (N, v, \overline{\mathcal{A}})$ $\cup$-stable cooperation structure

$$v^{\mathcal{C}} : 2^{\mathcal{C}} \longrightarrow \mathbb{R}, \quad v^{\mathcal{C}}(\mathcal{A}) = v^{\overline{\mathcal{A}}}(N)$$

POSITION VALUE

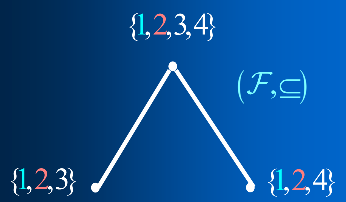
$$\pi_i(N, v, \mathcal{F}) = \begin{cases} \sum_{C \in \mathcal{C}_i} \frac{1}{|C|} \Phi_C(\mathcal{C}, v^{\mathcal{C}}), & \text{if } \mathcal{C}_i \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

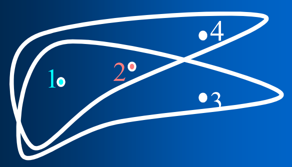
$$\mathcal{C}_i = \{C \in \mathcal{C} : i \in C\}$$

1/ Seller

2/ Stage agent

3,4/ Buyers





Example

$$N = \{1,2,3,4\}$$

$$\mathcal{F} = \{\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}, \qquad \mathcal{B} = \mathcal{C} = \{\{1,2,3\},\{1,2,4\}\}$$

$$v(S) = |S| - 1, \; v(\varnothing) = 0$$

$$v^{\mathcal{F}}(S) = \begin{cases} |S| - 1 & \text{if } S \in \mathcal{F} \\ 0 & \text{otherwise} \end{cases}$$

$$\mu(N, v, \mathcal{F}) = \left(\frac{13}{12}, \frac{13}{12}, \frac{5}{12}, \frac{5}{12}\right)$$

$\mathcal{A} \subseteq \mathcal{C}$	$\overline{\mathcal{A}}$	$C_{\overline{\mathcal{A}}}(N)$	$v^{\mathcal{C}}(\mathcal{A})$
$\{\{1,2,3\}\}$	$\{\{1,2,3\}\}$	$\{\{1,2,3\}\}$	2
$\{\{1,2,4\}\}$	$\{\{1,2,4\}\}$	$\{\{1,2,4\}\}$	2
$\{\{1,2,3\},\{1,2,4\}\}$	$\{\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}$	$\{\{1,2,3,4\}\}$	3

$$\pi(N, v, \mathcal{F}) = \left(1, 1, \frac{1}{2}, \frac{1}{2}\right)$$

RESTRICTED GAME

$$v^{\mathcal{F}} : 2^N \longrightarrow \mathbb{R},$$

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$$

CONFERENCE GAME

$(N, v, \mathcal{F}) \rightarrow (\mathcal{C}, v^{\mathcal{C}})$ conference game

$\mathcal{A} \subseteq \mathcal{C} \rightarrow (N, v, \overline{\mathcal{A}})$ $\cup$-stable cooperation structure

$v^{\mathcal{C}} : 2^{\mathcal{C}} \longrightarrow \mathbb{R}, \ v^{\mathcal{C}}(\mathcal{A}) = v^{\overline{\mathcal{A}}}(N)$

Theorem

Let $(N, v, \mathcal{F})$ be a union stable cooperation structure and $(\mathcal{C}, v^{\mathcal{C}})$ the associated conference game. Then

$$v^{\mathcal{F}}(S) = v^{\mathcal{C}}(\mathcal{C}_S)$$

where $\mathcal{C}_S = \{C \in \mathcal{C} : C \subseteq S\}$

- $C(v)=\left\{x\in\mathbb{R}^n : x(N)=v(N),\ x(S)\geq v(S), \forall S\subset N\right\}$

- Bondareva (1963), Shapley (1967):

A game (N,v) is balanced if and only if $C(v)\neq\varnothing$

- A game (N,v) is superadditive if

$$v(S\cup T)\geq v(S)+v(T),\quad \forall S,T\in 2^N, S\cap T=\varnothing$$

- A game (N,v) is convex if

$$v(S\cup T)+v(S\cap T)\geq v(S)+v(T),\quad \forall S,T\in 2^N$$

Theorem

Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. If $\left(\mathcal{C}, v^{\mathcal{C}}\right)$ non-negative and balanced, then $\left(N, v^{\mathcal{F}}\right)$ is balanced

Theorem

Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. If $\left(\mathcal{C}, v^{\mathcal{C}}\right)$ non-negative and superadditive, then $\left(N, v^{\mathcal{F}}\right)$ is superadditive

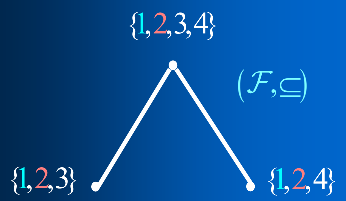
Theorem

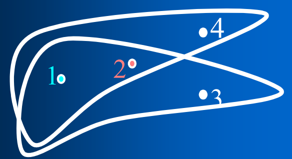
Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. If $\left(\mathcal{C}, v^{\mathcal{C}}\right)$ non-negative and convex, then $\left(N, v^{\mathcal{F}}\right)$ is convex

1/ Seller

2/ Stage agent

3,4/ Buyers





Example

$$N = \{1,2,3,4\}$$

$$\mathcal{F} = \{\, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\} \,\}$$

$$v(S) = \begin{cases} |S| & \text{if } |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

(N, v) is superadditive, so is $(N, v^{\mathcal{F}})$

The conference game $(\mathcal{C}, v^{\mathcal{C}})$ is not superadditive:

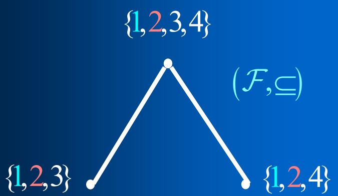
$$\mathcal{B} = \mathcal{C} = \{\, \{1,2,3\}, \{1,2,4\} \,\}$$

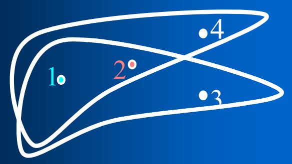
$$v^{\mathcal{C}}\left(\{\{1,2,3\},\{1,2,4\}\}\right) \leq v^{\mathcal{C}}\left(\{\{1,2,3\}\}\right) + v^{\mathcal{C}}\left(\{\{1,2,4\}\}\right)$$

1/ Seller

2/ Stage agent

3,4/ Buyers





Example

$$N = \{1,2,3,4\}$$

$$\mathcal{F} = \{\,\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\,\}$$

$$v(S) = \begin{cases} |S| & \text{if } |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

(N,v) is totally balanced, so is $(N,v^{\mathcal{F}})$

The conference game $(\mathcal{C},v^{\mathcal{C}})$ is not balanced:

$$\mathcal{B} = \mathcal{C} = \{\,\{1,2,3\},\{1,2,4\}\,\}$$

$$C(v^{\mathcal{C}}) = \left\{ y \in \mathbb{R}^{|\mathcal{C}|} : y_{\{1,2,3\}} + y_{\{1,2,4\}} = 4,\ y_{\{1,2,3\}} \geq 3,\ y_{\{1,2,4\}} \geq 3 \right\} = \varnothing$$

Corollary

Let $(N, v, \mathcal{F})$ be a union stable cooperation structure. If $(\mathcal{C}, v^{\mathcal{C}})$ non-negative and convex, then

$$\mu(N, v, \mathcal{F}) \in C(v^{\mathcal{F}})$$

$$C(v^{\mathcal{F}}) = \{x \in \mathbb{R}^n : x(N) = v^{\mathcal{F}}(N),\ \ x(S) \geq v(S),\ \forall S \in \mathcal{F}\}$$

Shapley (1971)

Lemma

(N, v) *convex* if and only if,

$$v(\bigcup_{i=1}^{k} T_i) - \sum_{i=1}^{k} v(T_i) \geq v(\bigcup_{i=1}^{k} S_i) - \sum_{i=1}^{k} v(T_i),$$

$$T_i \cap T_j = \varnothing, \quad i \neq j, \quad S_i \subseteq T_i, \quad i = 1, \ldots, k$$

Theorem

Let $(N, v, \mathcal{F}) \in USI^N$. If (N, v) is convex, then $(\mathcal{C}, v^{\mathcal{C}})$ is (non-negative) convex

Corollary

Let $(N, v, \mathcal{F}) \in USI^N$. If (N, v) is convex, then

$$\mu(N, v, \mathcal{F}) \in C(v^{\mathcal{F}})$$

$$\pi(N, v, \mathcal{F}) \in C(v^{\mathcal{F}})$$

$US^N = \{(N, v, \mathcal{F})$ union stable cooperation structures$\}$

<u>Allocation rule</u>: $\gamma : US^N \to \mathbb{R}^n$

Component-efficiency

$$\forall (N, v, \mathcal{F}) \in US^N, \quad \forall M \in C_{\mathcal{F}}(N),$$

$$\sum_{k \in M} \gamma_k \left(N, v, \mathcal{F} \right) = v(M)$$

Component-dummy

$$\forall i \notin \bigcup_{M \in C_{\mathcal{F}}(N)} M, \quad \gamma_i \left(N, v, \mathcal{F} \right) = 0$$

$$\gamma : US^N \to \mathbb{R}^n$$

ADDITIVITY

$$\gamma\left(N, v+w, \mathcal{F}\right) = \gamma\left(N, v, \mathcal{F}\right) + \gamma\left(N, w, \mathcal{F}\right)$$

SUPERFLUOUS SUPPORT

$$\gamma\left(N, v, \mathcal{F}\right) = \gamma\left(N, v, \overline{\mathcal{B} \setminus \{H\}}\right)$$

$$v^{\mathcal{C}}\left(\mathcal{A}\right) = v^{\mathcal{C}}\left(\mathcal{A} \setminus \{H\}\right), \ \forall \mathcal{A} \subseteq \mathcal{C}, \ \forall H \in \mathcal{C}$$

INFLUENCE

$$\gamma\left(N, v, \mathcal{F}\right) = \alpha I\left(N, \mathcal{F}\right)$$

$$I_i\left(N, \mathcal{F}\right) = \begin{cases} \sum_{C \in \mathcal{C}_i} \frac{1}{|C|}, & \text{if } \mathcal{C}_i \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$v^{\mathcal{C}}\left(\mathcal{A}\right) = f\left(|\mathcal{A}|\right), \quad \forall \mathcal{A} \subseteq \mathcal{C}$$

Theorem

The position value satisfies

- additivity
- superfluous support
- influence

Theorem

The Myerson value satisfies

- additivity
- superfluous support

$USI^N \subseteq US^N$ $(N, \mathcal{F})$ satisfying:

(1) For all $S, T \in \mathcal{F}$ with $|S \cap T| \geq 2$, we have $S \cap T \in \mathcal{F}$

(2) all non-unitary feasible coalition can be expressed in an unique way as union of non-unitary supports

Theorem

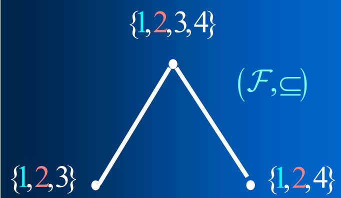
The position value is the unique allocation rule on USI^N satisfying

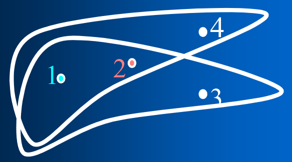
- additivity
- superfluous support
- influence $\left(\alpha = \dfrac{v^{\mathcal{C}}(\mathcal{C})}{|\mathcal{C}|} \right)$

1/ Seller

2/ Stage agent

3,4/ Buyers





Example

$$N = \{1, 2, 3, 4\}$$

$$\mathcal{F} = \{\{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}, \qquad \mathcal{B} = \mathcal{C} = \{\{1,2,3\}, \{1,2,4\}\}$$

$$v(S) = |S| - 1, v(\varnothing) = 0$$

$$v^{\mathcal{F}}(S) = \begin{cases} |S| - 1 & \text{if } S \in \mathcal{F} \\ 0 & \text{otherwise} \end{cases}$$

$$\mu(N, v, \mathcal{F}) = \left(\frac{13}{12}, \frac{13}{12}, \frac{5}{12}, \frac{5}{12}\right)$$

$$I(N, \mathcal{F}) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\left(\frac{13}{12}, \frac{13}{12}, \frac{5}{12}, \frac{5}{12}\right) \neq \alpha \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\gamma : US^N \to \mathbb{R}^n$$

ADDITIVITY

$$\gamma\left(N, v+w, \mathcal{F}\right) = \gamma\left(N, v, \mathcal{F}\right) + \gamma\left(N, w, \mathcal{F}\right)$$

SUPERFLUOUS SUPPORT

$$\gamma\left(N, v, \mathcal{F}\right) = \gamma\left(N, v, \overline{\mathcal{B} \setminus \{H\}}\right)$$

$$v^{\mathcal{C}}\left(\mathcal{A}\right) = v^{\mathcal{C}}\left(\mathcal{A} \setminus \{H\}\right),\ \forall \mathcal{A} \subseteq \mathcal{C},\ \forall H \in \mathcal{C}$$

POINT ANONYMITY

$$\gamma_i\left(N, v, \mathcal{F}\right) = \begin{cases} \alpha, & \text{if } i \in D \\ 0, & \text{otherwise} \end{cases}$$

$$v^{\mathcal{F}}\left(S\right) = f\left(\left|S \cap D\right|\right), \qquad D = \left\{ i \in N : \left|\mathcal{C}_i\right| > 0 \right\}$$

$$\gamma : US^N \to \mathbb{R}^n$$

ADDITIVITY

$$\gamma\left(N, v + w, \mathcal{F}\right) = \gamma\left(N, v, \mathcal{F}\right) + \gamma\left(N, w, \mathcal{F}\right)$$

SUPERFLUOUS SUPPORT

$$\gamma\left(N, v, \mathcal{F}\right) = \gamma\left(N, v, \overline{\mathcal{B} \setminus \{H\}}\right)$$

$$v^{\mathcal{C}}(\mathcal{A}) = v^{\mathcal{C}}\left(\mathcal{A} \setminus \{H\}\right),\ \forall \mathcal{A} \subseteq \mathcal{C},\ \forall H \in \mathcal{C}$$

POINT UNANIMITY

$$\gamma_i\left(N, v, \mathcal{F}\right) = \begin{cases} \alpha, & \text{if } i \in D \\ 0, & \text{otherwise} \end{cases}$$

$$v^{\mathcal{F}} = \beta u_D, \qquad D = \left\{ i \in N : \left|\mathcal{C}_i\right| > 0 \right\}$$

$USI^N \subseteq US^N \ (N,\mathcal{F})$ satisfying:

(1) For all $S,T \in \mathcal{F}$ with $|S \cap T| \geq 2$, we have $S \cap T \in \mathcal{F}$

(2) all non-unitary feasible coalition can be expressed in an unique way as union of non-unitary supports

Theorem

The Myerson value is the unique allocation rule on USI^N satisfying

- additivity
- superfluous support
- point unanimity $\left(\alpha = \dfrac{v^{\mathcal{F}}(\mathcal{D})}{|\mathcal{D}|}\right)$

$$\gamma : US^N \to \mathbb{R}^n$$

ADDITIVITY

$$\gamma(N, v+w, \mathcal{F}) = \gamma(N, v, \mathcal{F}) + \gamma(N, w, \mathcal{F})$$

SUPERFLUOUS PLAYER

$$\gamma(N, v, \mathcal{F}) = \gamma(N, v, \mathcal{F}_{N\setminus\{i\}})$$

$$v^{\mathcal{F}}(S) = v^{\mathcal{F}}(S \setminus \{i\}), \ \forall S \subseteq N$$

$$\mathcal{F}_{N\setminus\{i\}} = \{F \in \mathcal{F} : F \subseteq N \setminus \{i\}\}$$

POINT UNANIMITY

$$\gamma_i(N, v, \mathcal{F}) = \begin{cases} \alpha, & \text{if } i \in D \\ 0, & \text{otherwise} \end{cases}$$

$$v^{\mathcal{F}} = \beta u_D, \quad D = \{i \in N : |\mathcal{C}_i| > 0\}$$

Theorem

The Myerson value is the unique allocation rule on US^N satisfying

- additivity
- superfluous player
- point unanimity

Lemma

$\gamma : US^N \to \mathbb{R}^n$. If γ is an additive allocation rule satisfying superfluous player then

$$\gamma\left(N, v, \mathcal{F}\right) = \gamma\left(N, v^{\mathcal{F}}, \mathcal{F}\right), \quad \forall \left(N, v, \mathcal{F}\right) \in US^N$$

$$\gamma : US^N \to \mathbb{R}^n$$

ADDITIVITY

$$\gamma\left(N, v+w, \mathcal{F}\right) = \gamma\left(N, v, \mathcal{F}\right) + \gamma\left(N, w, \mathcal{F}\right)$$

STRONG SUPERFLUOUS SUPPORT

$$\gamma\left(N, v, \mathcal{F}\right) = \gamma\left(N, v, \overline{\mathcal{B} \setminus \{H\}}\right)$$

$$v^{\mathcal{F}}(S) = v^{\overline{\mathcal{B} \setminus \{H\}}}(S),\ \forall S \subseteq N,\ \forall H \in \mathcal{C}$$

POINT UNANIMITY

$$\gamma_i\left(N, v, \mathcal{F}\right) = \begin{cases} \alpha, & \text{if } i \in D \\ 0, & \text{otherwise} \end{cases}$$

$$v^{\mathcal{F}} = \beta u_D, \qquad D = \left\{ i \in N : \left| \mathcal{C}_i \right| > 0 \right\}$$

Theorem

The Myerson value is the unique allocation rule on US^N satisfying

- additivity
- strong superfluous support
- point unanimity

Lemma

$\gamma : US^N \to \mathbb{R}^n$. If γ is an additive allocation rule satisfying point unanimity then

$$\gamma\left(N, v, \mathcal{F}\right) = \gamma\left(N, v^{\mathcal{F}}, \mathcal{F}\right), \quad \forall \left(N, v, \mathcal{F}\right) \in US^N$$

$(N,\mathcal{F}) \in US^N$. A power measure is a function

$$\sigma : US^N \longrightarrow \mathbb{R}_+^N$$
$$(N,\mathcal{F}) \longrightarrow \sigma(N,\mathcal{F}) \in \mathbb{R}_+^N$$

$\sigma_i(N,\mathcal{F})$ non-negative power to player $i \in S$ in $\mathcal{F}$

A power measure is positive if

$\sigma_i(N,\mathcal{F}) > 0$ if and only if $i \in C,\ C \in \mathcal{C}$

U-STABLE SYSTEM

$$v^{\mathcal{F}} : 2^N \longrightarrow \mathbb{R},$$

σ a positive power measure

HARSANYI POWER SOLUTION

Vasil'ev (1982)
Vasil'ev (2003)

Derks, Haller and Peters (2000)

van den Brink, van der Laan and Pruzhansky (2011)

$$\varphi^{\sigma}: \quad US^N \longrightarrow \mathbb{R}^N$$

$$(N, v, \mathcal{F}) \in US^N \longrightarrow \varphi_i^{\sigma}(N, v, \mathcal{F}) \in \mathbb{R}^N$$

$$\varphi_i^{\sigma}(N, v, \mathcal{F}) = \sum_{\substack{T \subseteq N, i \in T \\ \sum_{j \in T} \sigma_j(N, \mathcal{F}_T) > 0}} \frac{\sigma_i(N, \mathcal{F}_T)}{\sum_{j \in T} \sigma_j(N, \mathcal{F}_T)} \Delta_{v^{\mathcal{F}}}(T)$$

$US^N = \{(N, v, \mathcal{F})$ union stable cooperation structures$\}$

Theorem

Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : US^N \to \mathbb{R}^n$ is an allocation rule satisfying additivity

$\gamma : US^N \to \mathbb{R}^n$

SUPERFLUOUS SUPPORT

$$\gamma(N, v, \mathcal{F}) = \gamma\left(N, v, \overline{\mathcal{B} \setminus \{H\}}\right)$$

$$v^{\mathcal{C}}(\mathcal{A}) = v^{\mathcal{C}}(\mathcal{A} \setminus \{H\}),\ \forall \mathcal{A} \subseteq \mathcal{C},\ \forall H \in \mathcal{C}$$

$US^N = \{(N, v, \mathcal{F})\ \text{union stable cooperation structures}\}$

$USI^N \subset US^N$

1. $\forall S, T \in \mathcal{F}$ with $|S \cap T| \geq 2$ we have $S \cap T \in \mathcal{F}$
2. All non-unitary feasible coalitions can be written in a unique way as a union of non-unitary supports

Theorem

Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : USI^N \to \mathbb{R}^n$ satisfies the superfluous support property on USI^N

$$I_i(N,\mathcal{F}) = \sum_{C\in\mathcal{C}_i} \frac{1}{|C|}$$

$$E_i(N,\mathcal{F}) = 1 \text{ if } \mathcal{C}_i \neq \emptyset$$

$$E_i(N,\mathcal{F}) = 0 \text{ if } \mathcal{C}_i = \emptyset$$

$$\mathcal{C}_i = \{C \in \mathcal{C} : i \in C\}$$

$$\gamma : US^N \to \mathbb{R}^n$$

INFLUENCE

$$\gamma(N, v, \mathcal{F}) = \alpha I(N,\mathcal{F})$$

$$I_i(N,\mathcal{F}) = \begin{cases} \sum_{C\in\mathcal{C}_i} \frac{1}{|C|}, & \text{if } \mathcal{C}_i \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$v^{\mathcal{C}}(\mathcal{A}) = f(|\mathcal{A}|), \quad \forall \mathcal{A} \subseteq \mathcal{C}$$

$$\gamma : US^N \to \mathbb{R}^n$$

WEAK INFLUENCE

$$\gamma\left(N, v, \mathcal{F}\right) = \alpha I\left(N, \mathcal{F}\right)$$

$$I_i\left(N, \mathcal{F}\right) = \begin{cases} \sum_{C \in \mathcal{C}_i} \frac{1}{|C|}, & \text{if } \mathcal{C}_i \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

$$v^{\mathcal{C}} = \beta u_{\mathcal{C}}, \quad \forall \mathcal{A} \subseteq \mathcal{C}$$

$\sigma-$ INFLUENCE PROPERTY

$$\gamma\left(N, v, \mathcal{F}\right) = \alpha \sigma\left(N, \mathcal{F}\right)$$

$$v^{\mathcal{C}} = \beta u_{\mathcal{C}}, \quad \forall \mathcal{A} \subseteq \mathcal{C}$$

$\sigma-$ INFLUENCE PROPERTY YIELDS THE WEAK INFLUENCE PROPERTY WHEN $\sigma = I$

$\gamma : US^N \to \mathbb{R}^n$

POINT ANONIMITY

$$\gamma_i\left(N, v, \mathcal{F}\right) = \begin{cases} \alpha, & \text{if } i \in D \\ 0, & \text{otherwise} \end{cases}$$

$$v^{\mathcal{F}}\left(S\right) = f\left(\left|S \cap D\right|\right), \qquad D = \left\{i \in N : \left|\mathcal{C}_i\right| > 0\right\}$$

POINT UNANIMITY

$$\gamma_i\left(N, v, \mathcal{F}\right) = \begin{cases} \alpha, & \text{if } i \in D \\ 0, & \text{otherwise} \end{cases}$$

$$v^{\mathcal{F}} = \beta u_D, \qquad D = \left\{i \in N : \left|\mathcal{C}_i\right| > 0\right\}$$

$\gamma : US^N \to \mathbb{R}^n$

$\sigma -$ POINT UNANIMITY

$$\gamma\left(N, v, \mathcal{F}\right) = \alpha\sigma\left(N, \mathcal{F}\right)$$

$$v^{\mathcal{F}} = \beta u_D$$

$\sigma -$POINT UNANIMITY YIELDS POINT UNANIMITY WHEN

$$\sigma = E$$

Proposition

Let σ be a positive power measure. The Harsanyi power solution $\varphi^{\sigma} : US^N \to \mathbb{R}^n$ satisfies σ-point unanimity on US^N

Lemma

(*i*) Let $(N,v,\mathcal{F}) \in US^N$. If $(N,v,\mathcal{F})$ is support unanimous, then $(N,v,\mathcal{F})$ is also point unanimous

(*ii*) Let $(N,v,\mathcal{F}) \in USI^N$. Then $(N,v,\mathcal{F})$ is support unanimous if and only if $(N,v,\mathcal{F})$ is point unanimous

Corollary

Let σ be a positive power measure. The Harsanyi power solution $\varphi^{\sigma} : US^N \to \mathbb{R}^n$ satisfies σ-influence property on US^N

Theorem

Let σ be a positive power measure. The Harsanyi power solution $\varphi^{\sigma}: USI^{N} \rightarrow \mathbb{R}^{n}$ is the unique allocation rule in USI^{N} satisfying

- additivity
- superfluous support property
- σ-influence property

Corollary

Let σ be a positive power measure. The Harsanyi power solution $\varphi^{\sigma}: USI^{N} \rightarrow \mathbb{R}^{n}$ is the unique allocation rule in USI^{N} satisfying

- additivity
- superfluous support property
- σ-point unanimity property

Corollary

Taking the power measure $\sigma = I$, it holds that

$$\varphi^{\mathrm{I}}\left(N, v, \mathcal{F}\right) = \pi\left(N, v, \mathcal{F}\right),$$

for all $\left(N, v, \mathcal{F}\right) \in USI^{N}$

Example $N=\{1,2,3,4\},\ \mathcal{F}=\{\{1,2\},\{1,2,3\},\{2,3,4\},N\},\ v=u_{\{1,2,3\}}$

$$\mathcal{B}=\mathcal{C}=\{\{1,2\},\{1,2,3\},\{2,3,4\}\},\quad v^{\mathcal{F}}=u_{\{1,2,3\}}$$

$$N=\{1,2\}\cup\{2,3,4\}=\{1,2,3\}\cup\{2,3,4\}$$

$$(N,v,\mathcal{F})\notin USI^{N}$$

$$\{1,2,3\}\cap\{2,3,4\}=\{2,3\}\notin\mathcal{F}$$

$$\mu(N,v,\mathcal{F})=\left(\frac{4}{12},\frac{4}{12},\frac{4}{12},0\right),\quad \varphi^{E}(N,v,\mathcal{F})=\left(\frac{4}{12},\frac{4}{12},\frac{4}{12},0\right)$$

$$\pi(N,v,\mathcal{F})=\left(\frac{11}{36},\frac{13}{36},\frac{10}{36},\frac{2}{36}\right),\quad \varphi^{I}(N,v,\mathcal{F})=\left(\frac{5}{12},\frac{5}{12},\frac{2}{12},0\right)$$

$$\mu(N,v,\mathcal{F})=\varphi^{E}(N,v,\mathcal{F}),\quad (N,v,\mathcal{F})\in US^{N}$$

$$\pi(N,v,\mathcal{F})\neq\varphi^{I}(N,v,\mathcal{F}),\quad (N,v,\mathcal{F})\in US^{N}$$

Corollary

(i) The position value $\pi : USI^N \to \mathbb{R}^n$ is the unique allocation rule on USI^N satisfying additivity, superfluous support property and I-point unanimity

(ii) The position value $\pi : USI^N \to \mathbb{R}^n$ is the unique allocation rule on USI^N satisfying additivity, superfluous support property and I-influence property

(iii) The Myerson value $\mu : USI^N \to \mathbb{R}^n$ is the unique allocation rule on USI^N satisfying additivity, superfluous support property and E-point unanimity

(iv) The Myerson value $\mu : USI^N \to \mathbb{R}^n$ is the unique allocation rule on USI^N satisfying additivity, superfluous support property and E-influence property

$\gamma : US^N \to \mathbb{R}^n$

CONNECTEDNESS

$$\gamma\left(N, v, \mathcal{F}\right) = \gamma\left(N, w, \mathcal{F}\right)$$

$$v(S) = w(S),\ \forall S \in \mathcal{F}$$

INESSENTIAL SUPPORT PROPERTY

$$\gamma\left(N, v, \mathcal{F}\right) = \gamma\left(N, v, \overline{\mathcal{B} \setminus \{H\}}\right)$$

$$v = cu_T,\ c \in R,\ T \in \mathcal{F},\ T \neq \emptyset$$

$$H \in \mathcal{C},\ H \not\subset T$$

Proposition

Let σ be a positive power measure. The Harsanyi power solution $\varphi^{\sigma}: US^{N} \to \mathbb{R}^{n}$ satisfies the inessential support property and connectedness on US^{N}

Theorem

Let σ be a positive power measure. The Harsanyi power solution $\varphi^{\sigma}: US^{N} \to \mathbb{R}^{n}$ is the unique allocation rule on US^{N} satisfying

- additivity
- σ-point unanimity property
- inessential support property
- connectedness

Corollary

The Myerson value $\mu : US^N \to \mathbb{R}^n$ is the unique allocation rule on US^N satisfying

- additivity
- point unanimity property
- inessential support property
- connectedness

Proposition

$\gamma : US^N \to \mathbb{R}^n$. If γ is an additive allocation rule satisfying superfluous player property then γ verifies connectedness

Theorem

Let σ be a positive power measure. The Harsanyi power solution $\varphi^\sigma : US^N \to \mathbb{R}^n$ is the unique allocation rule on US^N satisfying

- additivity
- σ-point unanimity property
- superfluous player property

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