Towards an evolutionary cooperative game theory

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Abstract

The idea behind evolutionary game theory is to interpret payoffs as fitness. Non-cooperative game theory has seen many interesting results and applications. The aim of this paper is to introduce a replicator-dynamics version for cooperative game theory. We generate payoffs, or fitness values, by way of the continuous Shapley value introduced by Aumann & Shapley (1974). We can show the usefullness of our approach by way of simple examples such as the apex game. (Also, we generate some general results.)

Keywords: Continuous Shapley value, replicator dynamics, asymptotic stability, apex game

JEL classification: C71

1. Introduction

Evolutionary models of various forms have been part and parcel of economics for a long time (see, for example, the articles collected by Witt 1993). A specific class of models have been developed within game theory. In usual parlance, evolutionary game theory (see, for example, Weibull (1995) or Samuelson (1997)) means evolutionary theory applied to non-cooperative games. The aim of this paper is to develop an evolutionary cooperative game theory where we concentrate on the simple transferable-utility case.

In terms of interpretation, evolutionary noncooperative game theory (ENGT) differs from evolutionary cooperative game theory (ECGT). ENGT builds on the idea that two players are drawn at random from a large population. They are programmed to play a certain (mixed) strategy and the strategy that does better than other strategies grows faster. In contrast, ECGT concern all the agents of all players at the same time – the whole economy, so to speak.

Cooperative game theory rests on two pillars. First, the economic (or political or sociological ...) situation is described by a player set N and a coalition function $v : 2^N \to \emptyset$. Subsets of N are called coalitions, with N being the grand coalition. For every coalition $K \subseteq N$, $v(K)$ is its worth that stands for the possibilities open to that coalition.

Consider, for example, the apex game h for $N = \{1, ..., 4\}$. It is defined by

$$h(K) = \begin{cases} 1, & 1 \in K \text{ and } K \backslash \{1\} \neq \emptyset \\ 1, & K = N \backslash \{1\} \\ 0, & \text{otherwise} \end{cases}$$

Thus, the apex player 1 needs one additional player to produce the worth 1. This worth can also be created by the coalition of the three less important players 2, 3, and 4.

Cooperative game theory's second pillar are solution concepts, the most famous being the core and the Shapley's (1953) value. From the point of view of applicability, the Shapley value has the advantage of producing unique payoffs for the n players. Thus, the coalition function is the input into a solution concept and the payoff vector the output. For the above apex game, the payoff vector is

$$Sh(h) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right).$$

Imbedding games like the apex game into an evolutionary setting, we interpret the payoffs as fitness. A player's success feeds into his proliferation. In order to model reproductive differences between players, we distinguish between players (like the four players in the apex game) and agents who take up the roles (or types) of these n players. If a role is particularly fruitful (in producing relatively high payoffs for the agents assuming that role), the relative number of agents assuming that role increases.

While the number of players is a natural number, it is best to deal with uncountably many agents, for example a continuum of agents. Therefore, we need an extended coalition function that is capable of dealing with non-integer players (agents). The Lovasz extension v^ℓ or the multi-linear (Owen) extension v_{mult} are suitable candidates. In this paper, we choose the Lovasz extension (see the conclusions).

Extensions of coalition functions cannot be an input for the (standard) Shapley value. Therefore, we use the continuous Shapley value introduced by Aumann & Shapley (1974). We thus obtain the payoff information seen as a fitness variable and can then define the replicator dynamics. Whenever an agent receives an above-average payoff, the population share of this agent (and of all other agents playing the same role) will increase – a standard result in replicator-dynamics research (for example Weibull 1995, chapter 3).

In the following section, we will formally introduce agents, the extended coalition functions, and the continuous Shapley value. We present the replicator dynamics in section 3.

For the apex game, we find two (up to symmetry) asymptotically stable population configurations:

- $x = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$ and
- $x = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

The first configuration obtains whenever the apex player's initial share is at least as high as the share of any of the weak players. In that case, the apex player teams up with the weak player who has the largest initial share. If, however, the apex player's initial share is lower than the shares of all the weak players, our replicator dynamics yields the second stable-stable population share – the apex player's share tends to zero.

Section 4 offers some conclusions.

2. Players and agents, extended coalition functions, and the continuous Shapley value

2.1. Coalition functions

Let N be the nonempty player set. A TU game is a pair (N, v) where v is a function $2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. v is called a game, a characteristic function or a coalition function. The set of all coalition functions on N is denoted by $V(N)$. A payoff vector x for N is an element of $\mathbb{R}^N$ or a function $N \to \mathbb{R}$.

For any nonempty coalition $T \subseteq N$, $u_T(S) = 1$, $S \supseteq T$; 0 otherwise, defines a game, called unanimity game. The set of unanimity games is a basis of $V(N)$ in the sense of linear algebra. I.e., for every $v \in V(N)$, there exist unique coefficients $m_v(T)$ such that

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} m_v(T) u_T$$

holds. The coefficients are given by

$$\begin{aligned} m_v &: 2^N \to \mathbb{R}, \\ & T \mapsto m_v(T) = \sum_{K \in 2^T \setminus \{\emptyset\}} (-1)^{|T|-|K|} v(K) \end{aligned}$$

and are known as Harsanyi dividends. They are best calculated by the well-known induction formula

$$\begin{aligned} m_v(\emptyset) &= 0, \\ m_v(S) &= v(S) + \sum_{K \subset S} (-1)^{|S|-|K|} v(K) \end{aligned}$$

holds.

Example 2.1. *Revisiting the apex game defined in the introduction, we obtain the Harsanyi dividends*

$$\begin{aligned} m_h(\emptyset) &= 0, \\ m_h(1) &= m_h(2) = 0, \\ m_h(1,2) &= h(1,2) = 1, \\ m_h(2,3) &= h(2,3) = 0, \\ m_h(2,3,4) &= h(2,3,4) = 1, \\ m_h(1,2,3) &= h(1,2,3) - h(1,2) - h(1,3) - h(2,3) = -1 \\ m_h(1,2,3,4) &= h(1,2,3,4) - h(1,2,3) - h(1,2,4) - h(1,3,4) - h(2,3,4) \\ & \quad + h(1,2) + h(1,3) + h(1,4) \\ &= 0 \end{aligned}$$

and can express the apex game (defined in the introduction) as a linear combination of some unanimity games to obtain

$$\begin{aligned} h(K) &= -u_{\{1,2,3\}}(K) - u_{\{1,2,4\}}(K) - u_{\{1,3,4\}}(K) + u_{\{2,3,4\}}(K) \\ &\quad + u_{\{1,2\}}(K) + u_{\{1,3\}}(K) + u_{\{1,4\}}(K). \end{aligned}$$

For example, letting $K := \{1,2\}$, *we find*

$$\begin{aligned} h(\{1,2\}) &= -u_{\{1,2,3\}}(\{1,2\}) - u_{\{1,2,4\}}(\{1,2\}) - u_{\{1,3,4\}}(\{1,2\}) + u_{\{2,3,4\}}(\{1,2\}) \\ &\quad + u_{\{1,2\}}(\{1,2\}) + u_{\{1,3\}}(\{1,2\}) + u_{\{1,4\}}(\{1,2\}) \\ &= -0 - 0 - 0 + 0 \\ &\quad +1 + 0 + 0 \\ &= 1. \end{aligned}$$

2.2. Agents

For present and future reference, we define

- $\mathbb{R}^N_+ := \left\{x \in \mathbb{R}^N : x_i \geq 0 \text{ for all } i \in N\right\}$,
- $\mathbb{R}^N_{++} := \left\{x \in \mathbb{R}^N : x_i > 0 \text{ for all } i \in N\right\}$,
- $\Delta := \Delta(N) := \left\{x \in \mathbb{R}^n_+ : \sum x_i = 1\right\}$ and
- $int(\Delta) := int(\Delta(N)) = \left\{x \in \mathbb{R}^n_{++} : \sum x_i = 1\right\}$.

Our economy is not populated by the n players but by intervals of agents, one interval for each player. Assume any vector $s = (s_1, ..., s_n) \in \mathbb{R}^n_+$ where s_i is the number of agents taking on player i's role. In case of $s \in \{0,1\}^N$, we identify s with the coalition

$$\mathbf{K}(s) := \{i \in N : s_i = 1\}.$$

Let λ be the Lebesgues-Borel measure on $\mathbb{R}$. We now choose n intervals $I_i \subseteq \mathbb{R}$, such that $\lambda(I_i) = s_i$ holds for every $i \in N$ and such that the intervals do not intersect. Thus, I_i stands for the agents representing player i and $I := \cup_{i \in N} I_i$ is the set of all agents. Somewhat loosely, we sometimes call an agent from I_i agent i. By $\mathcal{B}$ we mean the set of Borel sets of I. We now define μ^s_i by

$$\mu^s_i(K) := \lambda(K \cap I_i), K \in \mathcal{B}.$$

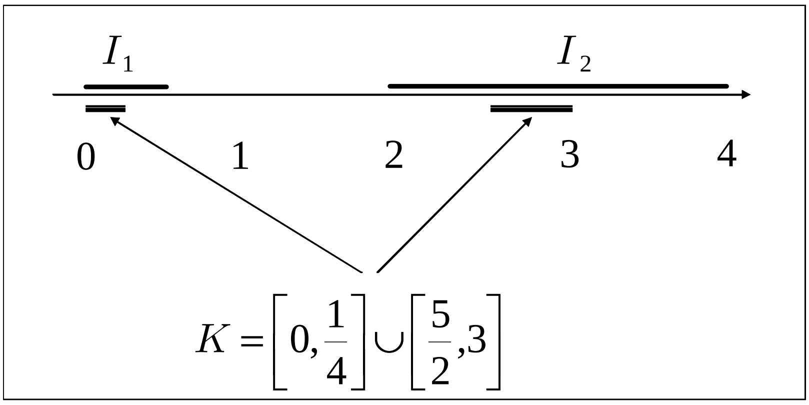

Figure 2.1: Attributing measures to types

It is easy to show that μ_i^s is a measure on $(I,\mathcal{B})$. Let $\mu^s = \prod_{i\in N}\mu_i^s : \mathcal{B}\to\mathbb{R}^n$, $K\mapsto(\mu_i^s(K))_{i\in N}$ be the Cartesian product of these measures. $\mu^s(K)$ distributes the agents in K among the n player types and attributes a measure to each player type.

Example 2.2. *Consider fig. We have* $N=\{1,2\}$, $s=\left(\frac{1}{2},2\right)$ *and intervals* $I_1=\left[0,\frac{1}{2}\right]$ *and* $I_2=[2,4]$. *For* $K:=\left[0,\frac{1}{4}\right]\cup\left[\frac{5}{2},3\right]\in\mathcal{B}$ *we obtain*

$$\begin{aligned}\mu_1^s(K) &= \lambda(K\cap I_1)=\lambda\left(\left[0,\frac{1}{4}\right]\right)=\frac{1}{4}\text{ and}\\ \mu_2^s(K) &= \lambda(K\cap I_2)=\lambda\left(\left[\frac{5}{2},3\right]\right)=\frac{1}{2}.\end{aligned}$$

2.3. Extensions

2.3.1. Definition and properties

We now need to extend our coaliton function v so as to accomodate players of any non-negative size. An extension of a coalition function v on N is a function

$$v^{ext}:\mathbb{R}_+^N\to\mathbb{R}$$

obeying

$$v^{ext}(s)=v(\mathbf{K}(s)),s\in\{0,1\}^N.$$

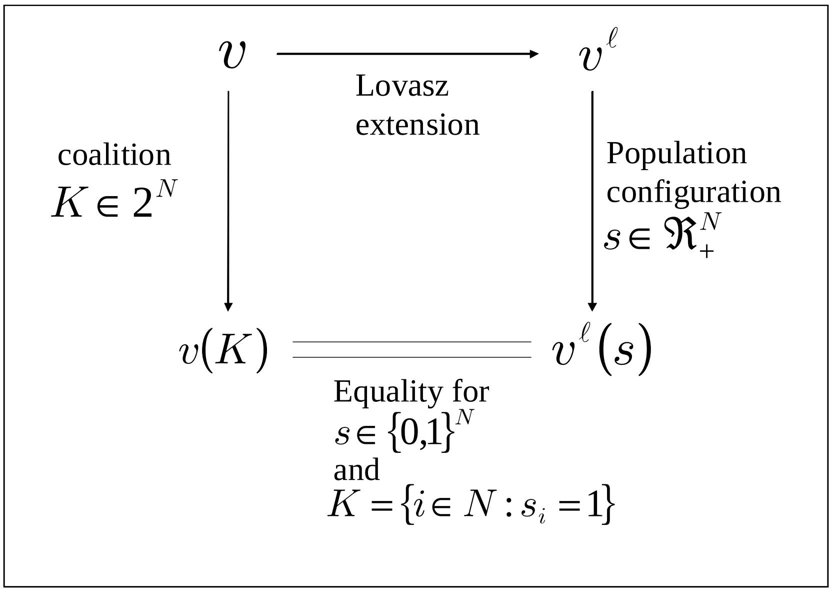

Figure 2.2: A coalition function and its Lovasz extension

Following Owen (1972), the so-called multilinear extension (MLE) is defined by

$$v^{MLE}(s) := \sum_{T \in 2^N \setminus \{\emptyset\}} m_v(T) \cdot \prod_{i \in T} s_i$$

while the Lovasz extension is given by

$$v^\ell(s) := \sum_{T \in 2^N \setminus \{\emptyset\}} m_v(T) \cdot \min_{i \in T} s_i. \tag{2.1}$$

Example 2.3. *Using the apex game and assuming $s_2 \leq s_3 \leq s_4$ without loss of* *generality, we calculate the apex game's Lovasz extension:*

$$\begin{aligned}
h^{\ell}(s) &= -\min_{i\in\{1,2,3\}} s_i - \min_{i\in\{1,2,4\}} s_i - \min_{i\in\{1,3,4\}} s_i + \min_{i\in\{2,3,4\}} s_i \\
&\quad + \min_{i\in\{1,2\}} s_i + \min_{i\in\{1,3\}} s_i + \min_{i\in\{1,4\}} s_i \\
&= \begin{cases} -3s_1 + s_2 + 3s_1, & s_1 \leq s_2 \leq s_3 \leq s_4 \\ -2s_2 - s_1 + s_2 + s_2 + 2s_1, & s_2 \leq s_1 \leq s_3 \leq s_4 \\ -2s_2 - s_3 + s_2 + s_2 + s_3 + s_1, & s_2 \leq s_3 \leq s_1 \leq s_4 \\ 2s_2 - s_3 + s_2 + s_2 + s_3 + s_4, & s_2 \leq s_3 \leq s_4 \leq s_1 \end{cases} \\
&= \begin{cases} s_2, & s_1 \leq s_2 \leq s_3 \leq s_4 \\ s_1, & s_2 \leq s_1 \leq s_3 \leq s_4 \\ s_1, & s_2 \leq s_3 \leq s_1 \leq s_4 \\ s_4, & s_2 \leq s_3 \leq s_4 \leq s_1 \end{cases}
\end{aligned}$$

These results also make intuitive sense. In case of $s_1 \leq s_2 \leq s_3 \leq s_4$, the maximal value $\min_{i\in\{2,3,4\}} s_i$ is achieved if the three small players cooperate. Player 1 cooperates with players 3 or 4 in the second line and with player 4 in the third line and fourth line.

According to Lovasz (1983) and Algaba, Bilbao, Fernandez & Jimenez (2004), the Lovasz extension obeys the following properties:

- (m.i) v^{ℓ} is non-negatively homogenous, i.e., $v^{\ell}(\lambda s) = \lambda v^{\ell}(s)$ for all $\lambda \geq 0$.
- (m.ii) $(v+w)^{\ell} = v^{\ell} + w_{\min}$.
- (m.iii) $(\lambda v)^{\ell} = \lambda v^{\ell}$ for all $\lambda \in \mathbb{R}$.
- (m.iv) v is supermodular iff v^{ℓ} is concave.

Concavity of v^{ℓ} means: For any $\alpha \in [0,1]$ and any $s, s' \in \mathbb{R}^n_+$, we have

$$\alpha v^{\ell}(s) + (1-\alpha) v^{\ell}(s') \leq v^{\ell}(\alpha s + (1-\alpha)(s'))$$

Furthermore, Wiese (2005) observed the following properties:

Lemma 2.4. *Let v be a (monotonic and non-trivial) coalition function. We obtain*

- *(m.v) $s \leq s' \Rightarrow v^{\ell}(s) \leq v^{\ell}(s')$ for all $s, s' \in \mathbb{R}^n_+$.*

- *(m.vi) If v is convex (supermodular), then*

$$v^{\ell}(s+s') \geq v^{\ell}(s) + v^{\ell}(s')$$

for all $s, s' \in \mathbb{R}^n_+$.

- *(m.vii) If v is additive, then*

$$v^{\ell}(s+s') = v^{\ell}(s) + v^{\ell}(s')$$

for all $s, s' \in \mathbb{R}^n_+$.

Remark 1. *Obviously, v^{ℓ} is continuous. Also, we have $v^{\ell}(0) = 0$.*

Remark 2. *Extensions of unanimity games are monotonic. So are combinations of unanimity games with positive coefficients (Harsanyi dividends). By definitional equation 2.1, any extended game v^{ℓ} is the difference of two monotonic games and hence of bounded variation.*

2.3.2. Differentiability

We will work with the Lovasz extension rather than the multilinear extension. For a unanimity game u_T, the Lovasz extension is given by

$$u_T^{\ell}(s) := \min_{i \in T} s_i.$$

Let $s_- := \min_T(s) := \min_{i \in T} s_i$ be the minimum player size of the T-players and let $T_- := \{j | s_j = s_-\}$ be the set of T-players with minimal size. Then, for any unanimity game u_T and any player $i \in T$ we find

$$\frac{\partial u_T^{\ell}(s)}{\partial s_i} = \begin{cases} 1, & T_- = \{i\} \\ 0, & i \notin T_- \end{cases}$$

but u_T^{ℓ} is not partially differentiable at s with respect to s_i in case of $i \in T_- \neq \{i\}$ (i is one of several players with minimal size).

For later purposes, we consider the following approximation by partially differentiable functions $u_T^{\ell,m} = \min_T^m : \mathbb{R}^N_+ \to \mathbb{R}$ which are defined, for all $\emptyset \neq T \subseteq N$ and $m \in \mathbb{N}$, by

$$\min_T^m(s) := \begin{cases} 0, & s_- = 0 \\ \frac{|T|^{\frac{1}{m}}}{\left(\sum_{i \in T} \frac{1}{s_i^m}\right)^{\frac{1}{m}}} = |T|^{\frac{1}{m}} \left(\sum_{i \in T} s_i^{-m}\right)^{-\frac{1}{m}}, & \text{else.} \end{cases} \tag{2.2}$$

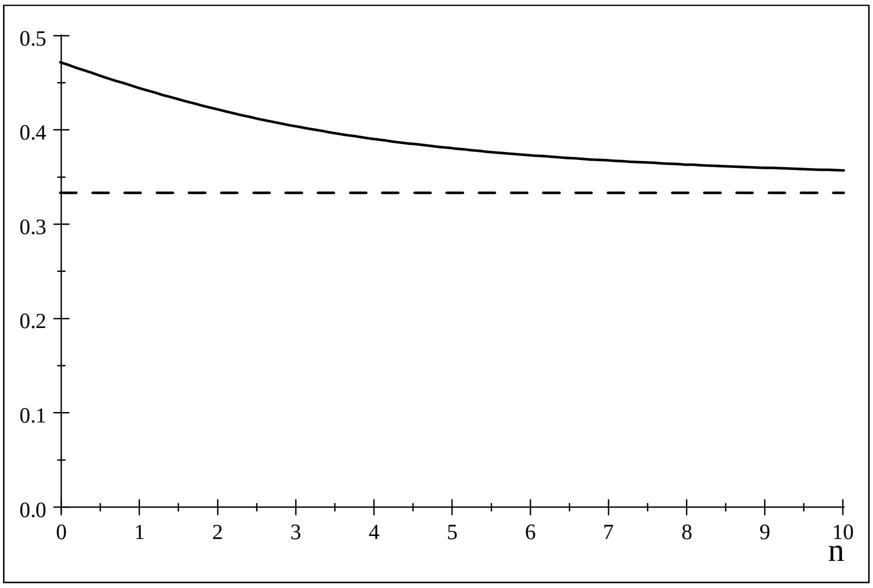

Figure 2.3: Approximation of Lovasz extension

Using $N = \{1,2,3\}$, $T = \{1,2\}$ and $s = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{6}\right)$, the approximation can be seen from the figure below where the dashed line is at $\frac{1}{3} = s_-$ and the solid line gets closer and closer to $\frac{1}{3}$ as m approaches infinity.

Indeed, by standard rules for limites, we can confirm $\lim_{m\to\infty} \min_T^m (s) = \min_T (s)$ for all $\emptyset \neq T \subseteq N$ and all $s \in \mathbb{R}_+^N$.

We denote by $v^{\ell,m}$ the m-th approximation of v^ℓ given by

$$v^{\ell,m}(s) := \sum_{\emptyset \neq T \subseteq N} \lambda_T(v) \cdot \min_T^m (s), \qquad m \in \mathbb{N},$$

which also is linear in $v \in V(N)$ and non-negatively homogenous in $s \in \mathbb{R}_+^N$.

2.4. Vector measure games and Shapley value

Before we can apply the continuous Shapley value, we need to define the appropriate vector measure games

$$\begin{aligned} v^{\ell,s} &: = v^\ell \circ \mu^s : \mathcal{B} \to \mathbb{R} \text{ and} \\ v^{\ell,m,s} &: = v^{\ell,m} \circ \mu^s : \mathcal{B} \to \mathbb{R} \end{aligned}$$

Given a coalition $K \in \mathcal{B}$, $\mu^s(K)$ specifies how to devide K among the n groups and how to measure these subgroups. v^ℓ or $v^{\ell,m}$ then yield the worth in accordance with the underlying TU game v.

We now apply the Aumann & Shapley (1974, Theorem B) diagonal formula (see also Neyman 2002, pp. 2141). Taking any coalition $K \in \mathcal{B}$ and any coalition function v, this formula yields

$$\mathrm{Sh}\, v^{\ell,m,s}(K) = \sum_{j=1}^{n} \mu_j^s(K) \int_0^1 \left. \frac{\partial v^{\ell,m}}{\partial s_j} \right|_{\tau s} d\tau$$

$\mathrm{Sh}\, u_T^{\ell,m,s}(K)$ is the payoff accruing to coalition K. The analogue of player j's marginal contribution in the discrete Shapley formula is the derivative of the coalition's worth with respect to the measure of agents of player type j. This derivative is evaluated at $\tau s = (\tau s_1, ..., \tau s_n)$. Thus, the formula looks at coalitions on the diagonal only. Remember that we have a continuum of agents. If we take a subset of agents by chance, it is likely that the composition in this subset (how many agents of player type 1, player type 2 etc.) will not deviate much from the composition in the overall population.

Letting $K := I_i$ and $v^{\ell,m} := u_T^{\ell,m}$, we obtain

Lemma 2.5. *We have*

$$\mathrm{Sh}\, u_T^{\ell,m,s}(I_i) = \begin{cases} 0, & i \notin T \\ 0, & s_- = 0 \\ |T|^{\frac{1}{m}}\, s_i^{-m} \left(\sum_{j \in T} s_j^{-m} \right)^{-\frac{m+1}{m}}, & i \in T \text{ and } s_- \neq 0 \end{cases}$$

and

$$\mathrm{Sh}\, u_T^{\ell,s}(I_i) := \lim_{m \to \infty} \mathrm{Sh}\, u_T^{\ell,m,s}(I_i) = \begin{cases} \frac{s_-}{|T_-|}, & i \in T_-, s_- \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Using the additivity of the value, one also obtains the payoffs $\mathrm{Sh}\, v^{\ell,s}(I_i)$ and $\mathrm{Sh}\, v^{\ell,m,s}(I_i)$, $i \in N$, $m \in \mathbb{N}$. One easily checks that $\mathrm{Sh}\, v^{\ell,s}(I_i)$ and $\mathrm{Sh}\, v^{\ell,m,s}(I_i)$ are homogenous of degree 1 with respect to s.

Definition 2.6. *Consider a coalition function $v \in V(N)$, a player $i \in N$ and a population configuration $s \in \mathbb{R}_+^N$ with $s_i > 0$. The averge payoff accruing to agents from I_i is also called agent i's payoff and is given by*

$$\mathrm{Sh}_i\left(v^{\ell,s}\right) := \frac{\mathrm{Sh}\, v^{\ell,s}(I_i)}{s_i}.$$

Example 2.7. *For any unanimity game, we find*

$$\mathrm{Sh}_i\left(u_T^{\ell,s}\right)=\begin{cases}\frac{1}{|T_-|}, & i\in T_-, s_-\neq 0\\ 0, & \textit{otherwise}\end{cases}$$

so that $N=T=\{1,2,3\}$ *yields*

$$\mathrm{Sh}_1\left(u_T^{\ell,s}\right)=\begin{cases}1, & s_1<\min(s_2,s_3)\\ \frac{1}{2}, & s_1=s_2<s_3\\ \frac{1}{2}, & s_1=s_3<s_2\\ \frac{1}{3}, & s_1=s_2=s_3\\ 0, & s_1>s_2 \text{ or } s_1>s_3\end{cases}$$

and, using example 2.1, the apex payoffs are given by

$$\begin{aligned}&\left(\mathrm{Sh}_1\left(h^{\ell,s}\right),\ \mathrm{Sh}_2\left(h^{\ell,s}\right),\ \mathrm{Sh}_3\left(h^{\ell,s}\right),\ \mathrm{Sh}_4\left(h^{\ell,s}\right)\right)\\ =&\begin{cases}(0,1,0,0), & s_1<s_2<s_3<s_4\\ \left(0,\frac{1}{2},\frac{1}{2},0\right) & s_1<s_2=s_3<s_4\\ \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) & s_1<s_2=s_3=s_4\\ \left(\frac{1}{2},\frac{1}{2},0,0\right) & s_1=s_2<s_3<s_4\\ \left(\frac{2}{3},\frac{1}{6},\frac{1}{6},0\right) & s_1=s_2=s_3<s_4\\ \left(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right) & s_1=s_2=s_3=s_4\\ (1,0,0,0), & s_2<s_1<s_3<s_4\\ (1,0,0,0) & s_2<s_1=s_3<s_4\\ \left(\frac{2}{3},0,\frac{1}{6},\frac{1}{6}\right) & s_2<s_1=s_3=s_4\\ (1,0,0,0) & s_2<s_1<s_3=s_4\\ (1,0,0,0), & s_2<s_3<s_1<s_4\\ (1,0,0,0) & s_2=s_3<s_1<s_4\\ \left(\frac{1}{2},0,0,\frac{1}{2}\right) & s_2\leq s_3<s_1=s_4\\ (0,0,0,1) & s_2<s_3<s_4<s_1\\ \left(0,0,\frac{1}{2},\frac{1}{2}\right) & s_2<s_3=s_4<s_1\\ \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) & s_2=s_3=s_4<s_1\\ (0,0,0,1) & s_2=s_3<s_4<s_1\end{cases}\end{aligned}$$

Theorem 2.8. *If* $i,j\in N$ *are symmetric in* (N,v) *and* $s_i=s_j$ *then* $\mathrm{Sh}\,v^{\ell,s}(I_i)=\mathrm{Sh}\,v^{\ell,s}(I_j)$ *and* $\mathrm{Sh}\,v^{\ell,m,s}(I_i)=\mathrm{Sh}\,v^{\ell,m,s}(I_j)$.

3. Replicator dynamics

3.1. Formula

Interpreting the agents' Shapley payoffs as fitness and assuming a constant birthrate β and a constant death rate δ, the evolution of s_i is defined by

$$\dot{s}_i = \left[\beta + \mathrm{Sh}_i\left(v^{\ell,s}\right) - \delta\right] s_i.$$

In terms of population shares

$$x_i := \frac{s_i}{\sum_{j=1}^n s_j}$$

we obtain the replicator dynamics

$$\dot{x}_i = \left(\mathrm{Sh}_i\left(v^{\ell,s}\right) - \sum_{j=1}^n \mathrm{Sh}_j\left(v^{\ell,s}\right) x_j\right) x_i$$

where the growth rate of a player's population share equals the difference of his agents' fitness and the average fitness of all agents.

3.2. Differential equation and discrete replicator dynamics

Standard methods do not guarantee the existence of a solution to the replicator equations. Therefore, we resort to discrte replicator dynamics which obviously exist. Assuming a starting point at time $t = 0$ and a population share vector $x(0) = (x_1(0), ..., x_n(0)) \in \Delta$, we define the discrete replicator dynamics by

$$x_i(t) = x_i(t-1) + x_i(t-1)\left[\mathrm{Sh}_i\left(v^{\ell,x(t-1)}\right) - \sum_{j=1}^n \mathrm{Sh}_j\left(v^{\ell,x(t-1)}\right) x_j\right], t \geq 1$$

While it is easy to check that $\sum x_i(t) = 1$ follows from $\sum x_i(t-1) = 1, t \geq 1$, we cannot, in general, exclude a negative population share. In order to avoid this problem and in order to smooth out the solution orbit, we introduce a (very small) step length $\sigma > 0$ and work with the replicator dynamics

$$x_i(t) = x_i(t-1) + x_i(t-1)\,\sigma\left[\mathrm{Sh}_i\left(v^{\ell,x(t-1)}\right) - \sum_{j=1}^n \mathrm{Sh}_j\left(v^{\ell,x(t-1)}\right) x_j\right], t \geq 1 \tag{3.1}$$

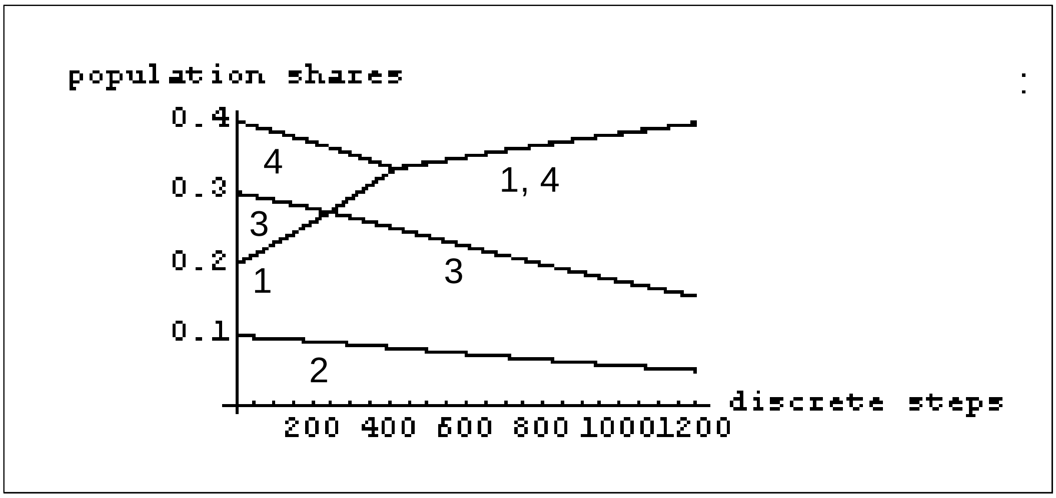

Figure 3.1: The apex player teams up with player 4

In a continuous case, σ would affect the velocity of change but not the solution orbit.

We now revisit the apex game. The initial population share vector $x(0) = \left(\frac{2}{10}, \frac{1}{10}, \frac{3}{10}, \frac{4}{10}\right)$ is used in fig. 3.1 where the plot builds on S =1200 steps with step length $\sigma = \frac{1}{600}$. In the beginning, only player 1's agent set grows. As soon as the sizes of player 1's agent set and player 4's agent set equal, both agent sets grow while the agent sets of players 2 and 3 tend towards zero.

Fig. 3.2 starts with the population configuration $\left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}\right)$. Finally, the three unimportant players grow until each reaches a population share of $\frac{1}{3}$.

Applying the formula

$$\text{number of time periods} = \text{number of steps times step length},$$

the above examples rest on 2 (physical) time periods, $2 = T = S \cdot \sigma = 1200 \cdot \frac{1}{600}$.

Definition 3.1. *Consider a coalition function $v \in V(N)$ and a starting population share vector $x(0) = (x_1(0), ..., x_n(0)) \in \Delta(N)$. The Euler replicator dynamic for T time periods is defined by the discrete replicator dynamics 3.1 obeying $0 \leq t \leq S$, $\sigma = \frac{T}{S}$ and $S \to \infty$.*

Definition 3.2. *A vector of population shares $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ is a steady state if there exists a population share vector $x(0) = (x_1(0), ..., x_n(0)) \in [(0,1)]^n$ such*

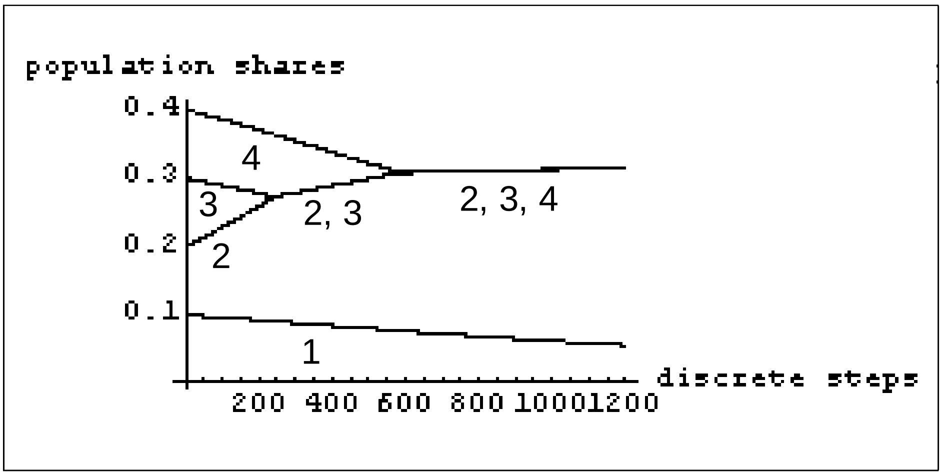

Figure 3.2: The three unimportant players trump the apex player

that the Euler replicator dynamics yields

$$\lim_{T\to\infty} x_i\,(t) = \hat{x}_i$$

for all $i = 1, ..., n$.

Definition 3.3. *A steady state* $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ *is called asymptotically stable if there exists some* $\varepsilon > 0$ *such that for all population vectors* $x\,(0)$ *obeying* $\|x\,(0) - \hat{x}\|_2 < \varepsilon$ *we have*

$$\lim_{T\to\infty} x_i\,(t) = \hat{x}_i.$$

Proposition 3.4. *Consider the apex game for four players and* $x_2\,(0) \leq x_3\,(0) \leq x_4\,(0)$ *without loss of generality. The dynamics of the apex game admit four steady states* $\hat{x} = (\hat{x}_1, ..., \hat{x}_4)$*:*

1. $x_1\,(0) < x_2\,(0) : \hat{x}_1 = 0, \hat{x}_2 = \hat{x}_3 = \hat{x}_4 = \frac{1}{3}$ *(steady state WWW: coalition of the weak)*

2. $x_1\,(0) \geq x_2\,(0)$ *and*

   - $x_3\,(0) < x_4\,(0) : \hat{x}_1 = \hat{x}_4 = \frac{1}{2}, \hat{x}_2 = \hat{x}_3 = 0$ *(steady state SW: the strong with one weak agent)*

- $x_3(0) = x_4(0)$ *and*
  - $x_2(0) < x_3(0)$ *and* $x_1(0) > x_2(0) : \hat{x}_1 = \hat{x}_3 = \hat{x}_4 = \frac{1}{3}, \hat{x}_2 = 0$ *(steady state SWW: the strong with two weak agents)*
  - $x_2(0) = x_3(0)$ *or* $x_1(0) = x_2(0) : \hat{x}_1 = \hat{x}_2 = \hat{x}_3 = \hat{x}_4 = \frac{1}{4}$ *(steady state SWWW: grand coalition)*

The steady states WWW and SW are asymptotically stable while SWW and SWWW are not.

4. Conclusion

In conclusion, we offer some remarks and point to future research. First, we find that evolutionary cooperative game theory (ECGT) is very attractive from an interpretational point of view. First of all, even the most basic models as they are presented in this paper belong to the (a) "playing the field" and the (b) polymorphic variety. (a) just results from the way the Shapley value is calculated and interpreted – an agent's payoff depends on the set-up of the economy as a whole. (b) is also the natural outflow from different players roles. In contrast, the most basic model of evolutionary noncooperative game theory (ENGT) builds on "pairwise contests" and a monomorphic population playing a symmetric game. Of course, more advanced ENGT models also deal with polymorphic playing-the-field situations.

Second, in this paper, we prefer the Lovasz extensions over the multi-linear extension for reasons given by Wiese (2005). If players (or agents) work together (in the framework of a unanimity or an apex game) and if the size of the agents is below 1, the multilinear extension, v_{MLE}, has a probabilistic interpretation (as noted by Owen (1972, p. 64)) – the players work together only if their time schedules happen to coincide. For example, two productive players in the unanimity game $u_{\{1,2\}}^{\{1,2\}}$ with $s = \left(\frac{1}{2}, \frac{1}{3}\right)$ can produce $\frac{1}{2} \cdot \frac{1}{3}$, only. It seems to us that (by appropriate coordination), the two agents should be able to produce the minimum of these two figures, $\frac{1}{3}$,. which is exactly what the Lovasz extension does. Also, consider $s = (2, 3)$. The multi-linear extension yields $2 \cdot 3 = 6$ whereas the minimum extension leads to 2.

However, we also admit that another reason strongly militates the multi-linear extension. With that extension, we find that the continuous Shapley values do not depend on s and generate boring results. The agents' shares always stay.

Third, in our model, we differentiate between players and agents who assume the role of players. It is the agents whose shares change. In contrast, one might envision a model where the players themselves grow or shrink. A suitable example is provided by firms. Depending on their profits they will grow in an organic fashion (rather than grow by mergers and acquisitions). This model is a special instance of dynamic cooperative games (see Filar & Petrosjan 2000). The basic idea is to define a sequence of games (in discrete or in continuous time) so that one TU game is determined by the previous one and by the payoffs achieved under some solution concept.

We now turn to future work in ECGT and note that the ESPCs or replicator dynamics are concerned with selection. Of course, mutation is the other evolutionary force to be reckoned with. It is concerned with the change of parameters rather than the selection pressures for a given set of parameters. Within a replicator dynamics, mutation can take different forms:

1. We may consider small changes of the coalition function v.

2. Other player types could be added with very small sizes such that the worths for the other players stays the same for a zero size of the new arrival.

3. Somewhat similar to the procedure used in ESPC, one could model independent small additions to, or subtractions from, the agent sizes.

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5. Appendix

.1. Proof of lemma 2.5

The 0 payoffs follow from eq. 2.2. In case of $s_- > 0$ and $i \in T$, the diagonal formula yields

$$
\begin{aligned}
\operatorname{Sh} u_T^{\ell,m,s}(I_i) &= \mu_i^s(I_i)\int_0^1 \left.\frac{\partial u_T^{\ell,m}}{\partial s_i}\right|_{ts} dt \\
&= s_i\,|T|^{\frac{1}{m}}\int_0^1 \left(-\frac{1}{m}\right)\left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{1}{m}-1}\cdot(-m)\left. s_i^{-m-1}\right|_{ts} dt \\
&= s_i\,|T|^{\frac{1}{m}}\int_0^1 \left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{m+1}{m}}\cdot \left. s_i^{-(m+1)}\right|_{ts} dt \\
&= s_i\,|T|^{\frac{1}{m}}\int_0^1 \left(\sum_{j\in T} (ts_j)^{-m}\right)^{-\frac{m+1}{m}}\cdot (ts_i)^{-(m+1)}\,dt \\
&= s_i s_i^{-(m+1)}\,|T|^{\frac{1}{m}}\int_0^1 \left(\sum_{j\in T} t^{-m}s_j^{-m}\right)^{-\frac{m+1}{m}}\cdot (t^m)^{-\frac{m+1}{m}}\,dt \\
&= s_i^{-m}\,|T|^{\frac{1}{m}}\int_0^1 \left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{m+1}{m}} dt \\
&= |T|^{\frac{1}{m}}\,s_i^{-m}\left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{m+1}{m}}
\end{aligned}
$$

We rewrite $\mathrm{Sh}\, u_T^{\ell,m,s}(I_i)$ for the third case to find

$$\begin{aligned}
|T|^{\frac{1}{m}} s_i^{-m} \left(\sum_{j\in T} s_j^{-m} \right)^{-\frac{m+1}{m}} &= s_i \, |T|^{\frac{1}{m}} \left(s_i^m\right)^{-\frac{m+1}{m}} \left(|T_-| \, s_-^{-m} + \sum_{j\in T\setminus T^-} s_j^{-m} \right)^{-\frac{m+1}{m}} \\
&= s_i \, |T|^{\frac{1}{m}} \left(|T_-| \, \frac{s_i^m}{s_-^m} + \sum_{j\in T\setminus T^-} \frac{s_i^m}{s_j^m} \right)^{-\frac{m+1}{m}} \\
&= \begin{cases} s_i \, |T|^{\frac{1}{m}} \left(|T_-| \, \underbrace{\frac{s_i^m}{s_-^m}}_{1} + \sum_{j\in T\setminus T^-} \underbrace{\left(\frac{s_i}{s_j} \right)}_{<1}^m \right)^{-\frac{m+1}{m}} & , \quad i \in T_- \\ s_i \, |T|^{\frac{1}{m}} \left(|T_-| \underbrace{\left(\frac{s_i}{s_-} \right)}_{>1}^m + \sum_{j\in T\setminus T^-} \frac{s_i^m}{s_j^m} \right)^{-\frac{m+1}{m}} & , \quad i \notin T_- \end{cases}
\end{aligned}$$

Standard rules for limites imply the desired results.