A general proof that diversification pays, even with contagion

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March 15, 2013

Abstract

The swiftness with which risk spreaded throughout the market lead to a shift to a more "connection-based" approach to financial regulation. This change in focus has implications on the desirability portfolio diversification from society's perspective. Indeed while the individually risk reducing effect of diversification has been known since Markowitz (1952), it also forms "connections" between investors through common asset holdings, who have been identified as a major carrier of contagion in the presence of fire sales. The paper takes a first step towards quantifying this contagion externality diversification and compares it to its individual risk-reducing effect. We do so through the distribution of the vector of investor wealth, where marginal densities capture the former effect, while the latter is embedded in the dependence structure. Covariance matrices between investors are obtained from a micro-founded model that captures the contagion steeming from cross asset holding, which translates into a linear system of N recurrence equations of price returns.

This paper is in its preliminary stages, comments are very welcome.

1 Introduction

When thinking about financial markets and systemic risk, one can find it useful to consider a group of climbers roped together at the top of a cliff. Each climber individually favors being roped as it lowers his chances of falling off, yet one climber tripping now threatens the stability of his neighbors. The effect of being roped on the probability that many or all climbers fall is thus a priori ambiguous.

Prior to the 2008 credit crunch, both market participants and regulatory instances seemed to favor the roped equilibrium, implicitly assuming that individual soundness leads to systemic soundness. Yet the swiftness with which risk spreaded throughout the market, largely unanticipated, lead to a shift to a more "connection-based" approach. Basel III will apply extra capital ratio requirement of up to 2.5% to well connected establishments. Measures of systemic importance that account for externalities, such as Covar (Adrian and Brunnermeier, 2011), or Shapley values (Tarashev et al., 2010) have recently gained in popularity.

This change in focus has implications on the desirability portfolio diversification from society's perspective. Indeed while the individually risk reducing effect of diversification has been known since Markowitz (1952), it also forms "connections" between investors through common asset holdings, who have been identified as a major carrier of contagion in the presence of fire sales (Shleifer and Vishny, 2011). The goal of this paper is thus to take a first step towards quantifying the contagion externality diversification may generate, and compare it to its individual risk-reducing effect, in order to get a primary assessment of diversification's total impact on systemic risk, and the factors on which this impact may depend.

We estimate both the individual risk reduction effect and the propagation externality in a unified framework by mapping each into two distinct components of systemic risk. An individually safer investor implies a lower probability that one investor goes brankrupt, while a higher contagion externality means a higher likelihood that a large number X of investors fall, conditional on one bankruptcy. The former affects the marginal distribution of a given investor's portfolio, while the latter will be contained within the dependence structure between the N investors.

In a nutshell, the paper proceeds in two steps:

1) setting up a model that captures the contagion steeming from cross asset holding. N regulation constrained long-term investors who hold from 1 to N assets interact with short term investors, impacting prices, which further tightens the constraint, and so on. Mathematically this translates into a linear system of N recurrence equations of price returns, in which the strength of the recurrence will depend on the magnitude of the constraint and the discount of the resulting supply/demand. Using the properties of circulant matrices, we are able to solve analytically the system, and obtain well-behaved N by N covariance matrices between assets and investors.

2) using the correlations between assets and investors to run a statistical analysis. The multivariate distribution of portfolio losses notably tells us, within a specified network, how the likelihood that a large number X of investors go bankrupt changes with the level of diversication n. We then specify a cost function which increases exponentially with the number of failures, defining a diversication threshold above which the benefits of individual soundness outweight the costs of increased connectivity.

We study two possible regimes: a "stable" one in which agents follow a constant linear rule, and a "panic" one in which they may change this rule in the presence of extreme selling movements. In the stable one, we show that while intermediate levels of failures are less likely with a higher degree of diversification, the probability that investors fail simultaneously is larger as a result of increased correlation between investors. In a context of high recurrence, this probability becomes non-trivial, so that low levels of or no diversification may become optimal for society. This finding echoes the case recently made against diversification by some researchers, such as Wagner (2010), Cont (2012) or Caccioli et al. (2012).

In the panic regime the discount applied by short-term investors depends on the magnitude of the first selling shock. We find this strongly enhance the desirability of higher levels of diversification as it spreads out sales accross markets, minimizing extreme movements and the probability of panic. Ultimately, the paper argues that over a reasonable set of parameters a high level of diversification should increase the resilience of financial markets.

The first step of the paper can be related to the literature on cross asset holding contagion and fire sales, which has uncovered many channels through which investors' actions may degenerate into systemic events. Schinasi, and Smith (1999) show that routine portfolio rebalancing brings contagion. The scope is increased when agents are subject ro rising risk aversion, or wealth effects as in Kyle and Xiong (1999). Pauzner and Goldstein (2004) point out that fire sales may also result from strategic risk.[1]

The second step is more linked to the literature of on the desirability of diversification, and more precisely its dependence upon the distribution of asset prices.

One approach, taken by Shaffer (1994) or Wagner, is to compare statistically a fully diversified situation to a fully undiversified one. Both authors show that the risk that all investors fail simultaneously is necessarily higher

[1] Others authors relate diversication to different amplication mechanisms, which deserve to be mentioned although they will not feature as such in this paper. First, market distress may lead to an even wider collapse if it turns into liquidity distress. Shin (2008) shows that for an investor debt is negatively correlated to the market value of his assets. Allen et al (2010) find that when investors need to roll debt over, being connected brings a negative reputation externality. Calvo and Mendoza (2000) argue that diversication lowers the incentive for investors to acquire information about securities before selling.

in the former than the latter. Our study confirms this fact but also goes more in depth by considering any level of diversification and any number of failures.

A popular approach deals with "fat tails", which have been known to mitigate the strength of the variance reducing effect since the work of Samuelson (1969). In particular, Ibrahimov et al. (2011) use an indicator of the tail behavior of returns to define a diversication threshold, similar to ours. They find that on a given parameter range there can exist a wedge between investors interest and society ones.

The crucial link between both strands of literature is the correlation structure between asset returns, which are the ouput of the former, and the input of the latter. In any contagion model the actions of investors endogenously create correlations between asset returns, as even 2 securities which are "fundamentally" independent can be linked through fire sales in a given network. By specifying a larger scale model [2] we obtain such endogenous correlations at a much broader level, and thus are able to draw systemic systemic conclusions from micro-founded analysis.

To the best of our knowledge 2 recent papers have used a similar approach. Danielsson, Shin, and Zigrand, define in 2011 a multivariate model which produces a matrix of covariance of unrestricted dimensions. The correlations obtained have a "fundamental" and an "endogenous risk" component, ie they result partly from "the actions of market participants which are hard wired in the system". Cont and Wagalath specify a similar but more aggregated model, which they calibrate to estimate the realized covariance matrix during well-know fire sales episodes, such as the aftermath of the collapse of Lehman brothers.

With respect to methodology, our study differs from Cont et al. through is micro foundations, and from Danielsson et al. in that their framework implies similar correlations accross all assets, while we specify a network which leads the correlations between two assets i and j to depend on the "distance" i-j. In spirit, these two papers' focus is on explaining the pattern of prices during crisis episodes, while our's is very much on the desirability of diversification from a systemic perpective.

Finally part of our analysis may be linked to network analysis of financial stability. Our results are consistent with a "robust yet fragile" financial system, which is in line with Nier et al. (2007) or Amini et al (2010). Caccioli et al. use network analysis to find that overlapping portfolios may enhance systemic when leverage is high, which is consistent with part of our analysis.

Section II will present the agents maximization problems, and how they meet to generate recurrence in asset prices. Section III solve this recurrence system, defines the realized covariances between assets and investors, and sets an value function for society. Section IV presents the results in the stable and panic regimes, before concluding.

2 Set-up

The model below is designed to generate contagion in its simplest form. Regulation induced constraints force long-term horizon investors to sell some of their holdings to short-term investors at a discount. Depending on the haircut on this sale long-term investors may reach a new equilibrium with lower quantities unharmed, or be wiped out by the price falls.

[2] The models usually feature only 2 assets, and diversication is defined as how evenly an investor spreads his wealth accross both. Lagunoff and Shreft (1999) try moving to a 3 asset case. They find that the scope for contagion is decreased.

2.1 The market

The investment period starts at t=0 and finishes at t=T, where each period $t \mapsto t+1$ may be seen as "a day" on the markets, and T is a large but finite number. Financial markets are composed of N risky assets, prices move following an arithmetic brownian motion, where the drift μ_i represents the fundamental evolution of asset i. Noise comes from three sources:

- a fundamental shock $\varepsilon_{i,T}^F \; N \sim (0, \sigma_i^F)$ that represents exogenous forces that may cause the drift to deviate from its fundamental return μ_i .

- a noise trading component $\varepsilon_{i,T}^{*N}$, $N \sim (0, \sigma_i^N)$ that follows a Ornstein-Uhlenbeck process: $\varepsilon_{i,t+1}^{*N} = -\lambda \sum_{j=0}^{j=t} (1-\lambda)^j \varepsilon_{i,t-j}^N + \varepsilon_{i,t+1}^N$. This means any supply shock by noise traders at a given t must be matched a corresponding increase in demand in subsequent periods, so that total noise trading returns to its zero mean, and prices to pre-shock level.

- a constrained trading component $\varepsilon_{i,T}^C$, which represents the variation in prices steeming from the selling decisions of constrained agents, and may exhibit auto and inter-correlation accross assets and time. Note that we use the terminology "cronstrained selling" more often, as we focus on fire sales.

Both trading components are linear function of the amounts demanded/supplied between t and t+1 so that $\varepsilon_{i,t+1}^N = f(\triangle q_{i,t}^N)$ and $\varepsilon_{i,t+1}^C = f(\triangle q_{i,t}^C)$. In the case of constrained selling the magnitude also depends on δ, the amount of constrained investors holding each asset.

Mathematically:

$$p_{i,t+1} = \mu_{i,t} + p_{i,t} + \varepsilon_{i,t+1}^F + \varepsilon_{i,t+1}^{*N} + \varepsilon_{i,t+1}^C \tag{1}$$

We assume these moments are the same accross assets and time $\mu_{i,t} = \mu_{j,t} = \mu$ and $\sigma_{i,t}^2 = \sigma_{j,t}^2 = \sigma^2$. For fundamental and noise trading shocks we also assume that assets are independent, so that the fundamental and noise trading covariance matrix at each period are $\sum_F = diag(\sigma_F^2)$ and $\sum_N = diag(\sigma_N^2)$. These assumptions will simplify presentation but also show how contagion may still occur through endogenous risk only, even though assets are symmetric and fundamentally independent.

This pattern implies two investment strategies: a long-term buy and carry strategy based on fundamental drifts, and a short-term one based on taking the other side of noise trading. In this model "long run" or "passive" investors focus on the former, and "short run" or "active" investors on the latter. Long-term investors trade risky assets and a riskless zero-bond whose price return is normalized to 0 $N \sim (0,0)$. Initially M long-term investors are endowed with "their" asset i=I, but can diversify their portfolio. $n \in [1, N]$ is the level of diversification.

This full specialization results from an ex-ante arbitrage by both agents. Short run investors pay a fixed cost at each t to be more efficient in monitoring price fluctuations and reacting swiftly. Such a market segmentation is close to that of Graham (1949) who differentiates an "active or enterprising approach to investing" from a "passive or defensive strategy that takes little time or effort but requires an almost ascetic detachment from the alluring hullabaloo of the market" (Zweig, 2003, p101).

The content of the covariance matrix of constrained selling shocks $\sum_c$ will depend on the amount of constrained selling, which in turns depends on δ. During "good times" this proportion will be small, so that σ^C will be moderate. Crucially, since the price impact of these sales is limited, the losses steeming from such sales and thus the autoregressive component are minimal. Taking this logic to its extreme, in BAU investors may approximate $\sum_c = diag(\sigma_C^2)$ and $E(\varepsilon_{i,t+1}^C \mid \varepsilon_{i,t}^C) = 0$. Figure 1 represents the BAU price dynamics:

However this approximation now longer is acceptable when δ and thus $\varepsilon_{i,t+1}^C$is high. Our model therefore applies

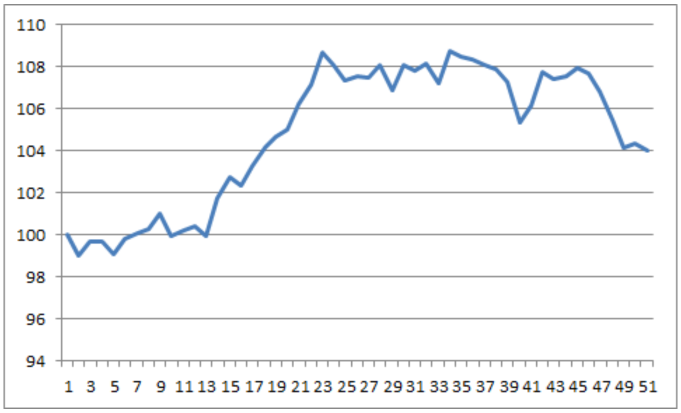

Figure 1: BAU Price evolution of an asset with moments $\mu = 0,1$ and $\sigma^2 = 1$

to a situation in which an extended period of BAU has led investors to partly ignore the latent endogenous risk from constrained selling, which has quietly built up. Both experience and theory provide us with reasons why this could happen: investors may find it hard to dissociate constrained selling from other sources of noise. The spillovers to other assets may be constantly changing and costly to estimate. More simply, high confidence during booms may lead them to disregard latent threats.

This particular set-up motivates the study of two versions of the model : a "stable regime" in which agents stick with their assessment that $\sum_c = diag(\sigma_N^2)$ throughout the period of the crisis, and a "panic regime" in which short-term agents realize the scope of fire sales when constrained selling rises sharply.

2.2 Investors

Both short term and long term investors have CARA utility, with risk aversion of τ_{st} and τ_{lt} respectively. The risk aversion is similar accross investors of the same horizon, and each agent is a price-taker. Both agents base their strategies on assessing, correctly, the fundamental evolution of assets. Active investors need this knowledge to assess how current prices deviate from their fundamental values, passive investors need it to evaluate the long-term returns. This means $E(\mu) = \mu$ and $E(\sigma^F) = \sigma^F$

We introduce some notation: $q_{i,I}$ represents the actual quantity of asset i held by a giv. $q_I = \Sigma_{i=1}^{i=N} q_{i,I}$ is the total investment in all risky assets by investor I. $q_i = \Sigma_{I=1}^{I=N} q_{i,I}$ is the total quantity of asset i in the economy, accross all investors. These quantities are the general elements of vectors $Q_{i,I}$, Q_I and Q_i respectively . P is the N vector of prices and fundamental values, where each row i represents a given asset i. Desired quantities for a given investor are noted q^*.

2.2.1 Short term investors

At each time t short-term investors want to take the other side of noise trading. They hold assets during $t*$ periods, the time it takes form noise trading shocks to die out, and pay ϵ to be able to track prices at each t. Finally they are free of regulation and have "deep pockets", so that they always hold the portfolio they desire $q_{i,t}^{*ST} = q_{i,t}^{ST}$. Mathematically:

$$Max\ E(-e^{-\frac{w_{I,t+t*}}{\tau_{st}}})$$

$$u/c \ w_{I,t+t*} = w_{I,t} + Q^{*\intercal}(P_{t+t*} - P_t) - \epsilon$$

Using the moment generating function [3] the vector of optimal quantities for a given investor is:

$$Q_t^{*st} = \tau_{st} \frac{E(P_{t+t*} - P_t)}{\sum_{\triangle P_t}}$$

The price dynamics (1) allow us to reexpress:
$E(P_{t+t*} - P_t) = E(\mu t * + \sum_{j=t+1}^{j=t+t*} \varepsilon_j^F + \sum_{j=t+1}^{j=t+t*} \varepsilon_j^{*N} + \sum_{j=t+1}^{j=t+t*} \varepsilon_j^C) = \mu t * + E(-\sum_{j=0}^{j=t} \varepsilon_j^{*N})$ [4]

Setting m the number of active investors operating on the market and differencing, the total demand/supply shock between two periods t and t+1 is $m\tau_{st}\frac{\mu t* + E(-\sum_{j=0}^{j=t+1} \varepsilon_j^{*N})}{\sum_{\triangle P_t}} - m\tau_{st}\frac{\mu t* + E(-\sum_{j=0}^{j=t} \varepsilon_j^{*N})}{\sum_{\triangle P_t}}$, so that:

$$\triangle q_{i,t}^{*st} == -m\tau_{st} \frac{E(\varepsilon_{i,t+1}^{*N})}{\sum_{\triangle P_t}}$$

The future shocks of any kind are impossible to predict, but $\varepsilon_{i,t+1}^{*N}$ is partly defined by the past noise trading shocks on security i, which are bound to revert. Though investors cannot directly observe these past shocks, they may use their knowledge of the fundamental value $P_t^F = P_0 + t\mu + \sum_{j=1}^{j=t} \varepsilon_{,j}^F$ to estimate them, as $P_t - P_t^F = \sum_{j=0}^{j=t} \varepsilon_j^{*N} + \sum_{j=0}^{j=t} \varepsilon_j^C$, so that $(P_{t+1} - P_{t+1}^F) - (P_t - P_t^F) = \varepsilon_{t+1}^{*N} + \varepsilon_{t+1}^C$.

In words, at each period short-term investors observe a certain deviation from fundamentals, which is can be attributed partly to noise traders. The estimation of this part for each asset will depend on the estimated correlation between random variables $\varepsilon_{i,t+1}^{*N}$ and $(p_{i,t+1} - p_{i,t+1}^F) - (p_{i,t+1} - p_{i,t}^F)$, which will be $\tilde{\rho} = \frac{\tilde{\sigma_N}}{\sqrt{\tilde{\sigma_N^2} + \tilde{\sigma_C^2}}}$ since ε_j^{*N} and ε_j^C are independent.

Therefore following a long BAU period we have $E(\varepsilon_{t+1}^{*N}) = \tilde{\rho}[(P_{t+1} - P_{t+1}^F) - (P_t - P_t^F)] = \tilde{\rho}(\triangle p_{i,t} - \mu - \varepsilon_{i,t+1}^F)$, and $E(\sum_{\triangle P_t}) = diag(\sigma_N^2 + \sigma_F^2 + \sigma_C^2) = diag(\sigma_{tot}^2)$. Note that as $\varepsilon_{i,t+1}^N = f(\triangle q_{i,t}^N)$ and $\varepsilon_{i,t+1}^C = f(\triangle q_{i,t}^C)$, $\tilde{\rho}$ will depend on the perception by short-term investors of the amount of noise trading compared to constrained selling, and hence on δ. Hence the possibility in section III that $\tilde{\rho}$ in response to large constrained supply shocks.

Fo now we may then reexpress excess demand for security i at a given date t:

$$\triangle q_{i,t}^{*st} = -\frac{\tilde{\rho} m \tau_{st}}{\sigma_{tot}^2}(\triangle p_{i,t} - \mu - \varepsilon_{i,t+1}^F) \tag{2}$$

Using the market-clearing condition $\triangle q_{i,t}^{ST} = -(\triangle q_{i,t}^N + \triangle q_{i,t}^C)$, where $\triangle q_{i,t}^C$ and $\triangle q_{i,t}^N$ are the change in quantities held by constrained long-term investors and noise traders respectively. and plugging the price dynamics (1) in (2):

$$\frac{\sigma_{tot}^2}{\tilde{\rho} m \lambda \tau_{st}}(\triangle q_{i,t}^N + \triangle q_{i,t}^C) = \varepsilon_{i,t+1}^C + \varepsilon_{i,t+1}^{*N} \tag{3}$$

$\triangle q_{i,t}^C = 0 \Longrightarrow \varepsilon_{i,t+1}^C = 0$. We then have $\frac{\sigma_{tot}^2}{\tilde{\rho} m \lambda \tau_{st}} \triangle q_{i,t}^N = \varepsilon_{i,t+1}^{*N}$, and:

$$\varepsilon_{i,t+1}^C = \frac{\sigma_{tot}^2}{\tilde{\rho} m \lambda \tau_{st}} \triangle q_{i,t}^C \tag{4}$$

[3] see appendix A for a step by step derivation
[4] appendix B provides a step by step computation

This means that the optimal trading rule of the short-term investors is to apply a given haircut on every unit of quantity sold to them regardless of who originated the selling, as s they are incapable of differenciating constrained selling from noise trading.

2.2.2 Long term investors

Maximization problem All M investors endowed with asset i only differ in their level of capital, we thus study the maximization problem of a representative investor I=i.

They want to hold assets until they mature at t=T and finance themselves by issuing equity and borrowing at the risk free rate. Formally:

$$Max\ E(-e^{-\frac{w_{I,M}}{\tau_{lt}}})$$

$$u/c\ \ w_{I,M} = w_{I,t} + Q_t^{*\intercal}(P_M - P_t)$$

$E(P_M - P_t) = E(P_t + (T-t)\mu + \sum_{j=t+1}^{j=M} \varepsilon_j^F + \sum_{j=t+1}^{j=M} \varepsilon_j^{*N} - P_t) = (T-t)\mu$ and variance $\sum_{\triangle P_t} = (T-t)\sigma_F^2 = (T-t)\sigma_F^2$, as long-term investors do not incur the fixed cost and are thus incapable of monitoring short-term deviations. Note that even without this monitoring cost, such deviations become negligible as the investment horizon increases. The optimal quantities are therefore:

$$Q_t^{lt*} = \frac{\tau_{lt}\mu}{diag(\sigma_F^2)}$$

The key element is that, as long as fundamental traders do not see any change in the fundamentals, their demands will remain unaltered. With similar moments accross assets, the desired quantity of each asset by the representative agent is $q_{i,I}^{lt*} = q_{j,I}^{lt*} = \frac{\tau_{lt}\mu_T}{\sigma_F^2}$. Therefore ideally the investor holds the same quantity of each of the asset featuring in his portfolio, and:

$$\frac{q_{i,I}}{q_{I,t}} = \frac{1}{n} \tag{5}$$

regulatory constraint For simplicity we assume that he number of constrained investors δ remains constant accross market and time.

Each passive investor is subject to a Cooke ratio type constraint: capital over risk weighted assets may not go below a given value α. Mathematically:

$$E(A_{I,t})q_{I,t+1} \leq \alpha K_I \tag{6}$$

Where $A_{I,T+1}$ is the risk weight on investor I. If a long term investor is considered too risky at time t, he is required to lower his total investment in risky assets at t+1, so that the risk-weighted assets are below total equity at the beginning of the following period. Risk weight are usually asset specific and the lag between two risk classes may be large. In this model we need a more continuous measure, which applies to the entire portfolio[5]. We thus

[5]we discuss this choice in section 2.4

use the ratio of the standard deviation of the portfolio over the expected value of the portfolio at maturity, which may be seen as the inverse Sharpe ratio of the portfolio, as the weighting scheme for market risk

$$E(A_{I,t+1}) = \frac{\sqrt{Q_{i,I,t}^T \sum_R Q_{i,I,t}}}{E(P_{I,T})Q_{i,I,t}}$$

Using the fact that investors will hold the same quantities of all n assets he has access to, since their fundamental moments are similar, we have $E(P_{I,T})Q_{i,I,t} = q_{i,I,t} \sum_i (p_{i,I,t} + (T-t)\mu)$, where $\sum_i p_{i,I,t}$ is the sum of the prices of the assets featuring in investor I's portfolio. The regulator's expectation of the evolution of prices of is thus in line with that of other agents. This will not hold for variances as his estimation, which we note σ_R may differ from that of investors, and he refuses to take into account the variance reducing effect of diversification. Mathematically any element $a_{i,j}$ of the correlation matrix $\sum_R$ is $a_{i,j} = cov(p_i, p_j) = \sigma_R \sigma_R = \sigma_R^2$.

The refusal by the regulator to factor in the gain in variance from diversication one of the well documented features of the Basel agreements (see Blundell- Wignall, Atkinson, 2010). One can be see it as a way for regulation put more weight on the undesirable outcomes. The fact that regulators monitor the health of long term investors at every t captures the pro-cyclical effect of marked-to-market based regulation of individuals with longer horizon, which has been identied as a major carrier of fire sales by Tirole (2008).

We can therefore re-express $E(A_{I,t+1})$:

$$E(A_{I,T+1}) = \frac{\sqrt{q_{i,I}^2 n^2 \sigma_R^2}}{q_{i,I} \sum_{i=.}(p_{i,I,t} + (T-t)\mu)} = \frac{n\sigma_R}{(T-t)n\mu + \sum_{i=.} p_{i,I,t}} \tag{7}$$

Plugging into inequality (6), :

$$q_{I,t+1} \leq \alpha K_I \frac{(T-t)n\mu + \sum_{i=.} p_{i,I,t}}{n\sigma_R}$$

Therefore, for a given investor operating on his constraint at t, the amount of constrained selling between t and t+1 will be $\triangle q_{I,t+1}^c = \alpha K_I \frac{(T-(t+1))n\mu + \sum_{i=1}^{i=N} p_{i,I,t}}{n\sigma^F} - \alpha K_I \frac{(T-t)n\mu + \sum_{i=1}^{i=N} p_{i,I,t}}{n\sigma^F} = \frac{\alpha K_I}{n\sigma^F} \sum_i (\triangle p_{i,I,t} - \mu) = \frac{\alpha K_I}{n\sigma^F} \sum_i \varepsilon_{i,t}$, where ε_i is the sum of all shocks on the securities featuring in a given investor's I portfolio at a given time t, may it be fundamental, from noise trading or constrained investors. Total selling for investors from asset class I will thus be this individual selling times the amount of constrained investors of type I, which is δ for each market. Using equation (5) we obtain $\triangle q_{i,I,t+1}^c$, the amount of security i sold at t+1 by investors of type I as a response to shocks between t+1 and t.

$$\triangle q_{i,I,t}^c = \frac{\delta \alpha K_I}{n^2 \sigma^F} \sum_i (\triangle p_{i,I,t} - \mu) \tag{8}$$

2.3 Network formation pattern, matrix form

In the previous section we have studied how each fundamental trader behaves with and without the constraint binding, but the systemic implications of this individual rule will depend on the pattern of asset holdings we specify. Figure 2 summarizes the network formation for the representative investor of each asset: each holding of asset i by investor I is a connection between I and the investors I=i who initially held the asset. Investors are nodes and asset holdings are the connections between them. The numbers on each link thus list the assets that I and I'

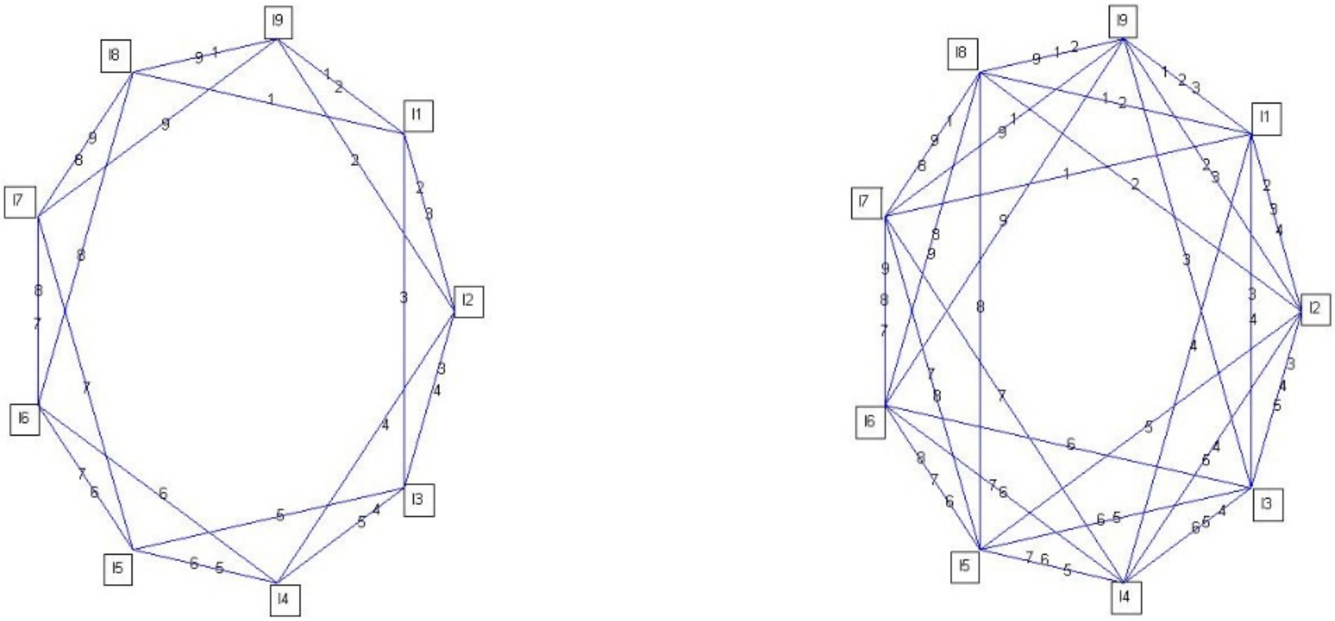

Figure 2: network formation

have in common, indicating of how closely related they are.

As obvious from figure 2, as the degree of diversication n increases, each long term investor I acquires the asset that is the closest neighbor to the right of the asset I=i he originated without featuring in his portfolio yet. For instance when n=3, I5 will hold assets 5,6,7. If n moves to 4, investor 5 will add assets 8 to his portfolio thereby connecting himself to investor I8, and creating a new connection with I5,I6,I7. If total number of asset N is 9, note also the peculiar investment profile of investor I9, which will hold 9,1,2. Although this network formation pattern is completely exogenous in our set-up, one could rationalized it using information costs.

The information may be expressed in matrix form. Each element $t_{I,i}$ of matrix T tells us whether investor I holds asset i or not. For instance if N=5 and n=3:

$$T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad T^{\intercal} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Each row I of the matrix $T(\triangle P_t - \mu)$ thus represents $\sum_i(\triangle P_{i,t} - \mu)$, the sum of all shocks to securities belonging to investor I. Therefore we may write $\triangle Q^c_{I,t}$, the (N,1) vector of constrained selling by each investor $\triangle q^c_{I,t}$:

$$\triangle Q^c_{I,t} = \frac{\delta \alpha K_I}{n^2 \sigma^F} T(\triangle P_{t-1} - \mu)$$

Where $r = \frac{\delta \alpha K_I}{\sigma^F}$ captures all parameters that impact on the strength of the regulatory constraint, to alleviate notation.

By the same token, each row i of the transpose $T^{\intercal}$ tells us if investor I holds asset i. Total supply/demand shock between T and T+1 may then be expressed as $\triangle Q^c_t = T^{\intercal} \triangle Q^c_{I,t}$. So that the vector of total long-term investors constrained demand/supply is:

$$\triangle Q_t^c = \frac{r}{n^2} T^\intercal T(\triangle P_{t-1} - \mu) \tag{9}$$

We give the model its final form by using (4) to introduce $\triangle Q_t^c$ in the price dynamics equation:

$$\triangle P_t - \mu = \frac{hr}{n^2} T^\intercal T(\triangle P_{t-1} - \mu) + \varepsilon_{i,t+1}^F + \varepsilon_{i,t+1}^{*N} \tag{10}$$

Where $h = \frac{m\lambda\tau_{st}}{\sigma_N^2 + \sigma_F^2}$ gathers all parameters that play on the haircut charged by short term investors . The end product of the model is thus a stochastic recurrence system that maps how price shocks spread through time and assets, in a situation in which passive investors operate on their constraint. Due to the fact that the matrix $T^\intercal T$ is circulant, this system will take us surprinsingly far in our study of systemic risk.

2.4 Assumptions

Before solving the model, it is appropriate to provide an honest discussion of 3 main assumptions underlying it:

1) *Stock prices are normally distributed.* The relevance of this postulate have already been discussed extensively (Andersen et al., 2001): while admitedly a poor description of reality, its merit is to be much easier to manipulate then alternative distributions. We have also chosen normality over log-normality, as the economic logic of the model dictates we work with price evolution rather than returns.

Nonetheless, perhaps the most common criticism attached to normal price returns is their incapacity to capture the actual thickness of the tails. Our model shows that under a particular set of circumstances the system endogenously generates a dependence structure between assets which make such "fat tails" appear. The paper thus not only avoid this pitfal, but participates to the research on asset prices distribution.

2) *Agents follow a linear rule.* On the supply side, this principle translates into the assumption that the regulator uses the inverse Sharpe ratio as weighting scheme. This method differs from common pratice amongst academics and practionners, which is to use value-at-risk (see for instance Danielsson et al., 2011). It is thus a bypass for the non-linear relationship implied by the regular VAR, though in essence it captures the same widely acknowledged effect of regulation on fire sales.

On the demand side, linearity comes from the fact that agents fail to perceive the auto regressive component from constrained trading. If they did see it, this would imply that both short-term and long-term agents would be selling more in the presence of fire sales. We may even witness selling from all agents , so prices would be ever falling. Such a set-up would also permit an analysis of systemic risk, where systemic crisis would be captured by a diverging system.

There are two reasons why we have constrained agents' perceptions to rule out this possibility. The first is that ever-falling prices do not represent the actual crisis dynamics, as in practice price fall heavily in the early stages but eventually converge to positive values. By maintaining this mopicity we ensure that a demand exist, may it be very weak. The second is that our system yields a converging distribution of prices falls that allows a more in depth statistical analysis, in which some investors may go bankrupt while others will survive.

Nonetheless, increased selling steeming from other factors than regulation is a reality during crisis episode. We could have extended the model in this direction, introducing speculators for instance, but to minimize modeling we have captured it as a lower demand from short-term investors who become suspicious through our "panic regime" in which ρ departs from 1 when sales go over a threshold, is thus a trick to capture these non-linearities using the demand side.

3) *Network formation.* Though circulant networks are well suited to mathematical analysis, the choice of this particular network formation is backed up, and was motivated, by economic intuition. Investors may expand their portfolio to the "neighboring security" due to information costs, herding mentality, or any other kind of home bias. However note that though such biases lead to circulant networks, the opposite is not true, so that the properties of circulant matrices may be applied to to wide variety of financial networks.

3 Solving

To convey maximum economic intuition while minimizing quantitative heavy lifting, we will study how a single stochastic shock spreads through time. We thus set all shocks to 0 past t=0 for all securities, solving a deterministic version of the model, in which the total losses will depend on the realization of the stochastic initial price fall. We also set these initial shocks to be fundamental ones, only in order not to overload the equations with the mean revertion component of noise trading. In this case equation (10) may be rewritten as

$$\varepsilon_t = (\frac{hr}{n^2} T^\intercal T)^t \varepsilon_0 \tag{11}$$

Where ε_t is the sum of all noise at time t, with $\varepsilon_0 = \varepsilon_0^F$ and $\varepsilon_t = \varepsilon_t^C$. The study of how the system reacts to a fundamental shock involves finding $D_{T^\intercal T} = diag(\phi_0, .., \phi_{N-1})$, the diagonal matrix similar to $T^\intercal T$, with ϕ_i the associated eigenvalues.

3.1 Solving

From the cyclic permutation matrix J to T Let J be the "cyclic permutation matrix", whose element $a_{i,j} = 0$ if $i \neq j-1$, $a_{i,j} = 1$ otherwise. For instance if N=5:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Taking J to the power n shifts the one-diagonal n-1 spots to the right. For instance :

$$J^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues of J are readily obtained from the observation that $J^N = I$. Using the Cayley-Hamilton theorem this implies that the characteristic polynomial of J is of the form $Q[x] = x^n - 1$, and its eigenvalues are the nth-roots of unity. Thus the matrix similar to J is $D_J = diag(\omega^0, \omega, ..., \omega^{N-1})$ where $\omega^k = e^{\frac{2i\pi k}{N}} = cos(\frac{2\pi k}{N}) + i\ sin(\frac{2\pi k}{N})$ and $\omega^{-k} = cos(\frac{2\pi k}{N}) - i\ sin(\frac{2\pi k}{N})$ according to Euler's identity. Note also that since $\omega^N = 1$, we have $\omega^{\alpha(N-k)} = \omega^{\alpha N - \alpha k} = \omega^{-\alpha k}$.

We now look for the change of basis matrix P such that $JP = PD_j$, which yields $x_3 = \omega^k x_2 = \omega^{2k} x_1$, etc. Setting $x_1 = \frac{1}{\sqrt{N}}$ we obtain an orthonormal base for the eigenvectors of the discrete inverse Fourier transform matrix of coefficient $\frac{1}{\sqrt{N}}$, whose general term is $p_{j,k} = \frac{\omega^{jk}}{\sqrt{N}}$, with $(j, k) \in \{0, 1, .., N-1\}^2$:

$$P = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & . \ . & 1 \\ 1 & \omega & . \ . & \omega^{N-1} \\ . & . & & . \\ . & . & & . \\ 1 & \omega^{N-1} & & \omega^{(N-1)^2} \end{pmatrix}$$

From the properties of J, matrix T may be expressed as a polynomial in J:

$$T = R(J) = \sum_{s=0}^{s=n-1} J^s$$

where n is the level of diversication. Since T = R(J) we have $T = R(PD_J P^{-1}) = PR(D_J)P^{-1}$. Therefore $D_T = R(D_J)$ similar to T. Note that any circulant matrix may be expressed as a polynomial in J, hence P is the change of basis matrix for all circulant matrices. Back to D_T, the general expression of T's eigenvalue ψ_k is therefore the inverse Fourier transform $\psi_k = R(\omega_k) = \sum_{s=0}^{s=n-1} \omega^{ks}$, which can be expressed as the sum a geometric series of n terms and common ratio $e^{\frac{2i\pi k}{N}}$, so that

$$\psi_k = \frac{1 - \omega^{kn}}{1 - \omega^k} = \frac{1 - cos(\frac{2\pi kn}{N}) - i\ sin(\frac{2\pi kn}{N})}{1 - cos(\frac{2\pi k}{N}) - i\ sin(\frac{2\pi k}{N})} \tag{12}$$

Except when k=0, in which case $\psi_0 = n$. Note that this expression also implies $\psi_{-k} = \psi_{N-k}$.

From T to $T^\intercal T$ To bridge these results to $T^\intercal T$ we use some of the numerous outstanding properties of P:

1) P is unitary: its inverse is equal to its conjugate transpose. $P^{-1} = P^\dagger$, whose general term is $\frac{\omega^{-jk}}{\sqrt{N}}$

2) $P = P^\intercal$ and $P^{-1} = (P^{-1})^\intercal$, so that $T^\intercal = P^{-1} D_T P$

3) The square of P and P^{-1} is a near reversal matrix noted V. For instance if N=5:

$$V = P^2 = P^{-2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We may then write $T^\intercal T = P^{-1} D_T P P D_T P^{-1} = P^{-1} D_T V D_T P^{-1}$. Pre-multiplying by PP^{-1} we get the desired result:

$$T^\intercal T = P(VD_T)^2 P^{-1} \tag{13}$$

$(VD_T)^2$ is the diagonal matrix similar to $T^{\intercal}T$. We may then re-express the system as:

$$\triangle P_t - \mu = (\frac{hr}{n^2})^t P(VD_T)^{2t}P^{-1}\varepsilon_0^F \tag{14}$$

General element of $\triangle P_t$ The element of $(VD_T)^2$ on row k>0 may be expressed as $\phi_k = \phi_{N-k} = \psi_k\psi_{-k} = \frac{1-\omega^{kn}}{1-\omega^k} \times \frac{1-\omega-^{kn}}{1-\omega^{-k}}$, which will be positive for any k. Using (12) and rearranging we obtain:

$$\phi_k = \phi_{N-k} = \frac{1-cos(\frac{2\pi kn}{N})}{1-cos(\frac{2\pi k}{N})}$$

If k=0, $\phi_0 = n^2$.

This yields the general expression of the evolution of the price of a given security k:

$$\triangle p_{k,t} - \mu = \frac{1}{N}\sum_{j=0}^{j=N-1}\varepsilon_{0,j}^F\sum_{q=0}^{q=N-1}\omega^{(k-j)q}(\phi_q)^t \tag{15}$$

Using Euler's identity, we deduce $\omega^{(k-j)(N-q)}\phi_{N-k} + \omega^{(k-j)q}\phi_k = 2cos(\frac{2i\pi(k-j)q}{N})\phi_k$, we may then express the evolution of a given asset k's price over time[6]:

$$\triangle p_{k,t} - \mu = (\frac{hr}{n^2})^t\frac{1}{N}\sum_{j=0}^{j=N-1}\varepsilon_{0,j}^F(n^{2t} + 2\sum_{q=1}^{q=(N-1)/2}cos(\frac{2\pi(k-j)q}{N})\left(\frac{1-cos(\frac{2\pi nq}{N})}{1-cos(\frac{2\pi q}{N})}\right)^t) \tag{16}$$

When N is odd [7].

Asset k depends on the evolution of all the other assets of the economy, to an extend that varies with the "distance" between k and each security, with the condition on the market hr, the completeness of the market N, and of course the level of diversification n.

Convergence of the system The system will converge if all the elements of $\frac{hr}{n^2}{D_T}^{2t}$ have a modulus below one. The mamixum eigenvalue of $T^{\intercal}T$ is $|\psi_0| = (\sum_{s=0}^{s=n-1}\omega^0)^2 = n^2$. Therefore the convergence condition for $\frac{hr}{n^2}{D_T}^{2t}$ is $\frac{hr}{n^2}n^2 < 1$ or:

$$hr < 1$$

This condition is necessary to study the distribution of the vector of price dynamics is well-behaved, for f $hr > 1$, we would have $lim_{t\to+\infty}var(\triangle P_{j,t}) = +\infty$.

3.2 Endogenous correlations

We study the distributions of the vectors $\triangle P_t - E(\triangle P_t)$, $\Sigma_{t=0}^{t=+\infty}(\triangle P_t - E(\triangle P_t))$, and $\frac{1}{n}T\Sigma_{t=0}^{t=+\infty}(\triangle P_t - E(\triangle P_t))$, which respectively refer to the vector of price changes at a given period t, accross time, and average portfolio return for each investor accross time. As all three vectors may be expressed as a linear combinaison of the initial normally

[6]see appendix C from matrix for a derivation through an N=4 example

[7]$\triangle p_{k,t} - \mu = (\frac{hr}{n^2})^t\frac{1}{N}\sum_{j=0}^{j=N-1}\varepsilon_{0,j}^F(2\sum_{q=1}^{q=N/2-1}cos(\frac{2\pi(k-j)q}{N})\left(\frac{1-cos(\frac{2\pi nq}{N})}{1-cos(\frac{2\pi q}{N})}\right)^t + n^{2t} + cos(\pi(k-j))\left(\frac{1-cos(n\pi)}{2}\right)^t)$ if N is even

distributed shocks, they must also be normally distributed. As we study deviations from the mean each stochastic vector's expected value is 0, so that we only need to find the realized covariance matrices to obtain the distributions.

In assets per period The covariance matrix at a given t is $\sum_t = E([\triangle P_t - E(\triangle P_t)][\triangle P_t - E(\triangle P_t)]^\intercal) = E([\triangle P_t - \mu][\triangle P_t - \mu]^\intercal)$, we thus have:

$$\sum_t = (\frac{hr}{n^2})^{2t} P(VD_T)^{2t} P^{-1} E(\varepsilon_0^F \varepsilon_0^{F\intercal}) P^{-1} (VD_T)^{2t} P \tag{17}$$

Where $\varepsilon_0^F \varepsilon_0^{F\intercal}$ is the fundamental covariance matrix, so that $E(\varepsilon_0^F \varepsilon_0^{F\intercal}) = diag(\sigma_F^2)$. Therefore: [8]

$$\Sigma_t = \sigma_F^2 (\frac{hr}{n^2})^{2t} P(VD_T)^{2t} V (VD_T)^{2t} P \tag{18}$$

The matrix is symmetric since $V = V^\intercal$ and $P = P^\intercal$. Post-multiplying by PP^{-1} we find $\Sigma_t = \sigma_F^2 (\frac{hr}{n^2})^{2t} P(VD_T)^{2t} V (VD_T)^{2t} VP$ so that $(VD_T)^{2t} V (VD_T)^{2t} V$ is the matrix similar to Σ_t. It is diagonal, and its eigenvalues are positive, so that Σ_t is definite positive.

This gives us the general expression of the covariance between two securities k and j:

$$cov(\triangle p_j, \triangle p_k)_t = \frac{\sigma_F^2}{N} (\frac{hr}{n^2})^{2t} (2 \sum_{q=1}^{q=(N-1)/2} cos(\frac{2\pi(k-j)q}{N}) \left(\frac{1 - cos(\frac{2\pi nq}{N})}{1 - cos(\frac{2\pi q}{N})} \right)^{2t} + n^{4t})$$

[9] As time increases the term $(\frac{hr}{n^2})^{2t}$ starts dominating, and correlations return to 0 as in BAU.

This result is in line the growing literature on endogenous risk: the actions of the market participants lead to correlations between assets that would not exist otherwise. One of the contributions of the paper is to make these correlations contingent on the nature on the links between investors who hold them in the financial network. While this paper is primarily concerned with systemic risk, it is interesting to note that from an investor's perspective the above implies that the act of diversifying lowers the case for diversification. Formally if we plugged the realized covariance matrix within the portfolio optimization problem, the optimal holdings of each asset would change, which in turn would impact the actual correlations.

In assets accross time Let us first we define the vector :

$$\Sigma_{t=0}^{t=+\infty} (\triangle P_t - E(\triangle P_t)) = P(\sum_{t=0}^{t=+\infty} (\frac{hr}{n^2})^t (VD_T)^{2t}) P^{-1} \varepsilon_0^F = P D_{tot} P^{-1} \varepsilon_0^F$$

Where the element of row k of $D_{tot} = \Sigma_{t=0}^{t=+\infty} [(\frac{hr}{n^2})^t (VD_T)^{2t}]$ is $\xi_k = \Sigma_{t=0}^{t=+\infty} (\frac{hr}{n^2} \phi_k)^t$, the sum of a geometric series of common ratio $(\frac{hr}{n^2} \phi_k)$, whose value is below one since hr<1. Therefore letting the number of periods tend

[8] $E(\sum_u) = (\frac{hr}{n^2})^{2t} P(VD_T)^{2t} P^{-1} E(\varepsilon_0^F \varepsilon_0^{F\intercal}) P(VD_T)^{2t} P^{-1} = (\frac{hr}{n^2})^{2t} P(VD_T)^{2t} P^{-1} \Sigma_F P(VD_T)^{2t} P^{-1} = (\frac{hr}{n^2})^{2t} P(VD_T)^{2t} P^{-1} diag(\sigma_i^{F^2}) P(VD_T)^{2t} P^{-1}$

$E(\sum_u) = \sigma^{F^2} (\frac{hr}{n^2})^{2t} P(VD_T)^{2t} P^{-1} I P^{-1} (VD_T)^{2t} P = \sigma^{F^2} (\frac{hr}{n^2})^{2t} P(VD_T)^{2t} V (VD_T)^{2t} P$

[9] In the case where $(N-1)/2$ is a whole number, ie N is odd. If N is even $cov(\triangle p_j, \triangle p_k)_t = \frac{\sigma_F^2}{N} (\frac{hr}{n^2})^{2t} (2 \sum_{q=1}^{q=(N/2-1)} cos(\frac{2\pi(k-j)q}{N}) \left(\frac{1-cos(\frac{2\pi nq}{N})}{1-cos(\frac{2\pi q}{N})} \right)^{2t} + n^{4t} + cos(\pi q) \left(\frac{1-cos(\frac{2\pi nq}{N})}{1-cos(\frac{2\pi q}{N})} \right)^{2t})$

to infinity:

$$\xi_k = \frac{1}{1-(\frac{hr}{n^2}\phi_k)} = \frac{1}{1-(\frac{hr}{n^2}\frac{1-cos(\frac{2\pi kn}{N})}{1-cos(\frac{2\pi k}{N})})} \tag{19}$$

With $\xi_0 = \frac{1}{1-hr}$ and $\xi_k = \xi_{N-k}$ since $\phi_0 = n^2$ and $\phi_k = \phi_{N-k}$.

The ex post covariance matrix is therefore:

$$\Sigma_{tot} = E((PD_{tot}P^{-1}\varepsilon_0^F)(PD_{tot}P^{-1}\varepsilon_0^F)^\intercal) = \sigma_F^2 PD_{tot}VD_{tot}P$$

The general element of the covariance[10] between two assets k and j is thus:

$$cov(\triangle p_j, \triangle p_k)_{tot} = \frac{\sigma_F^2}{N}((\frac{1}{1-hr})^2 + 2\sum_{q=1}^{q=(N-1)/2} cos(\frac{2\pi(k-j)q}{N})(\frac{1}{1-(\frac{hr}{n^2}\frac{1-cos(\frac{2\pi kn}{N})}{1-cos(\frac{2\pi k}{N})})})^2) \tag{20}$$

[11]

Investors correlation Investors' correlation is the covariance between the average return of portfolios accross active investors. The vector of per period average price return may be written as $I_t = \frac{1}{n}T\triangle P_t$, and the total price return over the period, on which we shall focus, is $I = \frac{1}{n}T\sum_{t=0}^{t=+\infty}\triangle P_t$, implying $I - E(I) = \frac{1}{n}T\sum_{t=0}^{t=+\infty}(\triangle P_t - E(\triangle P_t)$ After some straightforward algebra[12], we define the the matrix of covariance between investors :

$$\Sigma_I = \frac{\sigma_F^2}{n^2}P(D_TD_{tot})V(D_{tot}D_T)P$$

The general element of the covariance between two investors k and j is thus:

$$cov(I_k, I_j)_{tot} = \frac{\sigma_F^2}{N}((n\frac{1}{1-hr})^2 + 2\sum_{q=1}^{q=(N-1)/2} cos(\frac{2\pi(k-j)q}{N})\left((\frac{1-cos(\frac{2\pi kn}{N}) - i\,sin(\frac{2\pi kn}{N})}{1-cos(\frac{2\pi k}{N}) - i\,sin(\frac{2\pi k}{N})})(\frac{1}{1-(\frac{hr}{n^2}\frac{1-cos(\frac{2\pi kn}{N})}{1-cos(\frac{2\pi k}{N})})})\right)^2) \tag{21}$$

[13]

[10] A step by step derivation through the N=4 example is available in appendix matrix

[11] In the case where $(N-1)/2$ is a whole number, ie N is odd. If N is even $cov(\triangle p_j, \triangle p_k)_t = \frac{\sigma_F^2}{N}((\frac{1}{1-hr})^2 + 2\sum_{q=1}^{q=(N-1)/2} cos(\frac{2\pi(k-j)q}{N})(\frac{1}{1-(\frac{hr}{n^2}\frac{1-cos(\frac{2\pi kn}{N})}{1-cos(\frac{2\pi k}{N})})})^2 + cos(\pi q)(\frac{1}{1-(\frac{hr}{n^2}\frac{1-cos(\frac{2\pi kn}{N})}{1-cos(\frac{2\pi k}{N})})})^2)$

[12] $E([\frac{1}{n}T(\Sigma_{t=0}^{t=+\infty}(\triangle P_t - E(\triangle P_t))][\frac{1}{n}T(\Sigma_{t=0}^{t=+\infty}(\triangle P_t - E(\triangle P_t))]^\intercal)$
$= E(\frac{1}{n^2}TPD_{tot}P^{-1}\varepsilon_0^F\varepsilon_0^{F\intercal}P^{-1}D_{tot}PT^\intercal) = \frac{\sigma_F^2}{n^2}TPD_{tot}VD_{tot}PT^\intercal$
$= \frac{\sigma_F^2}{n^2}PD_TP^{-1}PD_{tot}VD_{tot}PP^{-1}D_TP = \frac{\sigma_F^2}{n^2}P(D_TD_{tot})V(D_{tot}D_T)P$

[13] In the case where $(N-1)/2$ is a whole number, ie N is odd. If N is even
$cov(\triangle p_j, \triangle p_k)_t = \frac{\sigma_F^2}{N}((n\frac{1}{1-hr})^2 + +2\sum_{q=1}^{q=N/2-1} cos(\frac{2\pi(N/2)q}{N})\left(\frac{1-cos(\frac{2\pi nq}{N})}{1-cos(\frac{2\pi q}{N})}\right)(\frac{1}{1-(\frac{hr}{n^2}\frac{1-cos(\frac{2\pi kn}{N})}{1-cos(\frac{2\pi k}{N})})})^2 + cos(\frac{2\pi(N/2)q}{N})\left(\frac{1-cos(\frac{2\pi nq}{N})}{1-cos(\frac{2\pi q}{N})}\right)(\frac{1}{1-(\frac{hr}{n^2}\frac{1-cos(\frac{2\pi kn}{N})}{1-cos(\frac{2\pi k}{N})})})^2$

4 Analysis

We discuss the impact of diversification on the likely number of investor bankruptcies for different parameter sets, we then specify an objective function for society in which the cost is an exponential function of the number of failures. Note we only study the distribution of investors because that of assets accross time is quite similar and less interesting from our perspective, while an analysis for per period distributions of prices, though interesting to study diffusion, would take us out of our path slightly.

4.1 Stable regime

4.1.1 Choice of parameters

The results will depend on 5 parameters: the fundamental moments of assets during BAU $\mu$and σ^2, the maximum loss K that investors can incur before bankruptcy, the number of assets N, and finally the conditions on the markets hr. We choose our baseline parameter set to fit the state of the financial markets coming in to the 2008 credit crunch. We normalize price and total quantity at t=0 to 1, which does not impact our results.

To estimate the fundamental moments we compute the average daily return and variance of an asset belonging to the S&P 500, from the first of July 1997 to the first of July 2007. This yields a fundamental daily price drift of $\mu = 0.0003$ and daily variance of $\sigma^2 = 0.000616$. We have chosen the S&P because of it is designed to provide a broader description of the investment opportunities than its counterparts, and equities are the asset class which suits our model the most. The period of measurement has been set to avoid the subprime crisis for we are interested in describing BAU, but includes the internet bubble, which we considered as an increased volatily episode rather than a systemic event.

An investor is bankrupt when his losses exceed his capital K. In our model this capital is defined a fraction of the risk-weighted portfolio, so that a natural mile stone is the Cooke ratio, which requires that a bank capital equal 8% of its risk-weighted assets.

To estimate N, the maximum level of diversification achievable by investors, we use a classic paper from Evans and Archer (1968). They estimate that diversification is no longer profitable past 10 securities, a belief shared amongst practionners. We thus set N=10.

The parameters mentioned have fairly straightforward implications: a more risky portfolio or a lower default threshold make each investor marginally more risky, while increasing the number of asset decreases the likelihood attached to any possible number of bankruptcies in a uniform manner. For conciseness we thus keep these parameters constant throughout the paper, multivariate distributions with different values for K' , $\sigma_I/E(I)$, and N are available on request.

With respect to market conditions hr, we study 3 scenarios:

- a "mild" one in which the constrained agents are forced to sell assets in fairly small quantities when a shock hits, and the demand by active investors of such assets is strong. $hr = 0.6$

- a "windy" scenario in which the quantities sold by constrained agents and the demand by active investors is are both moderate . $hr = 0.75$

- a "storm" scenario in which the procyclical effect of regulation is large, and the response by active investors is weak. $hr = 0.9$

This choice results primarily from the lack of data on fire sales by investors[14], but is also interesting as part of our analysis, as market conditions will have a key impact the desirability of diversification.

4.1.2 Distribution of investors' bankruptcy

Figure 3 shows the likelihood attached to any number of investors falling, from 0 to N=10, and for each level of diversification, from n=1 to n=10, in the hr=0.75 case.

The trade-off between the individual risk reducting effect and increased dependence appears clearly. The likelihood of going bust for a single firm is the highest for n=1, but as investors are independent in this case a large number of failures is unlikely. In the n=N case on the other hand, any investor has a low probability of going under, but all would fail if so. The intermediary levels of diversification are steps from one extreme to the other. In this hr=0.75 case, the most desirable level of diversification is not obvious: the likelihood of "intermediate" levels of failure falls quite rapidly, but we also observe a gradual detachment from 0 of the likelihood that every investor fail as n increases, reaching a non-trivial 0.005% for n=10

Figures 4,5,6 zoom on number of failures large enough to constitute a systemic event, with hr values of 0.6, 0.75, 0.9 respectively.

Figure 4 and 6 seem to give more clear-cut conclusions about the desirability of diversification. In the former losses from fire sales do not seem sufficient to bring down a diversified investor. The individually risk reducing effect dominates, and we should expect diversification to enhance the stability of the system. In the latter however, the large discount on asset sales renders the contagious externality of diversification more harmful, lowering the case for any level of diversification above 1.

Looking at dynamics, a striking feature is the rise of the odds of an "all fail" outcome, very sensitive to market conditions in the early levels of diversification. When hr=0.6 this likelihood when n=3 is only 1% of that when n=N, it moves to 7% if hr=0.75, and finally to 41% in the hr=0.9 case. Hence the relative risk of a perfectly diversified situation is not necessarily a rising function of hr.

4.1.3 Welfare

After discussing diversification's impact on the probabilities attached to any number of failures, we now weight such probabilities against the cost attached to each event.

If the cost to society increased in a linear fashion with the number of failures, diversification would unambiguously be desirable from society's perspective. Yet there are many theoretical reasons for which this cost may in fact grow exponentially with the numbers of defaults. In particular, the well-identified channels for contagion may be enhanced by bankruptcies. For instance, we expect surviving investors to become much more risk averse, reputation risk to skyrock, the liquidity constraint to tighten, etc. Bernanke (1983) also highlights that financial bankruptcies have a more than proportional impact on the real economy, through decreased money supply and increased cost of financial intermediation.

Perhaps due to this high variety of channels and non-linearity, there are to our knowledge no estimates of the exact cost to society of financial bankruptcies, and authors who want to model this cost have used different mathematical artifices. For instance Ibrahimov et al. define a time to recovery for the system, which depends of the number of defaults. We simply specify the following cost function for society:

[14] There has been empirical evidence on the presence of fire sales (see for instance Coval and Stafford (2007)), but it is hard to use it for calibration as researchers can only conjecture that a given sale has been made out of nessecity, and their impact on prices.

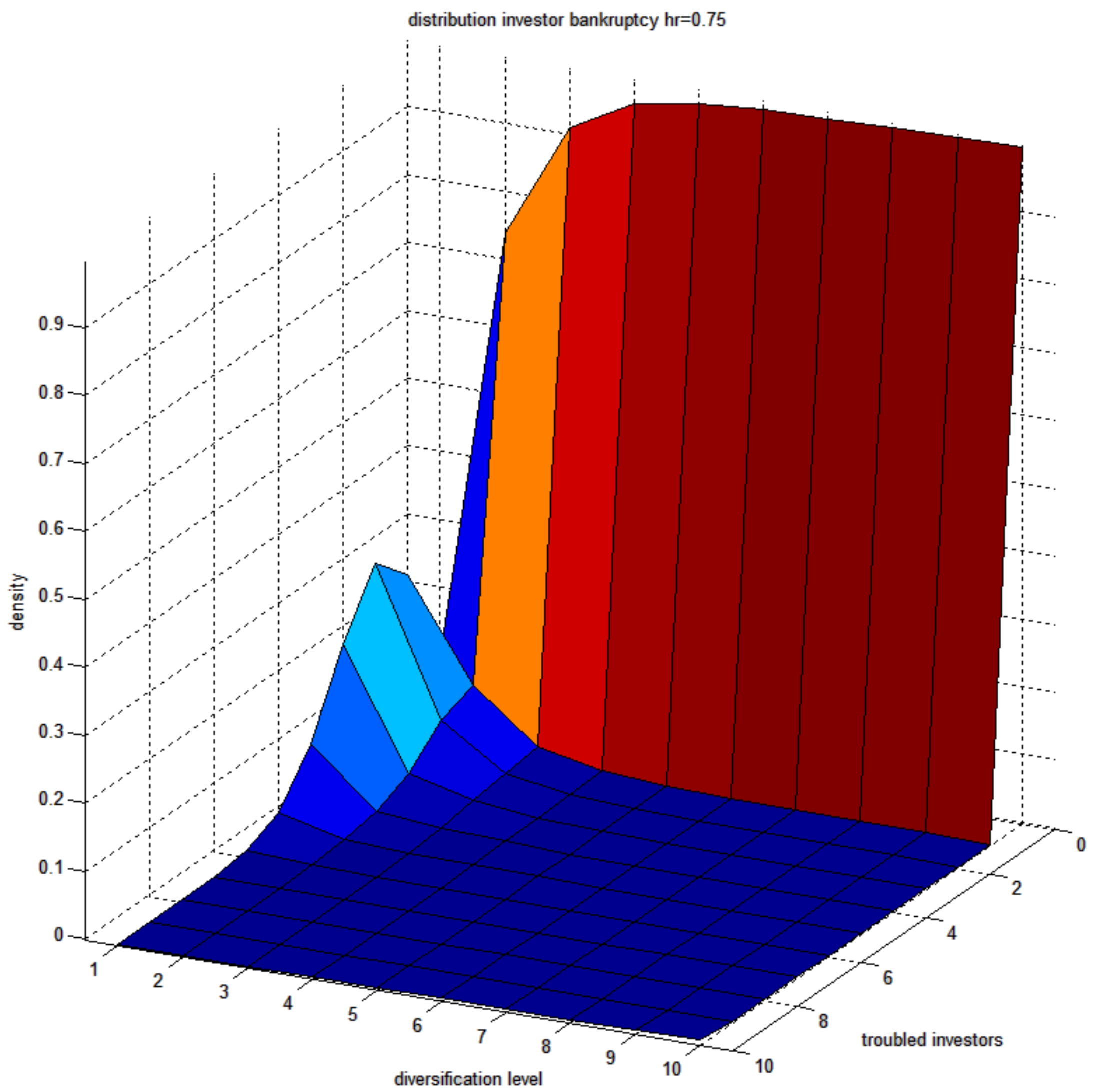

Figure 3: distribution of number of bankrupcy in stable regime with hr=0.75

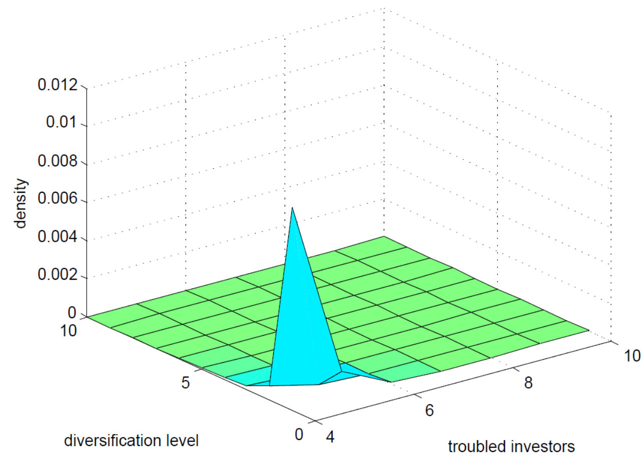

Figure 4: extreme bankruptcies odds, hr=0.6

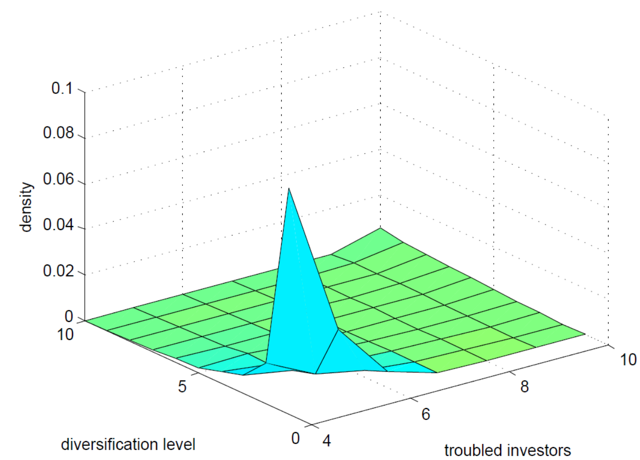

Figure 5: extreme bankruptcies odds, hr=0.75

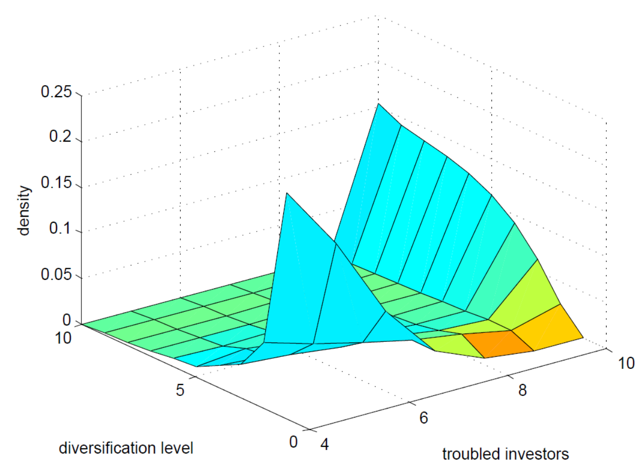

Figure 6: extreme bankruptcies odds, hr=0.9

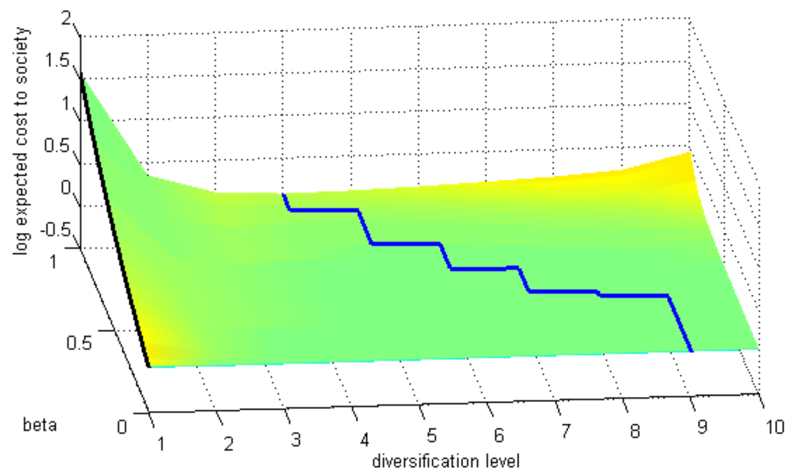

Figure 7: desirability of diversification with hr=0.6

$$C(\eta) = e^{\beta\eta}$$

Whereη, the number of failures, is a random variable, and β mitigates the severity of the increase in the cost to society of an additional failure. We show results for values of $\beta \in]0,1]$. To get an idea a the magnitude of how cost increases with the number of failures η: with $hr = 0.75$, $\beta = 1$ implies an average cost of a single bankruptcy is 810 times larger when $\eta = N$ than it is if $\eta = 1$.

The expected cost to society writes:

$$E(C) = \sum_{\eta=0}^{\eta=N} P(\eta)C(\eta)$$

Where the probalities $P(\eta = i)$ have been computed in last section, and of course depend on the level of diversification in the economy. The next 3 figures show this expected cost for our three values of hr, accross β and n. The blue line tracks the level of diversification in which the expected cost to society for a given (hr, β) couple in the lowest, the black line is the highest.

We find, as expected, that the expected systemic cost rises with β, and that a higher beta the unambiguously works against higher levels of diversification, in which the number of failures is higher conditional of one bankruptcy. Graphically this is shown by the left turns of the green line plotting the optimal diversification level.

Indivually, the impression left from the previous section remains. In the hr=0.6 case the contagion externality is too modest for diversification not to be desirable. However with large value of β the optimal level of diversification goes surprinsingly low, reaching n=4. In hr=0.75 the same logic applies, leading the optimal level to n=2, the point

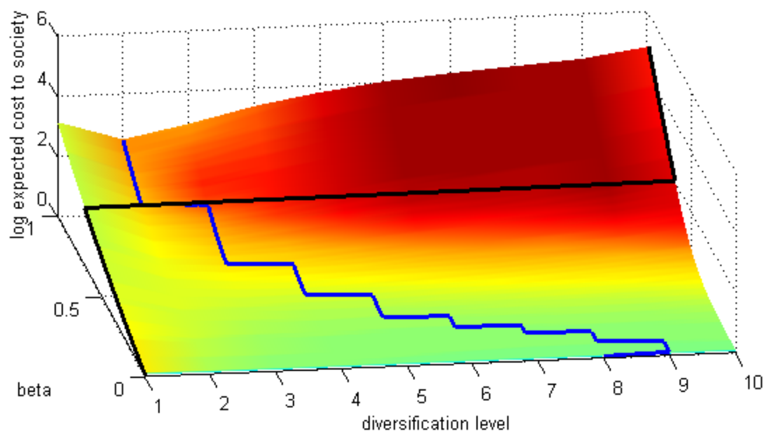

Figure 8: desirability of diversification with hr=0.75

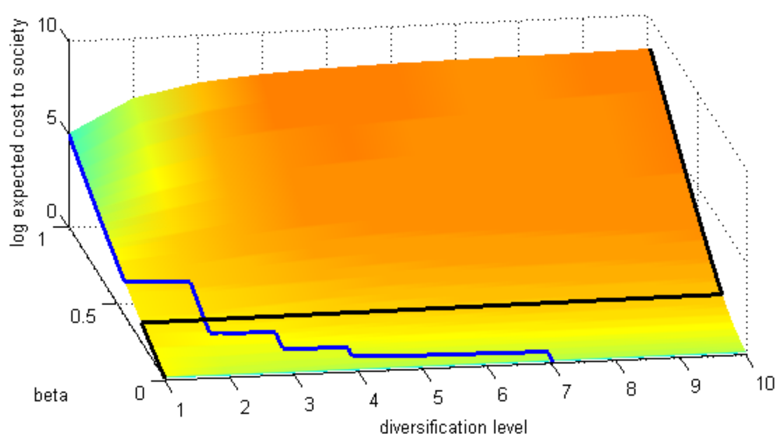

Figure 9: desirability of diversification with hr=0.9

at which the risk of a large number of failures, and in particular the all-fail outcome, is still acceptable in exchange for the huge private benefits of moving from n=1 to n=2. The contagion externality is larger, leading n=N to become the least desirable level for $\beta > 0.7$. In the last hr=0.9 situation, the scope for contagion is so high that no individual risk reduction justifies it past $\beta = 0.4$. The perfectly diversified situation becomes the worst possible situation rapidly, when $\beta > 0.2$.

$\beta = 0.7$ implies that the ratio of unit of failure in the "all fail" over that in the "one only" case is 54.5, while $\beta = 0.4$ implies a ratio of 3.65. As mentioned this measure is based more on theory than evidence, we leave it to the reader to assess where its true value would stand. In any case its seems fair to say that, in a situation in which financial shocks propagates linearly, there exist a reasonable set of parameters in which any level of diversification in the economy is dominated by a situation in which investors only trade their own assets, and complete diversification is generally not the optimal level for society.

4.2 Panic regime

4.2.1 Framework

We go back to the short-term investors maximization problem. Remember they observe $(P_{t+1} - P^F_{t+1}) - (P_t - P^F_t) = \varepsilon^{*N}_{t+1} + \varepsilon^C_{t+1}$ and use it to estimate $\varepsilon^{*N}_{i,t+1}$, by estimating the correlation $\widetilde{\rho} = \frac{\sigma_{\tilde{N}}}{\sqrt{\sigma^{\tilde{2}}_N + \sigma^{\tilde{2}}_C}}$ between both random variables. Initially $\widetilde{\rho}$ is high, as in the BAU period there are few constrained sales. We now allow them to revise their assessment to $\widetilde{\rho} \in]0,1]$ if total supply shock on any given market i at a given period t exceed a given threshold k*, significantly above what they expected from noise trading in BAU periods. In this case the optimal rule writes:

$$\triangle q^{*st}_{i,t} = -\frac{\widetilde{\rho} m \tau_{st}}{\sigma^2}(\triangle p_{i,t} - \mu - \varepsilon^F_{i,t+1})$$

Which yields $\varepsilon^C_{i,t+1} = h * \triangle q^C_{i,t}$, where $h* = \frac{\sigma^2}{\widetilde{\rho} m \tau_{st}}$ is the inverse of the derivative $\frac{d\triangle q}{dE(\triangle p)}$, representing the demand response to a rise in expected price return. A highly responsive demand implies lower discounts and scope for contagion, and conversely. As expected when active investors suspect fire sales the haircut charged rise. The system now writes:

$$\varepsilon_t = (\frac{hr}{n^2} T^\intercal T)^t \varepsilon_0 \, if \, \forall \triangle q_i : \mid \triangle q_i \mid > k*$$

$$\varepsilon_t = (\frac{h * r}{n^2} T^\intercal T)^t \varepsilon_0 \, if \, \exists \triangle q_i : \mid \triangle q_i \mid < k*$$

Where α is the probability that one or more market reach the sales threshold, and is given by the mutivariate cumulative normal distribution since sales depend linearly on the price shocks at t=0, ie $\alpha = P(\exists \triangle q_i : \triangle q_i < k*) = \Phi(k*, ..., k*)$. It will depend on the expected value and variance of sales at t=1, which are respectively 0 and $\Sigma_{\triangle Q} = E((\frac{r}{n^2} T^\intercal T \varepsilon_0)(\frac{r}{n^2} T^\intercal T \varepsilon_0)^\intercal) = \sigma^2_F (\frac{r}{n^2})^2 P(VD_T)^2 V(VD_T)^2 P$.

Before proceeding there are two things to note here. First this technique is consistent with our set-up since the sales are the largest at t=0, so though the discount depends on the size of the initial shock, it remains constant afterwards. In other words the initial shock "sets the tone" for the rest of the crisis episode. Second if we focus more on supply shocks by constrained investors, ie fire sales, the effect works symmetrically in the opposite direction, a unexpectedly large demand shock will lead short term investors a more than proportional price rise.

The use of Φ is not innocuous. To see why let us imagine an economy of two constrained investors and two assets. In scenario A the shocks at t=0 are $\varepsilon_1 = -1$ and $\varepsilon_2 = -1$, in scenario B $\varepsilon_1 = 0$ and $\varepsilon_2 = -2$. Though the aggregate shock is the same, the bivariate cumulative probabilities are respectively 2,52% in A and 1,14% in B, using a diagonal covariance matrix as investors do at t=0. The impact on investors confidence, and on the discount and on systemic risk, is thus higher in scenario B. Besides risk aversion, there are other reasons why A may be preferable to B: interagent lending is more likely to freeze in B due to accrued counterparty uncertainty, margin calls may be triggered between losses of -1 and -2, etc.

Back to mathematics, the variance of ε_t is now conditional on the value of $\triangle Q_0$, its expected value remains 0, while it's variance [15]is given by:

$$V(\varepsilon_t) = [(1-\alpha)(\frac{hr}{n^2})^{2t} + \alpha(\frac{hr*}{n^2})^{2t}]\sigma_F^2 P(VD_T)^{2t}V(VD_T)^{2t}P \tag{22}$$

Which implies the following variances for total deviation from mean accross time and investors respectively:

$$V(\sum_t \varepsilon_t) = \sigma_F^2[\alpha PD_{tot}^* VD_{tot}^* P + (1-\alpha)PD_{tot}VD_{tot}P]$$

$$\Sigma_I = \frac{\sigma_F^2}{n^2}[\alpha P(D_T D_{tot}^*)V(D_{tot}^* D_T)P + (1-\alpha)P(D_T D_{tot})V(D_{tot}D_T)P] \tag{23}$$

4.2.2 Distribution of investors' bankruptcy

We use the same parameter set as in the "stable" case, except hr now depends on the regime the system is in. In the last section we have seen hr=0.6 implies a very limited scope for contagion, hr=0.9 a large one. These values are thus natural candidates to estimate the no panic and panic case respectively. Regarding the panic threshold, similar to last section we try 3 values which representing respectively a low, moderate, and high tendency for short-term investors to panic, for which panic is triggered for initial shocks of respectively 0.05, 0.01 and 0.005, with h=1. This represents 5%, 1% and 0.5% of normalized total quantity of each asset. These values may appear low but one should remember our time unit is a "day".

Figure 10 to 12 summarizes our findings.

In figure 10, the extreme selling required to trigger panic is very unlikely to happen when n is large. This makes extreme failure, particularly the all fail outcome, very unlikely. The perfectly undiversified situation is relatively more dangerous as it may yield a considerable number of failures, with non trivial odds of as much as 80% of investors going under.

[15] $V(\varepsilon_t) = (V(\varepsilon_t/\exists\triangle q_i :\mid \triangle q_i \mid< k*) + [E(\varepsilon_t/\exists\triangle q_i :\mid \triangle q_i \mid< k*)]^2)P(\exists\triangle q_i :\mid \triangle q_i \mid< k*) + (V(\varepsilon_t/\forall\triangle q_i :\mid \triangle q_i \mid> k*) + [E(\varepsilon_t/\forall\triangle q_i :\mid \triangle q_i \mid> k*)]^2)P(\forall\triangle q_i :\mid \triangle q_i \mid> k*) - [E(\varepsilon_t)]^2$

Since $E(\varepsilon_t) = 0$ and $E(\varepsilon_t/\exists\triangle q_i : \triangle q_i < k*) = 0$, since the distribution of $\triangle Q$ is symmetric.

$V(\varepsilon_t) = \alpha\sigma_F^2(\frac{hr*}{n^2})^{2t}P(VD_T)^{2t}V(VD_T)^{2t}P + (1-\alpha)\sigma_F^2(\frac{hr}{n^2})^{2t}P(VD_T)^{2t}V(VD_T)^{2t}P$

It follows that $V(\sum_t \varepsilon_t) = \alpha\sigma_F^2 PD_{tot}^* VD_{tot}^* P + (1-\alpha)\sigma_F^2 PD_{tot}VD_{tot}P$

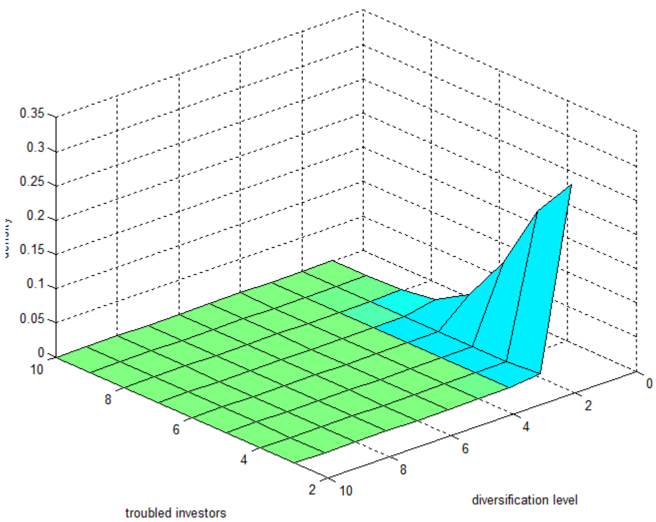

Figure 10: distribution of bankruptcy with low panic

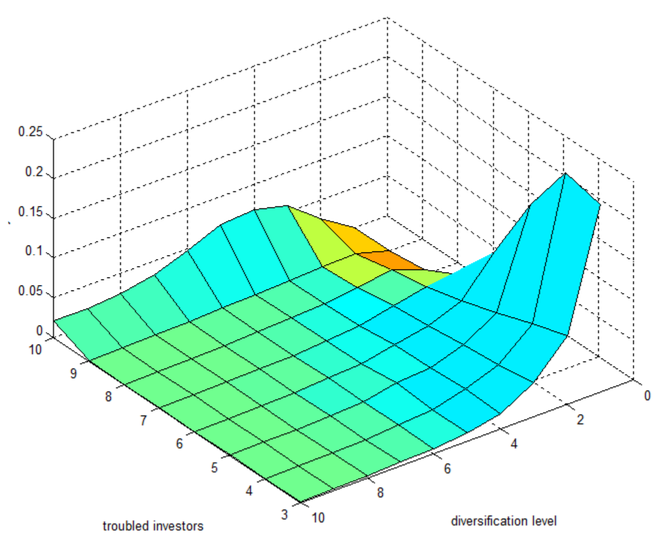

Figure 11: distribution of bankruptcy with moderate panic

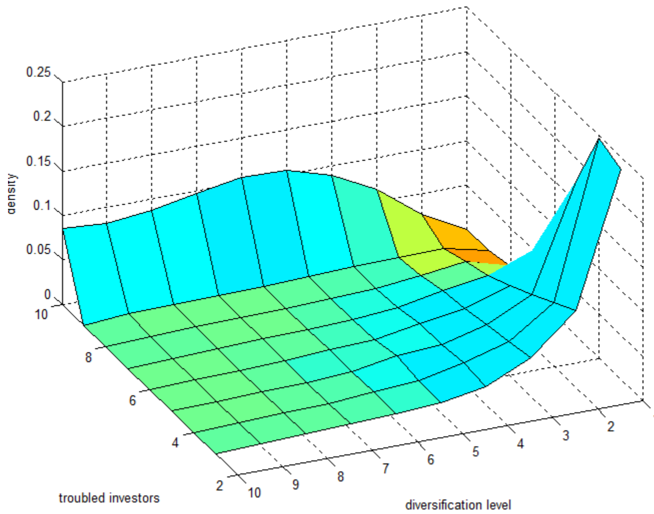

Figure 12: distribution of bankruptcy with high panic

The intermediate case brings new light to our results. Low levels of diversification which were an attractive option without panic now seem particularly harmful. The reason is that such levels are not efficient enough in smoothing the wealth shocks faced by portfolio holder, but they provide linkages through which shocks may spread accross assets. Further decreasing the threshold we observe convergence towards the hr=0.9 case. Yet even with this very low tolerance to fluctuations, the perfectly diversified situation is no longer the most dangerous situation, for any level of failure.

This intuition is confirmed by figures 13 on desirability, which shows that higher levels of diversification may no longer be the least desirable option, even with steeply increasing costs from mass failure.

4.2.3 Concluding remarks

Is diversification desirable to society? This paper first shows quite simply that the answer is not obvious and will depend the underlying scope for contagion on the markets, which itself depends on the tendency by investors faced with wealth shocks to sell and the discount they obtain on these sales. In a situation of high selling /low demand shocks may spread accross time and assets quickly, and diversification acts a gunpower which turns a sparkle into a widespread fire. With highly exponential costs a high level of diversification may become very undesirable to society, a point that has now been made by a number of researchers, though our approach is novel.

However supply overhang situations do not arise exogeneously. Amongst others, they may come from speculative selling or run for liquidity on the supply side, increased risk aversion or lower expected short-term returns on the demand side. Allowing for this latter possibility in our set-up changes our conclusions greatly. When panic is easily triggered, then only remains the higher spreading effect of diversification. However, when agents only panic when faced with extreme movements, the individually risk reducing impact of diversification as a direct negative impact on the scope for contagion. This effect has not been spotted, to our knowledge, in previous papers discussing the systemic impact of diversification.

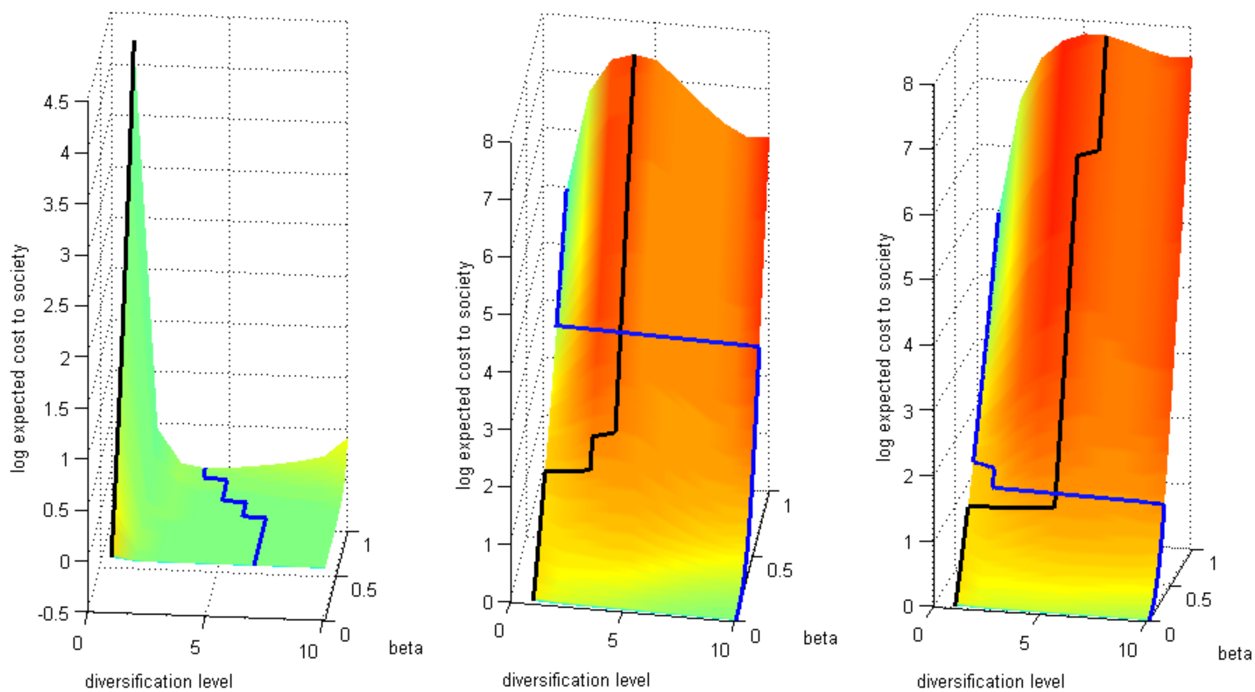

Figure 13: desirability of diversification in the panic regime, with low, moderate, and high scope for panic

We find that in this case it takes a surprinsingly low tolerance to economic fluctuations for an undiversified situation to be optimal. On the other hand low levels of diversification become very undesirable, as they creates linkages between agents while allowing these linkages to carry great risk through panic movements.

Let us look at the subprime crisis from this perspective. It is commonly accepted that the trigger of the crisis was the realization by banks that the credit backed assets, to which they were heavily exposed, were in fact much more closely related than they expected. As these correlations were high, banks were in essence holding a single credit backed asset, their actual diversification level was low. This lead the wealth shocks steeming from adverse movements on ABS to be much too large, triggering large fire sales to face short term commitments, but also instauring panic through increased counterparty risk and rising risk aversion.

According to our set-up two situations would have been preferable, either asset backed securities were very concentrated amongst few actors, which would have failed without impacting the healthy part of the system as much, or banks were more diversified and able to digest the losses from ABS markets. From this perspective the markets were in the moderate panic scenario plotted in figure 13.

Finally the paper also has implications on the new macropudential approach to regulation, and perticularly on the extra capital ratio requirement to well connected establishments that Basel III will apply. In our view such extra requirements are efficient in minimizing the extent of a crisis, conditional on a crisis occuring, as it may prevent a central actor from resorting to fire sales in a high discount environment. However the paper implies that a microprudential approach is also very much of order when trying to minimize to probability of a crisis, though it should avoid any pro-cyclicality at all cost.

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