A MODEL OF HIGH SKILL MIGRATION WITH PUBLIC EDUCATION

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Abstract
This paper proposes an original model of migration in professions where workers’ education is provided free of charge by governments. The analysis allows for a more or less substantial brain gain effect. We study a non-cooperative equilibrium where the poor country decides on the education level and the rich country decides on the quota of skilled migrants. We show that this game presents a single stable equilibrium with positive migration. Compared to the cooperative equilibrium, in the non-cooperative setting poor countries tend to under-invest in education. Whether migration is too strong or not sufficient depends on the size of the brain gain effect.

Keywords: Migration, Brain-gain, Public education, Human capital, Government.
JEL Classification: F22; I25; H11

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1 Introduction

Slovenia is one of the smallest Eastern European countries, but also one of the wealthiest. There were 6,346 doctors and dentists working in the public health system in 2010. Slovenia has two faculties of medicine (Ljubljana and Maribor). According to official estimates, additional 2,284 would be needed to "secure the professional development of the system". Facing a shortage of capacity of the two faculties, the Slovenian government decided to "import doctors", mainly from the Eastern neighbor countries. In January 2011, the Parliament passed a law that reduced the procedure time for recognition of professional qualifications of foreign doctors from around one to two years to only a month.\footnote{Based on information provided by Slovenian Government at www.ukom.gov.si.}

This anecdotal evidence is representative for an important trend in skilled migration. There are now between 192 and 216 million people living outside their place of birth (IOM, 2011; Ratha \textit{et al.}, 2011), which represents between 3 and 3.2 per cent of the world population. This means that roughly one of every thirty-five persons in the world is a migrant. Among these migrants, highly skilled workers represent an ever-growing share: looking at those living in OECD countries, their number increased by 70% during the 1990s; at the same time the number of low skilled immigrants increased by only 30%. Massive migration of high-skill workers is often referred to as the "brain drain" effect, a concept that emphasizes the migration-driven erosion of human capital in countries that need it mostly.

The Docquier, Lowell and Marfouk (2009) dataset provides emigration stocks and flows towards OECD countries from all the countries of the world at three educational levels. It shows that, over the last decades, the brain drain has increased in terms of stocks but not necessarily in terms of emigration rates. High income countries may provide more high-skill workers in magnitude, but higher emigration rates are found in middle-income countries. Data provided by Docquier, Lowell and Marfouk (2009) do not differentiate between various professions and occupations. In their review of the economic literature on skilled migration, Docquier and Rapoport (2012) provide some case studies on notorious examples of brain drain: African medical doctors, European scientists
and researchers and Indian IT specialists, showing that these high-skill migrants differ in many respects but also have many things in common.

The economic literature on the brain drain mainly focuses on issues such as the determinants of skilled workers migration (see Docquier, Lohest and Marfouk, 2007; Ortega and Peri, 2009; Beine, Docquier and Ozden, 2011; Grogger and Hanson, 2011; Belot and Hatton, 2012) and the impact of skilled emigration on origin countries (see Docquier and Rapoport (2012) for an extensive literature review on this topic). One segment of this literature builds on the idea that brain drain might come with some form of brain gain, accounting for the fact that a country’s pre-migration human capital stock is endogenous to the prospect and realization of migration. Indeed, if education is a necessary (but not sufficient) condition to migrate, and if foreign returns to human capital are higher than domestic returns, then, in the presence of migration opportunities, more people but also more talented people will invest in education, thus raising the average quality of educated individuals in the domestic country (see for instance Mountford, 1997 or Beine et al., 2001). Some empirical studies tend to corroborate these theoretical models (Beine et al., 2001; Beine et al., 2008; Chand and Clemens, 2008; Batista, Lacuesta and Vicente, 2011; Gibson and McKenzie, 2011; Cortes and Pan, 2012); they show that the brain gain effect exists, and that it is not negligible.

Most of the papers studying the migration of high-skill workers analyze the case where the cost of investment in human capital is incurred by individuals (see for instance Stark and Wang, 2002; Beine et al., 2008; Docquier and Rapoport, 2012). However, in many countries, tertiary education is mostly financed through public funds. For instance, in 2009, OECD countries spend on average 6.2% of their GDP on educational institutions and tertiary education accounts for nearly one-quarter of this expenditure, or 1.6% of OECD countries’ GDP. This represents on average US$ 13 728 per tertiary student.\(^2\) Expenditure per student by tertiary institutions increased by 5 percentage points from 2000 to 2005 and by 9 percentage points over the 2005-09 period. In 2009, public funding accounts for 73% of all funds for tertiary educational institutions on average in

\(^2\) Excluding activities peripheral to instruction such as research and development and ancillary services such as welfare services to students, OECD countries annually spend on average US$ 8 944 per student at the tertiary level.
OECD countries (it accounted for 78% in 1995, 77% in 2000 and 73% in 2005) (OECD, 2012; Figure 1). Among OECD countries, public funding accounts for more than 85% of all funding of higher education in European countries (Eurydice, 2011; Figure 2).

Looking at developing countries and in particular at African countries, the picture is not so different. African countries steadily spent on average 0.78% of their GDP on tertiary institutions between 1995 and 2010. This represents approximately 20% of Africa’s current public expenditure on education; this rate is comparable to the world average and higher than the corresponding rate of non-African developing countries (18%). The public resources per student in tertiary education have been decreasing over the last decade and amount to approximately US$ 2 000 in 2006. Public funds represent approximately 75% of national expenditure on higher education (this figure is...
based on a sample of 18 countries only). The share contributed by households varies widely, from less than 10 percent in Mali, Chad, and the Republic of Congo to about 60 percent in Uganda and Guinea-Bissau. External assistance to higher education represents another important source of funds for tertiary education in Africa. Between 2002 and 2006, external assistance to higher education in Sub-Saharan Africa amounted to about US$ 600 million annually. Note however that more than 70% of this amount was spent in donors’ universities to compensate them for the cost of educating African students (World Bank, 2010).

Thus, the bulk of tertiary education cost is supplied by governments in both developed and developing countries. Individuals only have to finance a small part of that cost; the main burden for students is thus their opportunity cost of postponing entering the job market. To our knowledge, only a few migration studies consider public financing of education. Stark and Wang (2002) as well as Docquier, Faye and Pestieau (2008) analyze the appropriate policy mix between public subsidy for education and migration opportunities, when individuals finance part of their education cost and decide on their level of human capital, and origin countries’ governments maximize a social
welfare function. In a theoretical model, Poutvaara (2008) emphasizes the influence of the brain drain on the type of education publicly financed (internationally applicable or country-specific), when individuals choose their study effort.

This paper aims to analyze skilled migration in a setup where (1) higher education is supplied by governments and (2) the opportunity to migrate might attract more talented persons to a given profession (brain gain effect). Unlike other papers, we do not focus on individual decisions concerning education, but on governments’ decisions either on the level of public investment in education or on the number of hosted skilled migrants. While the problem is stated in a general framework, it is probably most relevant when applied to civil service sectors with high worker cross-border mobility, such as health care, higher education or fundamental research. In these sectors governments would commit to deliver a given volume of activity, and most often will sponsor the education of the civil servants.

The problem is cast as a non-cooperative game between the origin ("poor") country which chooses the education level per worker, taking as given the number of high-skill emigrants, and the host ("rich") country which decides on the number of highly skilled immigrants, taking as given their human capital. It will be shown that this game presents a single, stable Nash equilibrium with a positive number of migrants and a positive investment in education in the poor country. Changes in parameter values will be related to documented trends such as the development of emerging economies, or a higher price for capital in the wake of the Great Recession. The outcome of the non-cooperative equilibrium will then be compared to the cooperative solution. As it will be shown, the nature of the cooperative equilibrium depends to a large extent on the strength of the brain gain effect.

In this model, we focus on the case where each country uses its own national resources to fund the public provision of tertiary education. However, as mentioned before, developed countries tend to finance part of the tertiary education provided by developing countries through the official development aid (and part of that aid is used by developed countries to educate students from developing countries in their own universities). This is one important limitation of our analysis. Future research might study the optimal education transfer policy from richer to poorer countries.
The paper is organized as follows. The next section introduces the main assumptions of the model and studies the behavior of each country in the non-cooperative setup; in particular, we study the existence and stability properties of the equilibrium, as well as the local effects of the parameters on equilibrium variables. Section 3 analyzes the cooperative equilibrium in a bargaining framework. The last section presents the conclusion and some policy implications.

2 The non-cooperative game

There are two countries, referred to as the "Rich" and the "Poor" country, as we assume that there is a substantial gap between their development levels. Both need to produce a predetermined flow of a specific public service (for instance health care, fundamental research, higher education). Production of the public service requires effective labor services provided by qualified civil servants (respectively doctors, nurses, researchers, professors) and a constant amount of physical capital per civil servant. Effective labor services are proxied by the product between the number of workers and their average human capital. In turn, human capital depends on workers’ education and on workers’ talent, heterogeneously distributed in the population of workers attracted to that profession. Education of civil servants is provided for by the government, free of charge for the individuals. We focus on professions where skills can be transferable from one country to another. For instance, a researcher in nuclear physics trained in India can find an equivalent job in the US. On the opposite, an Indian judge, trained in specific Indian law, cannot easily work as a judge in the US.

Civil servants trained in the poor country migrate to the rich country provided that they get a work permit. Implicitly, we make the standard assumption according to which, for civil servants in the poor country, utility from working in the rich country is higher than the cost of migrating (and the net utility when migrating is higher than the utility when staying). The rich country decides on the number of work permits.

Both countries aim at minimizing the investment cost of delivering the required amount of public services.

In a first step, the problem is studied as a non-cooperative game between the two countries.
More precisely, the rich country decides on the number of migrants it is willing to accept (work permits delivered), taking as given their human capital. The poor country will choose the education level per worker, taking as given the number of migrants. In the Cournot-Nash equilibrium, each country adopts its best strategy, given that the other country follows its best strategy.

2.1 The "poor" country

The poor country must provide a predetermined amount of a specific public service (health care for instance), denoted by $Y_P$. Following an approach introduced by the efficiency wage theory (Solow, 1979), we assume that the production function can be written:

$$Y = F_P(Nq),$$

(1)

where $F_P(\cdot)$ has the standard neoclassical properties, and $Nq$ is the volume of effective labor services, delivered by $N$ civil servants (e.g. doctors), having an average human capital $q$. Production also requires a fixed amount of physical capital per employed worker (such as real estate, various equipment, computers, etc.).

We denote human capital of individual $i$ by $q_i = q(e, t_i)$, where $e$ is the education received by an individual and $t_i$ is the individual’s talent or ability to transform education in valuable skills for "real work". We admit that $q_i = et_i$.

The total number of young people who graduate in the required profession is $N + M$, where $N$ are those who will get a job at home and $M$ are those who will migrate. The public education budget thus is $E \equiv e(N + M)$. Following Beine et al. (2001), we consider that the ability to migrate depends on some characteristic that is independent of acquired human capital (family wealth, the network of friends, etc.). Thus migrants are drawn at random from the population of graduates; the average human capital of the migrants will thus be identical to the average human capital of the left-home persons. In this case, the prospect of migration will raise average talent (and human capital) of both migrants and left-home workers.

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3 Other papers (Chiquiar and Hanson, 2005; McKenzie, Stillman and Gibson, 2010; Cortes and Pan, 2012) consider the issue of the self-selection of migrants, which implies that migrants are necessarily those individuals at the higher end of the human capital distribution. Probably, in real life, human capital has an impact on the realization of migration, but whether this impact is strong as compared to other factors, the question is open to debate.
We admit that individuals attracted by the perspective of getting a job as a civil servant are heterogenous with respect to talent \( t \). To keep the analysis as simple as possible, we assume that \( t \) is uniformly distributed on the interval \([0, t^{\text{sup}}]\), where \( t^{\text{sup}} \) denotes the upper bound of the talent distribution.

The recent literature on the brain drain / brain gain debate (Mountford, 1997; Beine et al., 2001; Docquier and Rapoport, 2012) points out that the "quality" of people attracted to a given profession increases with the probability to get a well-paid job abroad and with the expected wage abroad.\(^4\) We can thus consider that the upper bound of the talent distribution is positively related to \( M \). To keep the analysis as simple as possible, we assume that \( t^{\text{sup}} \) is a linear function in \( M \), \( t^{\text{sup}} = t_0 + \lambda M \), where \( \lambda > 0 \) is the sensitivity of talent to \( M \), and \( t_0 > 0 \). The average talent is thus: \( t_{av} = 0.5(t_0 + \lambda M) \). In turn, the average human capital of individuals trained in the poor country can be written:

\[
q = 0.5(t_0 + \lambda M) e. \tag{2}
\]

Since we assumed that migrants self-select on a different criterion than talent, it turns out that the average human capital of migrants and of the left-home persons is identical.

Given the production function (1), the amount of labor services required to achieve the production target \( Y_P \) is:

\[
Nq = F_P^{-1}(Y_P). \tag{3}
\]

Replacing \( q \) by expression (2), we get:

\[
0.5N(t_0 + \lambda M)e = F_P^{-1}(Y_P). \tag{4}
\]

The former equation indicates the pairs \((N, e)\) the poor country can choose in order to reach his service target, given \( M \) decided by the rich country. For instance, we can write:

\[
N = \frac{\psi^2}{e(t_0 + \lambda M)} \text{ with } \psi = \sqrt{2F_P^{-1}(Y_P)} \tag{5}
\]

or, alternatively, \( e \) as a function of \( N \).

\(^4\) The expected wage of a candidate to education being \((Nw_P + Mw_R)/(N + M)\), where \( w_P \) (\( w_R \)) is the wage for high-skilled workers in the poor (rich) country, with \( w_R > w_P \).
We assume that the poor country aims at achieving the service target $Y_P$ with the lowest investment cost in human and physical capital. The investment cost of obtaining any service target is:

$$C_P = cN + e(M + N),$$  \hspace{1cm} (6)

where $c$ is the (constant) per-capita cost of the physical capital required to equip domestic workers, and $e$ is the per-capita education cost, that applies to both future migrants and left-home persons.

So the decision problem of the poor country government is:

$$\min_{N,e} \{C_P = cN + e(M + N)\}$$  \hspace{1cm} (7)

with: $N = \frac{\psi^2}{c(t_0 + \lambda M)}$

In order to solve it, we introduce the constraint in the objective:

$$C_P = \left(\frac{c + e}{e}\right) \frac{\psi^2}{(t_0 + \lambda M)} + eM$$  \hspace{1cm} (8)

and we look for the minimum of $C_P$. The FOC $dC_P/de = 0$ leads to the optimal education level (denoted by $e_P$):

$$e_P = \frac{\psi \sqrt{c}}{\sqrt{M(t_0 + \lambda M)}}.$$  \hspace{1cm} (9)

We can easily check that $d^2C_P/de^2 > 0$.

The relationship between $e_P$ and $M$ is the best response function of the poor country, when the quota of migrants $M$ is chosen by the rich country. We have $\lim_{M \to 0} e_P = +\infty$, $\lim_{M \to +\infty} e_P = 0$, $\frac{de_P}{dM} < 0$ and $\frac{d^2e_P}{dM^2} > 0$. We will alternatively refer to the $e_P = e_P(M)$ function as the $PP'$ curve (we represent it in Figure 3).

Notice that changes in parameters shift the $PP'$ curve as indicated by the arrows in Figure 3.

Indeed, from expression (9), it can be seen that: $\frac{\partial e_P}{\partial \psi} > 0$, $\frac{\partial e_P}{\partial c} > 0$, $\frac{\partial e_P}{\partial e} < 0$ and $\frac{\partial e_P}{\partial \lambda} < 0$.

In line with intuitive reasoning, the required (optimal) education level increases if the production target is raised, or if talent is "scarcer". The required education level also increases if the cost of equipping a worker increases; in this case, it is better to employ less staff, but each of them must be more productive thanks to a better training.

Introducing equation (9) in equation (5), we can also determine the optimal number of staff.
needed at home for the production of the public service, \( N_P \):

\[
N_P = \frac{\psi}{\sqrt{c}} \sqrt{\frac{M}{I_0 + \lambda M}}.
\]  

(10)

where \( N_P \) is an increasing function in \( M \).

### 2.2 The "rich" country

As in the poor country, the rich country government must provide a given amount of public services \( Y_R \), at the lowest investment cost. At difference with the poor country, the rich country can use either locals trained locally, or migrants trained in the poor country. While the rich country can decide on the number of work permits \( M \), it will take the education level of migrants as given.

Production of the public service also requires a fixed amount of physical capital per worker.

We have denoted by \( M \) the migrants, and by \( q \) their average human capital. Denoting by \( Z \) the number of locals and by \( \theta \) their average human capital (exogenously given), the production function is:

\[
Y = F_R(Z\theta, Mq).
\]  

(11)

We want the production function to allow for partial complementarity and partial substitution between the two types of effective labor services, mostly explained by coordination problems. For instance, when the group is dominated by a category of persons (be them locals or foreigners), it is beneficial to integrate a different one, since it comes with complementary skills, but this will affect the internal cohesion of the group. It is also reasonable to assume that the production function features constant returns to scale. A convenient form, that has these properties and also allows to reach analytical solutions, is the Cobb-Douglas function:

\[
Y = (Z\theta)^{0.5} (Mq)^{0.5}.
\]  

(12)

For the rich country, the education cost of a migrant is assumed to be nil. Let \( w \) be the cost of training a domestic worker, and \( c \) be the cost of physical capital per worker (for simplicity, it is assumed to be the same as in the poor country). We denote the relative cost of employing a worker trained in the rich country compared to a worker trained in the poor country by \( \sigma = \frac{w + c}{c} = \frac{w}{c} + 1 > 1 \).
The investment cost \( C_R \), whatever the production target, can be written as:

\[
C_R = (w + c)Z + cM. \tag{13}
\]

The rich country decision problem is:

\[
\begin{align*}
\min_{Z,M} \{ C_R = (w + c)Z + cM \} \\
\text{with: } (Z\theta)^{0.5} (M\theta)^{0.5} = Y_R \\
\text{with: } q = 0.5(t_0 + \lambda M) e
\end{align*} \tag{14}
\]

Note that the rich country has no impact on the education level \( e \), but can have an impact on the human capital of the migrants, anticipating that higher migration opportunities should attract better people to that profession in the poor country.

Following the standard resolution steps of such a standard optimization problem (see Appendix A), we first obtain a relationship between the optimal numbers of migrants and locals:

\[
Z = \left( \frac{0.5}{\sigma} \right) \frac{M(t_0 + \lambda M)}{0.5t_0 + \lambda M}. \tag{15}
\]

We introduce \( Z \) and \( q \) (Eq. 2) in the production function and set it equal to the production target. This equation implicitly defines the optimal number of migrants \( (M_R) \), given \( e \) and the various parameters:

\[
Y_R = \left[ \left( \frac{0.5}{\sigma} \right) \frac{M(t_0 + \lambda M)}{0.5t_0 + \lambda M} \right]^{0.5} [0.5M (t_0 + \lambda M) e]^{0.5}. \tag{16}
\]

This expression is equivalent to:

\[
G(M) = \frac{b}{e} \tag{17}
\]

where we define:

\[
G(M) = \frac{[M(t_0 + \lambda M)]^2}{0.5t_0 + \lambda M} \tag{18}
\]

\[
b = (2Y_R)^{0.5} \frac{\sigma}{\theta} \tag{19}
\]

**Proposition 1** There is a single \( M_R \), such that \( G(M_R) = \frac{b}{e} \).

**Proof.** It can be seen that \( G(0) = 0 \) and \( \lim_{M \to +\infty} G(M) = +\infty \). Furthermore, \( G(M) \) is a monotonously increasing function in \( M \):

\[
G'(M) = \frac{M(t_0 + \lambda M)}{(0.5t_0 + \lambda M)^2} \left[ 3\lambda^2 M^2 + 3t_0 \lambda M + (t_0)^2 \right] > 0, \ \forall M > 0 \tag{20}
\]
Since $\frac{b}{c} > 0$, $\exists ! M_R > 0$ such that $G(M_R) = \frac{b}{c}$. ■

**Proposition 2** $M_R$ is a decreasing function in $e$.

**Proof.** By differentiating equation (17) we get:

\[ G'(M_R) dM_R = -\frac{b}{e^2} de \]  
\[ \frac{dM_R}{de} = -\frac{b}{e^2 G'(M)} < 0 \]  

The optimal quota of migrants $M_R$ is a decreasing function in the education level $e$ such as decided by the poor country. The relationship between $M_R$ and $e$ can be interpreted as the best response function of the rich country, when the poor country decides on the level of $e$.

Since $M_R(e)$ is monotonous on the interval $[0; +\infty]$, it admits a reciprocal $e_R(M)$ that we will refer to as the $RR'$ curve (Figure 3):

\[ e_R(M) = \frac{b}{G(M)} = \frac{b}{0.5t_0 + \lambda M} \]  

The function $e_R(M)$ is decreasing in $M$ and convex, with $\lim_{M \to 0} e_R = +\infty$, $\lim_{M \to +\infty} e_R = 0$, $\frac{de_R}{dM} < 0$ and $\frac{d^2 e_R}{dM^2} > 0$.

Changes in parameters shift the $RR'$ relationship as indicated by the arrows in Figure 3. Since $b = \frac{(2Y_R)^2}{\theta} \left(1 + \frac{w}{c}\right)$, it turns out that $\frac{\partial e_R}{\partial Y_R} > 0$, $\frac{\partial e_R}{\partial w} > 0$, $\frac{\partial e_R}{\partial \lambda} < 0$, $\frac{\partial e_R}{\partial \theta} < 0$. We also can show that $\frac{\partial e_R}{\partial t_0} = -\frac{b M (0.5t_0 + 1.5 \lambda M)^2}{[M(t_0 + \lambda M)^3]} < 0$ and $\frac{\partial e_R}{\partial \lambda} = -\frac{b M (t_0 + \lambda M)^3}{(t_0 + \lambda M)^2} < 0$.

Eventually, the optimal number of locals $Z_R$ can be determined by introducing $M_R$ in equation (15).

### 2.3 The Nash Equilibrium

A Cournot-Nash equilibrium of this game is a pair $(e^*, M^*)$ satisfying simultaneously Equation (9) and Equation (23). If such a pair exists, the poor country decides on the optimal education level taking as given the quota of migrants decided by the rich country, and the rich country optimally decides on the number of migrants taking as given their education level such as chosen by the poor country. The total number of workers trained by the poor country and the number of local
workers employed by the rich country are also obtained as equilibrium values (Equations 10 and
15).

The equilibrium condition is:

\[ e_P(M) = e_R(M) \] (24)

\[ \frac{\psi \sqrt{c}}{\sqrt{M(t_0 + \lambda M)}} = \frac{b(0.5t_0 + \lambda M)}{[M(t_0 + \lambda M)]^2}. \] (25)

We define:

\[ J(M) = \frac{[M(t_0 + \lambda M)]^3}{(0.5t_0 + \lambda M)^2}. \] (26)

After some calculations, the equilibrium condition (24) can be written as:

\[ J(M) = \left( \frac{1}{c} \left( \frac{b}{\psi} \right) \right)^2 \] (27)

where we recall that \( b = \frac{(2Y_P)^2}{3} \left( 1 + \frac{w}{c} \right) \) and \( \psi = \sqrt{2F^{-1}_P(Y_P)} \). This condition implicitly defines
the equilibrium number of migrants, denoted by \( M^* \).

**Proposition 3** There is a single, positive, finite number of migrants satisfying the Cournot-Nash equilibrium condition (27).

**Proof.** It can be seen that \( J(0) = 0 \) and \( \lim_{M \to +\infty} J(M) = +\infty \). The function \( J(M) \) is monotonously increasing in \( M \):

\[ J'(M) = \frac{[M(t_0 + \lambda M)]^2 \left( (t_0 + 2\lambda M)^2 + 0.5(t_0)^2 \right)}{(0.5t_0 + \lambda M)^3} > 0, \forall M > 0. \] (28)

Since \( \frac{1}{c} \left( \frac{b}{\psi} \right)^2 > 0 \), thus \( \exists! M^* > 0 \) such that \( J(M^*) = \frac{1}{c} \left( \frac{b}{\psi} \right)^2 \). \( \blacksquare \)

The fact that the solution to Equation (27) is single tells us that the \( PP' \) curve and the \( RR' \) curves cross only once in the interval \([0; +\infty]\).

The equilibrium education level is then determined by:

\[ e^* = e_P(M^*). \] (29)

Figure 3 shows the best response function of the poor country \( PP' \) curve and of the rich country \( RR' \) curve. Point \( A \) is the Cournot-Nash equilibrium.

We can also show that:

**Proposition 4** The Cournot-Nash equilibrium of this game is dynamically stable.
**Proof.** We consider the sequence of adjustments where the poor country chooses $e_t$ given $M_{t-1}$, and the rich country chooses $M_t$ given $e_t$, and so on. The equilibrium is dynamically stable if starting from any outside point $(e, M)$, the system converges toward the equilibrium solution $(e^*, M^*)$. The stability condition is $e_R(M) > e_P(M) \forall M < M^*$ and $e_R(M) < e_P(M) \forall M > M^*$.

We know that 

$$\left( \frac{e_R(M)}{e_P(M)} \right)^2 = \frac{1}{e} \left( \frac{b}{\sigma} \right)^2 \left( \frac{0.5t_0 + \lambda M}{M(t_0 + \lambda M)} \right)^2 = \frac{J(M^*)}{J(M)}.$$ 

Since $J(M)$ is increasing in $M$, this proves the stability condition. \[\blacksquare\]

### 2.4 Comparative statics

We recall the compact notations of the parameters: $\psi = \sqrt{2F_P^{-1}(Y_P)}$ and $b = (2Y_R)^2 \left( \frac{w+c}{\sigma} \right)$. 

Table 1 summarizes the equilibrium values for the main endogenous variables.

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5 The rationale would not change if we allow for a delay in adjusting migration quotas as well.
From Figure 3 or total differentiation of the equilibrium conditions, we can determine the impact of parameter changes on the following key variables: the number of migrants, the education level in the poor country, the number of locals hired to produce the public service in the poor country, and the number of skilled locals hired in the rich country (see Appendix B). Table 2 summarizes the main findings.

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</table>

Table 2: Comparative statics.

Note that the local effect of the sensitivity of talent to migration opportunities (λ) on the equilibrium number of migrants (M*) is: \( \frac{dM^*}{dλ} = \frac{(0.5t₀ - λM^*)^{2} (M^*)}{4(λM^*)^2 + 4t₀λM^* + 1.5(t₀)^2} \) (see Appendix B). Thus the equilibrium number of migrants \( M^* \) increases with \( λ \) if and only if \( λM^* < 0.5t₀ \), i.e. when the incentive effect of migration is not "too high"; in the opposite case, \( M^* \) decreases with \( λ \).

Some variations in parameters can be related to global trends and policy changes.

- For instance, if a poor country is developing, it is very probable that the need for public
services will increase, that is \( dY_P > 0 \). As expected, the education level per civil servant would then increase. However, in equilibrium, the number of migrants would decrease, which, to some extent, would reduce workers’ human capital, and would require a stronger education per worker.

- After a financial crisis such as the 2007-2009 Great Recession, a higher cost of physical capital per worker \((c)\) would raise the education level in the poor country, because the poor country would like to hire less people but better trained. Yet, since in the rich country the relative price of people trained at home compared to migrants \((\sigma)\) declines, the rich country would host less migrants. The human capital of migrants would further decline, and reinforce the need to raise the education level in the poor country.

- If subject to increased tensions related to public debts and public deficits rich countries decide to shift some of the cost of training civil servants from the government to individuals themselves, this can be interpreted as a reduction in \(w\). According to our model, we should observe a decrease in skilled international migration and an increase in the education of workers in the poor country that would tend to offset a weaker brain gain effect.

As is often the case with non-cooperative equilibria, when decision makers play "one against the other", the outcome is Pareto-inefficient. In other words, should they negotiate on the key decision variables (in our case, \(e\) and \(M\)), the resulting cooperative equilibrium would bring about an improvement in welfare for both parties.

The next section investigates this cooperative equilibrium. In particular, we would like to know in what direction endogenous variables move when we go from the non-cooperative to the cooperative equilibrium. The analysis will emphasize the key role of the brain gain effect: the type of the cooperative equilibrium depends to a large extent on the sensitivity of talent to migration (as captured by our parameter \(\lambda\)).
3 The cooperative equilibrium

Starting from the Cournot-Nash equilibrium, the two countries might strive to work out a Pareto-efficient allocation of resources by bargaining about migration to the rich country and education in the poor country. Let us denote by $C^*_P$ and $C^*_R$ the investment costs in the poor (respectively rich) country as evaluated at the Nash equilibrium values $e^*$ and $M^*$ (see Table 2), and let us assume that should the negotiations fail, the non-cooperative solution prevails. In this case, the Nash solution to the negotiation problem would be obtained by maximizing the product of each country’s net gain of cooperating raised respectively to the power $\rho$ and $1-\rho$, with $\rho \in [0,1]$ being representative of the relative bargaining power.\(^6\)

\[
\begin{align*}
\max_{e,M}\left\{[(cN + e(M + N)] - C^*_P\}^{1-\rho} \left\{[w + c]Z + cM\} - C^*_R\right\}^\rho \\
\text{with: } Y_P = F_P(Nq) \\
\text{with: } Y_R = F_R(Z\theta, Mq) \\
\text{with: } q = 0.5(t_0 + \lambda M)\epsilon
\end{align*}
\]

The solution will be denoted by $(\hat{e}, \hat{M})$. It is difficult to solve this non-linear problem and find an explicit solution. One way to circumvent the analytical complexity of this problem is to focus on the isocost curves; this approach will allow us to provide a graphic solution.

3.1 Poor country isocost curves

Taking into account the production target constraint in the poor country, the total investment cost in physical and human capital can be written as a function in $e$ and $M$:

\[
C_P(e, M) = (c + e)N + eM
\]

\[
= \frac{(c + e)}{e} \left(\frac{\psi^2}{(t_0 + \lambda M)} + eM\right)
\]

An isocost curve is made of all pairs $(e, M)$ verifying condition: $C_P(e, M) = ct$. The shape of such isocost curves can be inferred from analyzing the derivative $\frac{dM}{de} C_P = ct$. Total differentiation of

\(^6\) In an extreme but plausible case, the rich country has all the power, $\rho = 1$. 

17
the isocost condition leads to:

$$dM \left[ \frac{(c + e)}{e} \frac{\psi^2}{(t_0 + \lambda M)^2} \lambda - e \right] = \frac{de}{e} \left[ M - \frac{c}{e^2} \frac{\psi^2}{(t_0 + \lambda M)^2} \right]$$

$$\left[ \frac{dM}{de} \right]_{C_p=ct} = \frac{M - \frac{c}{e^2} \frac{\psi^2}{(t_0 + \lambda M)^2} \lambda - e}{e}$$

(32)

Notice that $$\left[ \frac{dM}{de} \right]_{C_p=ct} = 0$$ implies $$M = \frac{c}{e^2} \frac{\psi^2}{(t_0 + \lambda M)^2}$$ or $$e = \sqrt{\frac{\psi^2}{M(t_0 + \lambda M)}} = e_P(M)$$. In other words, if the isocost curve is represented in the referential $$(Oe, OM)$$, we know that the curve will have a zero slope right on the best response function of the poor country. In the opposite referential $$(OM, Oe)$$, the slope of the isocost curve is infinite, i.e., the curve is tangent to a vertical line at the crossing point with the best response function.

In the neighborhood of $$e_P(M)$$, the sign of the denominator of expression (32) depends on the strength of the brain gain effect:

$$\frac{(c + e_p(M))}{e_p(M)} \frac{2F_{P}^{-1}(Y_{P})}{(t_0 + \lambda M)^2} \lambda - e_P(M) \geq 0 \iff \lambda \geq \frac{c}{e_p(M)} \frac{t_0}{M}$$

(33)

In particular, these inequalities hold for the Cournot-Nash equilibrium values $$(e^*, M^*)$$ which also depend on $$\lambda$$. So, equation:

$$\lambda = \frac{c}{e^*(\lambda)} \frac{t_0}{M^*(\lambda)}$$

(34)

implicitly defines a fixed point $$\lambda_0 > 0$$ such that:

- For a weak (or inexistent) brain gain effect ($$0 \leq \lambda < \lambda_0$$), in the neighborhood of $$(e^*, M^*)$$ we have $$\left[ \frac{dM}{de} \right]_{C_p=ct} > 0$$ for $$e < e^*$$, and $$\left[ \frac{dM}{de} \right]_{C_p=ct} < 0$$ for $$e > e^*$$.
- For a strong brain gain effect ($$\lambda > \lambda_0$$), in the neighborhood of $$(e^*, M^*)$$ we have $$\left[ \frac{dM}{de} \right]_{C_p=ct} > 0$$ for $$e > e^*$$, and $$\left[ \frac{dM}{de} \right]_{C_p=ct} < 0$$ for $$e < e^*$$.

In order to draw correct families of isocost curves, we use a numerical simulation. We use the same $$(OM, Oe)$$ referential as for the non-cooperative equilibrium. Figure 4 represents a family of isocost curves obtained for a small $$\lambda$$ ($$\lambda = 0.005$$), Figure 5 represents a family of isocost curves obtained in the case of a larger $$\lambda$$ ($$\lambda = 0.05$$). On both graphs we also represent the curve $$PP'$$, i.e.

---

7 By definition of the non-cooperative equilibrium, $$e^* = e_P(M^*)$$.

8 Parameter values are: $$F_{P}^{-1}(Y_{P}) = 105$$, $$Y_{R} = 120$$, $$t = 2$$, $$c = 1$$, $$\theta = 4$$, $$w = 3$$. 
the best response function of the poor country (in the non-cooperative equilibrium). As it can be seen, isocost curves present an infinite slope (they are tangent to a vertical line) at points where they cross the best response function. But they have a "D form" in the first case, and a "reverse D form" in the second case.

In case of a small brain gain effect, isocost curves located toward the NW correspond to lower investment costs for the poor country.

In case of a strong brain gain effect, isocost curves located toward the SE correspond to lower investment costs for the poor country.

3.2 Rich country isocost curves

In the same way, the total investment cost in physical and human capital in the rich country can be written as a function of $e$ and $M$.

First, we can write $Z$ as a function of $M$ and $e$, by taking into account the target production
condition and the definition of \( q \):

\[
(Z\theta)^{0.5}(Mq)^{0.5} = Y_R \iff Z = \frac{(Y_R)^2}{0.5\theta M(t_0 + \lambda M)e}
\]  

(35)

The investment cost of producing \( Y_R \) can be written:

\[
C_R(e, M) = 2\left(\frac{Y_R}{\theta e M(t_0 + \lambda M)}\right)^2(w + c) + eM
\]  

(36)

Isocost curves are defined by: \( C_R(e, M) = ct \). Total differentiation leads to:

\[
\left[ \frac{de}{dM} \right]_{C_R=ct} = c - \frac{2(Y_R)^2(w + c)}{\theta e} \frac{(t_0 + 2\lambda M)^2}{M^2(t_0 + \lambda M)^2}
\]  

(37)

The isocost curve has a zero slope right for \( M = M_R(e) = G^{-1}(\frac{\lambda}{\theta}) \) as defined in the previous section. Furthermore, since \( G(M) \) is increasing in \( M \), we can show that \( \left[ \frac{de}{dM} \right]_{C_R=ct} > 0 \) for \( G(M) > \frac{b}{e} \iff M > M_R(e) \); and \( \left[ \frac{de}{dM} \right]_{C_R=ct} < 0 \) for \( G(M) < \frac{b}{e} \iff M < M_R(e) \).

In other words, rich country isocost curves have a U-shape in the referential \((OM, Oe)\), with the troughs on the best-response function of the rich country. In Figure 6 we represent the \( RR' \) best response curve of the rich country and a family of isocost curves.
Isocost curves located toward the North of the \((OM, Oe)\) plane correspond to lower investment costs.

![Rich country isocost curves](image)

Figure 6: Rich country isocost curves

### 3.3 Two types of cooperative equilibria

As shown before, the form of the isocost curves of the poor country depends on the strength of the brain gain effect (as captured by \(\lambda\)). As a consequence, we have two types of cooperative equilibria, depending on this key parameter.

Economists still debate whether the brain gain effect exists at all, or is positive but week, or quite strong. As it will be shown, in our model, there is no structural difference between the extreme zero impact assumption and the positive but moderate brain gain assumption. A structural difference can be revealed in the case of a strong brain gain effect.
3.3.1 Cooperative equilibrium: the case of a weak (or inexistent) brain gain effect

Figure 7 displays the non-cooperative equilibrium \((e^*, M^*)\) at the crossing point between the two best response functions, and the two isocost curves, one for the poor country and another for the rich country, that correspond to the constant investment cost \(C_P\) and respectively \(C_R\).

Figure 7: The region of the cooperative equilibrium - Small brain gain effect

Whatever the cooperative equilibrium \((\hat{e}, \hat{M})\), we know that it will be located in the region between the two isocost curves, thus would require higher education for (all) locals in the poor country, and less migration (reduced migration quotas in the rich country).

**Proposition 5** *In the case of a relatively small (or inexistent) brain gain effect, without coordination the poor country tends to under-invest in education, and the rich country tends to hire too many migrants.*

**Proof.** Cf. graphic analysis of the cooperative solution. ■

The region of Pareto-efficient contracts is made up of the locus where the isocost curves of the two countries are tangent (i.e., have the same slope) : 

\[
\frac{de}{dM} \bigg|_{C_R=ct} = \frac{de}{dM} \bigg|_{C_P=ct}.
\]
How the gain of moving from the non-cooperative to the cooperative equilibrium is split between the two countries depends on their relative negotiation power. If the rich country has the whole negotiation power, the equilibrium solution will be the point where the highest isocost curve of the rich country is tangent to the isocost curve $C_P(e, M) = C_P$ of the poor country; this requires very high education per capita. If we move toward the center of the cooperative region and the equal split of the gains from cooperation, the required education level is smaller. The number of migrants is less sensitive to the negotiation power.

3.3.2 Cooperative equilibrium: the case of a strong brain gain effect

In the opposite case of a strong brain gain effect, the isocost curves of the poor country have the opposite shape. Figure 8 shows non-cooperative equilibrium $(e^*, M^*)$ at point A and the two isocost curves that correspond to this equilibrium education and migration levels.

![Figure 8: The region of the cooperative equilibrium - Large brain gain effect](image)

This time the Pareto ameliorating region is quite narrow (as compared to the small brain gain case). In other words, gains from cooperation can only be modest. We can however notice that:
Proposition 6 In the case of a relatively large brain gain effect, without coordination the poor country tends to under-invest in education, and the rich country tends to impose too much restrictions on migration.

Proof. Cf. graphic analysis of the cooperative solution.

From the analysis of the two cases, we can infer without ambiguity that in a non-cooperative setting, poor countries tend to under-invest in education of their civil servants (doctors, teachers, researchers, etc) - whatever the strength of the brain gain effect.

4 Conclusion

This paper has developed an original model of skilled migration, where migrants education is supplied for free by the government of the migrants’ origin country. Such a framework is most relevant to analyze the migration of civil servants with high cross-border mobility such as doctors, nurses, teachers, researchers, etc.

In our model, governments in both origin and host countries aim at delivering a predetermined amount of civil services, at the lowest investment cost. The problem is analyzed as a game between the poor country that decides on the education level of the civil servants and the rich country that decides how many such migrants it wants to host. The model can take into account the potentially favorable impact of migration on attracting highly talented people towards that profession, i.e. the brain gain effect. Despite the relatively high analytical complexity of the problem, the paper presents a neat solution with positive migration and positive investment in education. Comparative statics allow to comment on possible trends in migration; in particular, in this model, economic growth in poor countries would bring about less migration, and additional education per worker; if rich countries decide to shift the education burden on private individuals, the two trends mutually reinforce each other.

We then analyze the cooperative equilibrium, where the two countries can negotiate on the Pareto-efficient education-migration bundle. We show that the nature of the cooperative equilibrium depends to a large extent on the strength of the brain gain effect. Recent empirical studies tend to show that this effect may be significant. When the sensitivity of talent to migration opportunities is relatively small (not different from the case where the effect does not exist at all), in
the non-cooperative equilibrium the poor country tends to invest too little in education, and the rich country tends to attract too many migrants. In the opposite case of a strong brain gain effect, the poor country still neglects education, while the rich country does not host enough migrants.

Moving from the non-cooperative organization of the international relations on migration and education to a cooperative organization would make both rich and poor countries better-off. However, if the brain gain effect is relatively strong, as documented by the empirical literature, then this paper suggest that the expected gain from cooperation might be modest. Expected gains from cooperation would be much more important if the brain gain effect is small.

References


### Appendix A. Cost minimization: the rich country problem

We introduce the second constraint in the former, and write a Lagrangian:

\[ L = (w + c)Z + cM - \eta \{ F_R(Z\theta, Mq(M)) - Y_R \} \tag{38} \]

We denote the marginal productivities by \( F_{R1} \) et \( F_{R2} \):

\[ F_{R1}(Z\theta, Mq) = \frac{dF_R}{d(Z\theta)} = \frac{0.5}{Z\theta} F_R(Z\theta, Mq) \tag{A.39} \]

\[ F_{R2}(Z\theta, Mq) = \frac{dF_R}{d(Mq)} = \frac{0.5}{Mq} F_R(Z\theta, Mq). \tag{A.40} \]
First order conditions $dL/dZ = 0$ and $dL/dM = 0$ lead to:

$$w + c = \eta F_{R1}(Z\theta, Mq) \quad (A.41)$$

$$c = \eta \left( q(M) + M \frac{\partial q}{\partial M} \right) F_{R2}(Z\theta, Mq). \quad (A.42)$$

Replacing $q(M)$ and $\frac{\partial q}{\partial M}$ by their expressions, we get:

$$w + c = \eta F_{R1}(Z\theta, Mq) \quad (A.43)$$

$$c = \eta e (0.5t_0 + \lambda M) F_{R2}(Z\theta, Mq). \quad (A.44)$$

With notation $\frac{w+c}{c} = \sigma$, we get the ratio:

$$\frac{F_{R1}(Z\theta, Mq)}{F_{R2}(Z\theta, Mq)} = \frac{\sigma e}{\theta} (0.5t_0 + \lambda M). \quad (45)$$

Replacing $F_{R1}(Z\theta, Mq)$ and $F_{R2}(Z\theta, Mq)$, we get:

$$\frac{Mq}{Z\theta} = \frac{\sigma e}{\theta} (0.5t_0 + \lambda M). \quad (46)$$

Thus:

$$Z = \frac{1}{\sigma e} \frac{Mq}{(0.5t_0 + \lambda M)} \quad (47)$$

But $q = 0.5(t_0 + \lambda M)e$. It finally turns out that:

$$Z = \frac{0.5 M(t_0 + \lambda M)}{\sigma} \frac{1}{0.5t_0 + \lambda M} \quad (48)$$

This is the formula presented in the main text as equation (15).

**B Appendix B. Comparative statics calculations**

**B.1 The main equilibrium equations**

By totally differentiating the equilibrium condition

$$J(M^*) = \frac{1}{c} \left( \frac{b}{\psi} \right)^2 \quad (B.49)$$

$$\frac{[M^* (t_0 + \lambda M^*)]^3}{(0.5t_0 + \lambda M^*)^2} = (2Y_R)^4 \left( \frac{1}{\psi} \right)^2 \frac{(w + c)^2}{\theta^2 c^3} \quad (B.50)$$

we get:

$$\Sigma \left\{ 4 (\lambda M^*)^2 + 4t_0 \lambda M^* + 1.5 (t_0)^2 \right\} dM^* + (0.5t_0 + 2\lambda M^*) M^* dt_0 + (\lambda M^* - 0.5t_0) (M^*)^2 d\lambda \right\}$$

$$= \Pi \left[ -\frac{2}{\psi} d\psi + \frac{4}{Y_R} dY_R - \frac{2}{\theta} d\theta + \frac{2}{w + c} dw - \frac{3w + c}{c(w + c)} dc \right] \quad (B.51)$$
with $\Sigma = \frac{(M^*(t_0 + \lambda M^*))^2}{(t_0 + \lambda M^*)^2}$ and $\Pi = \frac{1}{\psi} \left( \frac{\psi}{\psi} \right)^2 = (2Y_R)^4 \left( \frac{1}{\psi} \right)^2 \frac{(w+c)^2}{w}.

Differentiating condition $e^* = \psi \sqrt{c} [M^*(t_0 + \lambda M^*)]^{-1/2}$, we get:

$$de^* = \frac{1}{\sqrt{c}M^*(t_0 + \lambda M^*)} \left\{ cd\psi + 0.5\psi dc - \frac{0.5\psi e}{M^*(t_0 + \lambda M^*)} \left[ M^* dt_0 + (M^*)^2 d\lambda + (t_0 + 2\lambda M^*) dM^* \right] \right\}$$

Differentiating the equilibrium value of $N^* = \frac{\psi}{\sqrt{c}} \frac{M^*}{t_0 + \lambda M^*}$, we get:

$$dN^* = \sqrt{\frac{M^*}{c(t_0 + \lambda M^*)}} \frac{d\psi}{\psi} + \frac{0.5\psi}{[c(t_0 + \lambda M^*)]^2} \sqrt{c(t_0 + \lambda M^*)} \frac{dt_0}{M^*} \left\{ \frac{c}{(0.5\lambda M^*)^2} \left[ (\lambda M^*)^2 d\lambda - M^* (t_0 + \lambda M^*) (35) \right] \right\}$$

Finally, differentiating the equilibrium value of $Z^* = \frac{0.5\psi}{w+c} M^*(t_0 + \lambda M^*)$, we get:

$$dZ^* = \frac{0.5}{(w+c)^2} \frac{M^*(t_0 + \lambda M^*)}{0.5\lambda M^*} [wdc - cdw] + \frac{0.5\psi}{(w+c) (0.5\lambda M^*)^2} \left\{ 0.5\lambda (M^*)^2 dt_0 - 0.5t_0 (M^*)^2 d\lambda + \left[ (\lambda M^*)^2 + t_0 \lambda M^* + 0.5(t_0)^2 \right] \right\}$$

**B.2 Partial derivatives of equilibrium values (Table 2)**

<table>
<thead>
<tr>
<th>of:</th>
<th>with respect to: $\theta$</th>
<th>with respect to: $w$</th>
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<tbody>
<tr>
<td>$M^*$</td>
<td>$- \frac{2}{\psi} \frac{(2Y_R)^4 (w+c)^2}{\psi^2 (\psi)^2} &lt; 0$</td>
<td>$\frac{2}{\psi} \frac{(2Y_R)^4 (w+c)^2}{(\psi)^2} &gt; 0$</td>
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<td>$e^*$</td>
<td>$- \frac{0.5\psi \sqrt{c(t_0 + 2\lambda M^<em>)}}{M^</em>(t_0 + \lambda M^<em>)} \frac{dM^</em>}{dt_0} &gt; 0$</td>
<td>$- \frac{0.5\psi \sqrt{c(t_0 + 2\lambda M^<em>)}}{M^</em>(t_0 + \lambda M^<em>)} \frac{dM^</em>}{dw} &lt; 0$</td>
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<tr>
<td>$N^*$</td>
<td>$- \frac{0.5\psi}{[c(t_0 + \lambda M^<em>)]^2} \sqrt{c(t_0 + \lambda M^</em>)} \frac{dM^*}{dt_0} &lt; 0$</td>
<td>$- \frac{0.5\psi}{[c(t_0 + \lambda M^<em>)]^2} \sqrt{c(t_0 + \lambda M^</em>)} \frac{dM^*}{dw} &gt; 0$</td>
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<tr>
<td>$Z^*$</td>
<td>$\frac{0.5c[(\lambda M^<em>)^2 + t_0 \lambda M^</em> + 0.5(t_0)^2]}{(w+c)(0.5\lambda M^<em>)^2} \frac{dM^</em>}{dt_0} &lt; 0$</td>
<td>$\frac{0.5cM^<em>(t_0 + \lambda M^</em>)}{(w+c)(0.5\lambda M^<em>)} \left{ \frac{(w+c)(\lambda M^</em>)^2 + t_0 \lambda M^* + 0.5(t_0)^2}{(w+c)^2 (0.5\lambda M^<em>)^2} \frac{dM^</em>}{dw} - 1 \right}$</td>
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<td>$e^*$</td>
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<td>$N^*$</td>
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<tr>
<td>$Z^*$</td>
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<td>( M^* )</td>
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<tr>
<td>(- \frac{(0.5t_0 + 2aM^<em>)M^</em>}{4(\lambda^2a^2)^2 + 4t_0\lambda M^* + 1.5(t_0)^2} )</td>
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with respect to: \( t_0 \)

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<th>( Z^* )</th>
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<td>( \frac{(0.5t_0 - \lambda M^<em>)(M^</em>)^2}{4(\lambda^2a^2)^2 + 4t_0\lambda M^* + 1.5(t_0)^2} )</td>
<td>(- \frac{\psi\sqrt{c}}{\sqrt{M^<em>(t_0 + \lambda M^</em>)}} \left( \frac{4(\lambda^2a^2)^2 + 4t_0\lambda M^* + 1.5(t_0)^2}{M^*} \right) )</td>
<td>(- \frac{0.5\psi\epsilon_0(M^<em>)^2}{\sqrt{\epsilon(t_0 + \lambda M^</em>)}} \left( \frac{4(\lambda^2a^2)^2 + 4t_0\lambda M^* + 1.5(t_0)^2}{M^<em>} \right) \frac{dM^</em>}{dt_0} )</td>
<td>(- \frac{0.5\psi\epsilon_0(M^<em>)^2}{(w+c)(0.5t_0 + \lambda M^</em>)^2} \left( \frac{(\lambda M^<em>)^2 + 2.5t_0\lambda M^</em> + 2(t_0)^2\lambda M^* + 0.5(t_0)^2}{4(\lambda^2a^2)^2 + 4t_0\lambda M^* + 1.5(t_0)^2} \right) )</td>
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with respect to: \( \lambda \)

<table>
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<tr>
<td>( \frac{2}{J(M^*)} (2\gamma_0)^{(w+c)^2} )</td>
<td>(- \frac{\sqrt{c}}{\sqrt{M^<em>(t_0 + \lambda M^</em>)}} \left( 1 - \frac{0.5\psi\epsilon_0(M^<em>)^2}{\lambda^2a^2} \right) \frac{dM^</em>}{d\psi} )</td>
<td>(- \frac{0.5\psi\epsilon_0(M^<em>)^2}{(M^</em>(t_0 + \lambda M^<em>))^{3/2}} \frac{dM^</em>}{d\lambda M^*} )</td>
<td>(- \frac{64\gamma_0^2}{J(M^*)} \left( \frac{(w+c)^2}{\gamma_0^2} \right) )</td>
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with respect to: \( \psi \) \((\text{with } \frac{d\phi}{dY_R} = \frac{F_{\psi^{-1}}(Y_R)}{\sqrt{2F_{\psi^{-1}}(Y_R)} > 0})\)

<table>
<thead>
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<th>( M^* )</th>
<th>( e^* )</th>
<th>( N^* )</th>
<th>( Z^* )</th>
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<tbody>
<tr>
<td>(- \frac{2}{J(M^*)} (2\gamma_0)^{(w+c)^2} )</td>
<td>(- \frac{\psi\sqrt{c}}{\sqrt{M^<em>(t_0 + \lambda M^</em>)}} \left( 1 - \frac{0.5\psi\epsilon_0(M^<em>)^2}{\lambda^2a^2} \right) \frac{dM^</em>}{d\psi} )</td>
<td>(- \frac{0.5\psi\epsilon_0(M^<em>)^2}{(M^</em>(t_0 + \lambda M^<em>))^{3/2}} \frac{dM^</em>}{d\lambda M^*} )</td>
<td>(- \frac{0.5\psi\epsilon_0(M^<em>)^2}{(w+c)(0.5t_0 + \lambda M^</em>)^2} \frac{dM^*}{d\psi} )</td>
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with respect to: \( Y_R \)