THE GEOMETRY OF VOTING POWER: WEIGHTED VOTING AND HYPER-ELLIPSOIDS

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Abstract. In cases where legislators represent districts that vary in population, the design of fair legislative voting rules requires an understanding of how the number of votes cast by a legislator is related to a measure of her influence over collective decisions. We provide three new characterizations of weighted voting, each based on the intuition that winning coalitions should be close to one another. The locally minimal and tightly packed characterizations use a weighted Hamming metric. Ellipsoidal separability employs the Euclidean metric: a separating hyperellipsoid contains all winning coalitions, and omits losing ones. The ellipsoid's proportions, and the Hamming weights, reflect the ratio of voting weight to influence, measured as Penrose-Banzhaf voting power. In particular, the spherically separable rules are those for which voting powers can serve as voting weights.

1. Introduction: Simple games and weighted voting

In *yes-no voting* several players vote for or against a proposal. Collective approval or disapproval is then determined by some decision rule, as modelled by a *simple game* $G = (N, \mathcal{W})$. Here N is a finite set of *players* (*voters*), and $\mathcal{W}$ is a collection of subsets of N that is *monotonic*: whenever $C \in \mathcal{W}$ and $C \subseteq D$, $D \in \mathcal{W}$. If a vote is taken, C is the resulting set of yes-voters, and the rule specifies collective approval then C is a *winning coalition* and we place $C \in \mathcal{W}$; if it specifies disapproval then C is a *losing coalition* and $C \in \mathcal{L}$, $\mathcal{W}$'s complement.

Example 1 (**Mathematics Department**) Professors R and B cast 1 vote each, Professor G (the Department's Chair) casts 2 votes, and approval of a proposal requires a minimum of 3 votes in favor. The associated simple game is $G_{Math} = (N, \mathcal{W}) = (\{R, B, G\}, \{\{R, G\}, \{B, G\}, \{R, B, G\}\})$. This is an example of a *weighted voting rule*, and G_{Math} is a *weighted simple game* (defined below) with *voting weights* $w_R = w_B = 1$, $w_G = 2$, and *quota* 3.

Definition 1.1. *A simple game $G = (N, \mathcal{W})$ for which $N = \{1, 2, \ldots n\}$ is weighted if there exists a vector of real number voting weights $\mathbf{w} = (w_1, \ldots w_n)$ together with a real number quota q such that a coalition C is winning precisely when the total weight $\mathbf{w}(C) = \sum_{i \in C} w_i$ of its members meets or exceeds quota. In this case, we say that $\mathbf{w}$ and q jointly realize G as a weighted game.*

We'll use $[q; w_1, w_2 \ldots, w_n]$ to specify weights and quota that realize a weighted game. Note that one can tell whether a game is weighted without referring to a quota – if $\mathbf{w}(C) > \mathbf{w}(C')$ holds for each pair $C \in \mathcal{W}$, $C' \notin \mathcal{W}$, then some quota

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can always be inserted between the weight of the heaviest losing coalition and that of the lightest winning coalition.

Example 2 (**Two couples on vacation together**) The Romans a and b and the Greeks γ and δ agree that any proposed detour from their planned vacation route requires approval of at least one person from each couple.

The corresponding game G_{Couple} is nonweighted, for if the two winning coalitions $C_1 = \{a, \delta\}$ and $C_2 = \{\gamma, b\}$ swap δ for b, the resulting coalitions $C_1' = \{a, b\}$ and $C_2' = \{\gamma, \delta\}$ are both losing. Yet every weighted game is *swap robust* – a one-for-one trade between two winning coalitions must leave at least one of them still winning. This is because any such exchange that lowers C_1's total weight must raise C_2's total by a corresponding amount.

Swap robustness is necessary for weightedness, but something stronger is needed for a full characterization. Several such characterizations appear in the literature, including *asummability* [9], the *greater-than-half property* [8], and *trade robustness* [27]. These properties form a related family, the geometric member of which is the separating hyperplanes characterization discussed in Sec. 3 (see [29] for a fuller discussion of this family).

The characterizations we introduce here seem entirely different. The geometric member of this new clan uses separating hyper-ellipsoids that are "aligned" – obtained by stretching a hypershere along each coordinate axis. Moreover, these stretch factors embody an unsuspected relationship between voting weight and influence, as measured by the Penrose-Banzhaf index of voting power.

Barthelemé and Monjardet [2] pioneered the application to voting theory of the two-step method we use for the characterization via ellipsoids – first convert Hamming distance to squared Euclidean distance in the hypercube, and then apply Huyghens theorem on the mean. It has also been exploited in [30], [31], and [4]. This approach has the potential to transform any result that entails minimizing a sum of Hamming distances, and deserves to be better known.

2. Weight VS Influence: Penrose-Banzhaf voting power

Interest in the theory of weighted voting has been sparked by its history as a real voting method. John Banzhaf's seminal 1965 article, "Weighted voting doesn't work" [1] originated as a critique of a draft written by one of his professors, made while he was a student at Columbia Law School.[1] In the period $1962 - 1964$ the U.S. Supreme Court had issued a series of decisions that established a precedent favoring the *one person, one vote* principle. Several states had been using voting districts of greatly varying population size; the court struck down these rules, on the grounds that the influence of an individual vote might depend heavily on the district within which it was cast. The professor's draft argued that one way to implement the court's principle (short of continually redrawing district lines so as to equalize district populations) would be to use weighted voting, with each representative given a voting weight proportional to the population of her district.

Banzhaf saw a flaw in this reasoning. Voting weight can be wildly disproportionate to influence. For example $[51; 49, 49, 2]$ and $[2; 1, 1, 1]$ represent the *same*

[1] We are borrowing heavily from Felsenthal and Machover [10], which devotes a chapter to recounting a much more complete history of weighted voting in the US, and of associated court cases. A later article [12] by these authors takes up the history of voting power more broadly.

voting rule – majority rule for 3 voters. Yet voter 1 cannot have 24.5 times as much influence as voter 3 and simultaneously have the same influence ... at least *one* of these weight vectors must misrepresent influence.[2] Our reasoning here contains an implicit assumption that a voter's "influence" can be meaningfully measured by a single number, but otherwise does not turn on any precise definition of "influence," resting instead on *wiggle room* – the sometimes considerable extent to which voting weights can vary while representing the same game.[3] However, a precise definition seems necessary if we wish to address two related issues:

QUESTION 1 "Is there always *some* choice of voting weights that perfectly reflects influence?"

QUESTION 2 "How can we choose voting weights for legislators in a representative assembly so that they appropriately reflect population differences among the districts represented?"[4]

So, how should we measure the influence of a voter in a simple game? In [1], Banzhaf argues that we should count instances in which a voter is *critical* or *decisive*, swinging the outcome of the collective decision. The resulting index (defined below) has come to be known as *Banzhaf voting power*. However, in 1946 the statistician Lionel Penrose had proposed the identical measure in an article [24] that did not attract much attention at the time. Penrose's work was later pointed out by Morriss [23], then by Felsenthal and Machover [10], and later by others.

Definition 2.1. *Let $G = (N, \mathcal{W})$ be a simple game, with $i \in N$. Then $\mathcal{W}_i$ denotes $\{C \in \mathcal{W} \mid i \in C\}$, $\mathcal{W}_{\neg i}$ denotes $\{C \in \mathcal{W} \mid i \notin C\}$, $\mathcal{W}_{crit\,i}$ denotes $\{C \in \mathcal{W}_i \mid C - \{i\} \notin \mathcal{W}\}$, $\mathcal{W}_{notcrit\,i}$ denotes $\{C \in \mathcal{W}_i \mid C - \{i\} \in \mathcal{W}\}$, and $\eta_i = |\mathcal{W}_{crit\,i}|$ – the raw Penrose-Banzhaf voting power of i – counts the number of winning coalitions in which voter i is critical.*

The following lemma provides one of several well-known, equivalent expressions for η_i. The proof follows immediately from the bijection between the sets $\mathcal{W}_{notcrit\,i}$ and $\mathcal{W}_{\neg i}$ given by $C \mapsto C \setminus \{i\}$.

Lemma 2.2. *Let $G = (N, \mathcal{W})$ be a simple game, with $i \in N$.*
Then $\eta_i = |\mathcal{W}_i| - |\mathcal{W}_{\neg i}|$.

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[2]In this particular example it is easy to see that the three voters play symmetric roles; every permutation of N is an automorphism of the simple game, and so one can argue that any reasonable measure of influence should assign equal amounts to the three voters. The example roughly models the situation prevailing in the Israeli Knesset during the late 1980s. Both Labor and Likud held large blocs of seats, but neither could form a ruling (majority) coalition without cutting deals with some small ultra-orthodox religious parties. These small parties wielded considerable bargaining power in the negotiations. The resulting controversy helped drive a change in Israel's parliamentary system to one in which the prime minister was elected by direct popular vote (see [3]), but that change was abolished in 2001.

[3]One alternatives is to measure influence with an interval of numbers; see [28].

[4]By "appropriately reflect population differences" we do not necessarily mean directly in proportion to population. If we view such a representative assembly as a two-tier voting rule in which individual citizens, when they elect their representatives, are in effect voting on the legislation itself, then there is an argument (which goes back to the original articles of Penrose and Banzhaf) that equalizing the influence of these citizens requires the voting powers of representatives to be in proportion to the *square roots* of their district populations.

It is typical to divide the *raw* Penrose-Banzhaf voting power η_i by 2^{n-1} (which allows a probabilistic interpretation[5]) or by $\sum_{i=1}^{n} \eta_i$ (to normalize, although this approach has been sharply criticized). However, we'll be concerned only with the relative proportions of voting power among the voters, so we'll identify η_i itself with i's *Penrose-Banzhaf voting power*.

The measure of Penrose and Banzhaf is not the only one possible, nor even the first to attract widespread attention – that distinction belongs to the alternative approach of Lloyd Shapley and Martin Shubik [26]. Our exclusive focus, in this paper, on Penrose-Banzhaf voting power is due to the role that it plays in the theorems of sections 4 and 6, rather than any inclination to carry a spear for either side in the voting power wars.

Before [1] even appeared, Banzhaf learned of a weighted voting case that was to be heard by the the New York Court of Appeals, involving weighted voting systems proposed in Washington and Saratoga counties, which would have used voting weights proportional to population. Banzhaf submitted an *amicus curae* brief, which included the galley proofs of his article. The court found Banzhaf's reasoning to be compelling, judging the proposed rules to be unconstitutional and referring to Banzhaf's work in their decision.

One might think that Saratoga's fate had settled the issue, at least for counties within New York State. But recent events show that not to be the case. The county of Schenectady adjoins that of Saratoga, lying just to its south.[7] On May 10, 2011, the 15-person Schenectady County Legislature voted to change from the then current "one-legislator-one-vote" rule, to a weighted rule (effective in 2012), in which the voting weights range from a low of 0.9048 to a high of 1.0799. These weights add to 15 and are chosen to be proportional, based on the 2010 Census, to the population share of each representative (equal to the district population divided by the number of representatives from that district, with 4 districts and from 3 to 5 representatives per district). Population shifts had forced some change to the old system, because the variance among population shares would otherwise have exceeded a state-mandated cap.

However, with the new weights, the heaviest 7 legislators have a total weight of 7.4052 (less than half of 15), while the lightest 8 weigh 7.5948. Most issues use weighted majority rule; that is, the quota is 7.5 (or 7.5001 – with these weights, it doesn't matter). Thus, although the new system presumably satisfies the state mandate and the old one would have violated it, the two systems are identical for such issues, with all legislators having equal influence. Certain budgetary matters, however, require a two-thirds (weighted) majority for approval, and here the story is quite different. With a quota of 10 the effect of the newly-approved weights is substantial; the heaviest legislator has a bit more than 19% more voting weight (or population-share) than the lightest, but has almost 77% more Penrose-Banzhaf voting power!

A recent series of expansions of the European Union has also played a role in inspiring research on voting power ([11], [15], [16], [18], and [19] are a sample). The

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[5]Consider the probability distribution $p^\star$ that assigns equal probability to every coalition D of $N \setminus \{i\}$. Then $\frac{\eta_i}{2^{n-1}}$ is the probability that D is losing and $D \cup \{i\}$ is winning. We can obtain $p^\star$ by assuming that each voter independently tosses a fair coin to decide whether to vote "yes" or "no."

[7]Union College (the home institution of one of the authors of the paper you are now reading) lies in Schenectady County.

EU's Council of Ministers has used weighted voting (as well as more complex rules that incorporate a weighted-voting component); each such expansion results in a sometimes spirited debate over changes to the previous voting weights.

Before moving on, it seems appropriate to situate the role of this paper within the context limned by the questions posed at the start of this section. For Question 1, as applied to either of the two particular power indices mentioned here, the simple answer is "No – we cannot always find a weight assignment that accurately reflects influence." A smallest counterexample is provided by the weighted game $[3; 2, 1, 1, 1]$;[8] both the Penrose-Banzhaf and Shapley-Shubik indices assign to the heavy player exactly thrice the voting power of a light player. No weighted representation can assign that much weight to the heavy player, because her weight alone would then match that of the winning coalition consisting of the three light players, yet she does not, alone, form a winning coalition. Such an example demonstrates that the mismatch between weight and influence goes beyond what can be explained in terms of wiggle-room alone.

A follow-up question might then be:

QUESTION 1B: For which weighted voting rules does some choice of voting weights accurately reflect influence?

There is a small literature ([7], [20], [21], and [22]) on "Penrose's limit theorem" (originally a conjecture of L. S. Penrose), showing that under certain circumstances, voting power tends to approximate voting weight when the number of voters is sufficiently large. With this important exception, however, 1B has received no attention in the literature, suggesting that it has not been seen as interesting. Our characterization of such rules (Section 6) in terms of spherical separability (for the Penrose-Banzhaf case) argues that that view might be up for revision.

Question 2 (tying the matter of choosing a measure of voting power with that of designing a fair voting rule) is, of course, very broad – arguably, it lies behind almost every one of the many papers that have been written on voting power. For the reader interested in learning more about the measurement of voting power, we recommend the books by Felsenthal and Machover [10] and by Laruelle and Valenciano [17].

3. Geometrization and hyperplane separation

Once we identify the set N of voters with an initial segment $\{1, 2, \ldots, n\}$ of natural numbers, each subset $X \subseteq N$ corresponds to a *characteristic vector* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for which $x_i = 1$ if $i \in X$ and $x_i = 0$ if not. Each such vector is *boolean* (it consists of 0s and 1s) and provides the coordinates of a vertex of the unit (hyper)cube in $\mathbf{R}^n$ – the *n-cube*. In this way, we can label each n-cube vertex with a corresponding coalition of voters. In terms of Example 1 (sec.1), if we identify voters R, B, and G with the x, y, and z axes respectively, then we obtain the labeled 3-cube of Fig. 1.

Notice that the shaded plane M of this figure, with equation $x + y + 2z = 2.5$, slices the cube so that winning coalitions are all strictly to one side of M and losing coalitions are all strictly to the other side – M is a *separating hyperplane* for the simple game G_{Math} of Example 1.

[8]We are grateful to Josep Freixas amd Moshé Machover for pointing out this example.

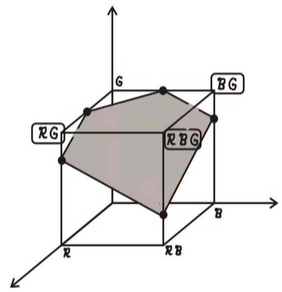

Figure 1. The 3-cube, with a separating hyperplane for G_{Math}

The weight of coalition RG, for example, is $3 = \mathbf{w}(R) + \mathbf{w}(G) = 1 + 0 + 2 = (1, 1, 2) \cdot (1, 0, 1) = (w_1, w_2, w_3) \cdot (x_1, x_2, x_3) = \mathbf{w} \cdot \mathbf{x}$, the dot product of the weight vector $\mathbf{w}$ with the characteristic vector $\mathbf{x}$ of the coalition RG. Each winning coalition Y of G_{Math} similarly satisfies $\mathbf{w} \cdot \mathbf{y} \geq 3 > 2.5$, while each losing coalition Z makes $\mathbf{w} \cdot \mathbf{z} \leq 2 < 2.5$. This explains why the plane M with equation $\mathbf{w} \cdot \mathbf{x} = 2.5$ strictly separates G_{Math}.[10] A straightforward generalization of this example leads to the following:

Theorem 3.1. *A simple game is weighted if and only if the winning and losing coalitions can be strictly separated by a hyperplane. In particular, a vector is normal to some strictly separating hyperplane if and only if serves as a vector of voting weights for the game.*

While our minds are on Fig. 1 it seems appropriate to note a key relationship between the Hamming and Euclidean metrics for n-cube vertices – a very simple observation that we will need for our final characterization in Sec. 6. The *Hamming* distance between boolean vectors $\mathbf{x}$ and $\mathbf{y}$ is equal to number of coordinates at which the vectors differ: $\mathcal{H}(\mathbf{x}, \mathbf{y}) = |\{i : x_i \neq y_i\}|$. For subsets X and Y of our voter set N, we will abuse terminology by using "Hamming distance between coalitions" or "$\mathcal{H}(X, Y)$" to refer to the Hamming distance $\mathcal{H}(\mathbf{x}, \mathbf{y})$ between their respective characteristic vectors, which coincides with the cardinality $|X \Delta Y|$ of the coalitions' symmetric difference. For such characteristic vectors $\mathbf{x}$, $\mathbf{y}$ the squared Euclidean distance $\|\mathbf{x} - \mathbf{y}\|^2$ is a sum of terms $\|x_i - y_i\|^2$, each of which has value 0 (when $x_i = y_i$) or 1 (when $x_i \neq y_i$), and this sum has the same value as $\mathcal{H}(\mathbf{x}, \mathbf{y})$:

Proposition 3.2. *Hamming distance between two coalitions equals squared Euclidean distance between their respective characteristic vectors:* $\mathcal{H}(X, Y) = \|\mathbf{x} - \mathbf{y}\|^2$.

4. Weighted Hamming distance and two characterizations of weighted voting

Throughout this section:

- $N = \{1, 2, \ldots, n\}$
- $G = (N, \mathcal{W})$ is a simple game

[10]Weaker forms of hyperplane separation also do service in the mathematics of voting. *Weak separability* allows the hyperplane to intersect the two sets being separated, while *neat separability* demands that any such intersection be the same for both sets; these notions play roles in the theory of *roughly weighted* simple games (see [13], [14] and [29]), and of *generalized scoring rules* for multicandidate elections (see [5], [30] and [31]), respectively.

- $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ is the vector of raw Penrose-Banzhaf voting powers for the voters of G
- $\mathbf{b} = (b_1, b_2, \dots, b_n)$ is a vector of nonnegative real numbers (the *Hamming weights*)
- and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ is a vector of nonnegative real numbers (the *voting weights*).

We will write $\mathbf{w} = \eta\mathbf{b}$ when $w_i = \eta_i b_i$ holds for each i. When E and F are subsets of N, their *weighted Hamming distance* is given by $\mathcal{H}_{\mathbf{b}}(E, F) = \sum_{i \in E \Delta F} b_i$. When F is a subset of N and $\mathcal{V}$ is a set of subsets of N we similarly define $\mathcal{H}_{\mathbf{b}}(\mathcal{V}, F) = \sum_{E \in \mathcal{V}} \mathcal{H}_{\mathbf{b}}(E, F)$ and $\mathcal{H}_{\mathbf{b}}(\mathcal{V}) = \sum_{E,F \in \mathcal{V}} \mathcal{H}_{\mathbf{b}}(E, F)$.

We'll say that G is *locally Hamming minimal via* $\mathbf{b}$ if whenever $E \in \mathcal{W}$, $E' \notin \mathcal{W}$, and $\mathcal{W}' = \mathcal{W} \setminus \{E\} \cup \{E'\}$ ($\mathcal{W}$ and $\mathcal{W}'$ are "neighbors") we have

$$\mathcal{H}_{\mathbf{b}}(\mathcal{W}) < \mathcal{H}_{\mathbf{b}}(\mathcal{W}') + \mathcal{H}_{\mathbf{b}}(E, E'),$$

and that G is *locally Hamming minimal* if it is locally Hamming minimal via some vector $\mathbf{b}$. The game G is *tightly Hamming packed via* $\mathbf{b}$ if every coalition in $\mathcal{W}$ is closer to $\mathcal{W}$ than is any coalition not in $\mathcal{W}$ – for every $E \in \mathcal{W}$ and $E' \notin \mathcal{W}$

$$\mathcal{H}_{\mathbf{b}}(\mathcal{W}, E) < \mathcal{H}_{\mathbf{b}}(\mathcal{W}, E')$$

– and is *tightly Hamming packed* if it is tightly Hamming packed via some $\mathbf{b}$.
Our goal is the following theorem:

Theorem 4.1. *For $G = (N, \mathcal{W})$ any simple game, the following are equivalent:*

(i) *G is locally Hamming minimal,*
(ii) *G is tightly Hamming packed,*
(iii) *G is weighted.*

The proof is immediate from:

Proposition 4.2. *For $G = (N, \mathcal{W})$ a simple game, $\mathbf{b}$ a vector of Hamming weights, and $\mathbf{w} = \eta\mathbf{b}$ the corresponding vector of voting weights, the following are equivalent:*

(i′) *G is locally Hamming minimal via $\mathbf{b}$,*
(ii′) *G is tightly Hamming packed via $\mathbf{b}$,*
(iii′) *G is weighted via $\mathbf{w}$.*

Recalling that a simple game is weighted via weight vector $\mathbf{w}$ if and only if $\mathbf{w}(E') < \mathbf{w}(E)$ holds whenever E is a winning coalition and E' is losing coalition, we see that proposition 4.2, in turn, follows immediately from:

Lemma 4.3. *For $G = (N, \mathcal{W})$ a simple game, $E \in \mathcal{W}$, $E' \notin \mathcal{W}$, $\mathbf{b}$ any vector of Hamming weights, and $\mathbf{w} = \eta\mathbf{b}$ the corresponding vector of voting weights,*[11] *the following are equivalent:*

(i″) $\mathcal{H}_{\mathbf{b}}(\mathcal{W}) < \mathcal{H}_{\mathbf{b}}(\mathcal{W}') + \mathcal{H}_{\mathbf{b}}(E, E')$,
(ii″) $\mathcal{H}_{\mathbf{b}}(\mathcal{W}, E) < \mathcal{H}_{\mathbf{b}}(\mathcal{W}, E')$,
(iii″) $\mathbf{w}(E') < \mathbf{w}(E)$.

[11] While the reader should think of $\mathbf{b}$ and $\mathbf{w}$ as *potential* choices for Hamming weights and voting weights, respectively, we are not assuming in this preamble that they are *good* choices (that witness local Hamming minimality or tight Hamming packedness, or that provide a weighted representation for the game); the only actual assumption here is that their components are nonnegative reals, and that $\mathbf{w} = \eta\mathbf{b}$.

The proof of lemma 4.3 will use the following:

Definition 4.4. *For $G = (N, \mathcal{W})$ a simple game, and $X \subseteq N$:*

- $X^C = N \setminus X$, *X's complement in N*
- $Agr_i(\mathcal{W}, X) = \{Y \in \mathcal{W} : i \in Y \Leftrightarrow i \in X\}$
- $Dis_i(\mathcal{W}, X) = \{Y \in \mathcal{W} : i \in Y \Leftrightarrow i \notin X\}$

Proof. For $(i'') \Leftrightarrow (ii'')$: Let $\mathcal{P} = \mathcal{W} \setminus \{E\}$ and $\mathcal{W}' = \mathcal{P} \cup \{E'\}$, so that $\mathcal{W}$ and $\mathcal{W}'$ are neighbors. Then

$$\begin{gathered} \mathcal{H}_{\mathbf{b}}(\mathcal{W}) < \mathcal{H}_{\mathbf{b}}(\mathcal{W}') + \mathcal{H}_{\mathbf{b}}(E, E') \\ \Updownarrow \\ \mathcal{H}_{\mathbf{b}}(\mathcal{P}) + \mathcal{H}_{\mathbf{b}}(\mathcal{P}, E) < \mathcal{H}_{\mathbf{b}}(\mathcal{P}) + \mathcal{H}_{\mathbf{b}}(\mathcal{P}, E') + \mathcal{H}_{\mathbf{b}}(E, E') \\ \Updownarrow \\ \mathcal{H}_{\mathbf{b}}(\mathcal{P}, E) < \mathcal{H}_{\mathbf{b}}(\mathcal{P}, E') + \mathcal{H}_{\mathbf{b}}(E, E') \\ \Updownarrow \\ \mathcal{H}_{\mathbf{b}}(\mathcal{W}, E) < \mathcal{H}_{\mathbf{b}}(\mathcal{W}, E'). \end{gathered}$$

For $(ii'') \Leftrightarrow (iii'')$:

$$\begin{gathered} \mathcal{H}_{\mathbf{b}}(\mathcal{W}, E) < \mathcal{H}_{\mathbf{b}}(\mathcal{W}, E') \\ \Updownarrow \\ \sum_{Z \in \mathcal{W}} \mathcal{H}_{\mathbf{b}}(Z, E) < \sum_{Z \in \mathcal{W}} \mathcal{H}_{\mathbf{b}}(Z, E') \\ \Updownarrow \\ \sum_{Z \in \mathcal{W}} \sum_{i \in Z \Delta E} b_i < \sum_{Z \in \mathcal{W}} \sum_{i \in Z \Delta E'} b_i \\ \Updownarrow \\ \sum_{i \in N} b_i |Dis_i(\mathcal{W}, E)| < \sum_{i \in N} b_i |Dis_i(\mathcal{W}, E')| \end{gathered}$$

Breaking each sum in four parts, this is equivalent to

$$\begin{aligned} &\sum_{i \in E \cap E'} b_i |Dis_i(\mathcal{W}, E)| + \sum_{i \in E^C \cap E'^C} b_i |Dis_i(\mathcal{W}, E)| \\ &\qquad + \sum_{i \in E \setminus E'} b_i |Dis_i(\mathcal{W}, E)| + \sum_{i \in E' \setminus E} b_i |Dis_i(\mathcal{W}, E)| \\ &\qquad\qquad < \sum_{i \in E \cap E'} b_i |Dis_i(\mathcal{W}, E')| + \sum_{i \in E^C \cap E'^C} b_i |Dis_i(\mathcal{W}, E')| \\ &\qquad\qquad\qquad + \sum_{i \in E \setminus E'} b_i |Dis_i(\mathcal{W}, E')| + \sum_{i \in E' \setminus E} b_i |Dis_i(\mathcal{W}, E')| \end{aligned}$$

Eliminating equal terms from both sides then yields

$$\begin{aligned} &\sum_{i \in E \setminus E'} b_i |Dis_i(\mathcal{W}, E)| + \sum_{i \in E' \setminus E} b_i |Dis_i(\mathcal{W}, E)| \\ &\qquad < \sum_{i \in E \setminus E'} b_i |Dis_i(\mathcal{W}, E')| + \sum_{i \in E' \setminus E} b_i |Dis_i(\mathcal{W}, E')| \end{aligned}$$

$$\Updownarrow$$

$$\sum_{i\in E'\setminus E} b_i\left(|Dis_i(\mathcal{W},E)|-|Dis_i(\mathcal{W},E')|\right) < \sum_{i\in E\setminus E'} b_i\left(|Dis_i(\mathcal{W},E')|-|Dis_i(\mathcal{W},E)|\right)$$

$$\Updownarrow$$

$$\sum_{i\in E'\setminus E} b_i\left(|Agr_i(\mathcal{W},E')|-|Dis_i(\mathcal{W},E')|\right) < \sum_{i\in E\setminus E'} b_i\left(|Agr_i(\mathcal{W},E)|-|Dis_i(\mathcal{W},E)|\right)$$

For the next equivalent expression, we add equal quantities to both sides, getting

$$\sum_{i\in E'\setminus E} b_i\left(|Agr_i(\mathcal{W},E')|-|Dis_i(\mathcal{W},E')|\right) + \sum_{i\in E\cap E'} b_i\left(|Agr_i(\mathcal{W},E')|-|Dis_i(\mathcal{W},E')|\right)$$
$$< \sum_{i\in E\setminus E'} b_i\left(|Agr_i(\mathcal{W},E)|-|Dis_i(\mathcal{W},E)|\right) + \sum_{i\in E\cap E'} b_i\left(|Agr_i(\mathcal{W},E)|-|Dis_i(\mathcal{W},E)|\right)$$

These sums collapse to

$$\sum_{i\in E'} b_i\left(|Agr_i(\mathcal{W},E')|-|Dis_i(\mathcal{W},E')|\right) < \sum_{i\in E} b_i\left(|Agr_i(\mathcal{W},E)|-|Dis_i(\mathcal{W},E)|\right)$$

$$\Updownarrow$$

$$\sum_{i\in E'} b_i\left(|\mathcal{W}_i|-|\mathcal{W}_{\neg i}|\right) < \sum_{i\in E} b_i\left(|\mathcal{W}_i|-|\mathcal{W}_{\neg i}|\right)$$

$$\Updownarrow$$

$$\sum_{i\in E'} b_i\eta_i < \sum_{i\in E} b_i\eta_i$$

$$\Updownarrow$$

$$\mathbf{w}(E') < \mathbf{w}(E).$$

□

The "extra" term $\mathcal{H}_{\mathbf{b}}(E,E')$ in the local Hamming minimality characterization (i) is disguised (if only lightly) in the characterization via tight packing; the sum $\mathcal{H}_{\mathbf{b}}(\mathcal{W},E)$ includes a term $\mathcal{H}_{\mathbf{b}}(E,E)$ that is zero, whereas the corresponding term $\mathcal{H}_{\mathbf{b}}(E,E')$ in $\mathcal{H}_{\mathbf{b}}(\mathcal{W},E')$ is not zero. But where did the extra term go in the final equivalent condition (iii''): $\mathbf{w}(E') < \mathbf{w}(E)$? The last few lines of the preceding proof suggest it has been completely absorbed into the choice of weight vector $\mathbf{w}$ – if $\mathcal{W}$ and $\mathcal{W}'$ were *both* weighted, then some alternative choice $\mathbf{w}'$ of weight vector would make $\mathbf{w}'(E) < \mathbf{w}'(E')$. It seems that while the choice of $\mathbf{w}$ can thus absorb this extra term, $\mathbf{b}$ cannot fully do so, presumably because some information lies within the η_i terms that are built in to $\mathbf{w}$ but not into $\mathbf{b}$.

5. Euclidean distance and a theorem of Huyghens

A theorem due to Christiaan Huyghens (1629 - 1695) is crucial in transforming the characterizations of the previous section into terms involving ellipsoids in Euclidean space. Huyghens (also spelled *Huygens* or *Hugens*) was a Dutch polymath and contemporary of Newton, known for inventing the pendulum clock, and proposing the wave theory of light, among other things. Our interest is in the following:

Theorem 5.1. Huyghens′ Theorem *Let* $S = \{\mathbf{s}_i\}_{i=1}^r$ *be a finite sequence of points in* $\mathbf{R}^d$, $\mathbf{q} = \left(\frac{\sum_{j=1}^r \mathbf{s}_j}{r}\right)$ *be the mean location of the points of* S*, and* $\mathbf{y}$ *be any point in* $\mathbf{R}^d$*. Then* $\sum_{j=1}^r \|\mathbf{y}-\mathbf{s}_j\|^2 = r\|\mathbf{y}-\mathbf{q}\|^2 + \sum_{j=1}^r \|\mathbf{q}-\mathbf{s}_j\|^2$. (†)

Christiaan Huyghens, the astronomer,
by Caspar Netscher (ca 1637 - 1684)

Fig. 2 illustrates the theorem for the case $r = 3$.

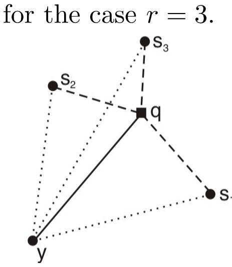

Figure 2. Huyghens' Theorem for three points in the plane.

Notice that if each triangle $\triangle\mathbf{yqs}_j$ in this figure were right (with right angle at $\mathbf{q}$) then equation (†) would follow immediately by applying the Pythagorean theorem separately to the triangles and adding the three resulting equations:

$$\|\mathbf{y}-\mathbf{s}_j\|^2 = \|\mathbf{y}-\mathbf{q}\|^2 + \|\mathbf{q}-\mathbf{s}_j\|^2.$$

Of course, the triangles are not actually right, but the underlying idea is correct, as we'll see shortly.

First, however, we observe that the theorem has an intimate relationship with the following proposition, whose well-known proof we skip:

Proposition 5.2. *Let* $S = \{\mathbf{s}_i\}_{i=1}^r$ *be a finite sequence of points in* $\mathbf{R}^d$*. Then the mean location* $\mathbf{q}$ *of the points of* S *is the point* $\mathbf{y}$ *in* $\mathbf{R}^d$ *that minimizes the sum* $\sum_{j=1}^r \|\mathbf{y}-\mathbf{s}_j\|^2$ *of squared Euclidean distances to the points of* S*.*

We'll use 5.2 in the proof of Huyghens' theorem (5.1), but it also can be seen as a corollary of 5.1, as follows: the $\mathbf{y}$ that minimizes the left side of eqn.(†) is the one that minimizes the right side, by minimizing the distance from $\mathbf{y}$ to $\mathbf{q}$. More important is that this same argument actually establishes a corollary of Huyghens' Theorem stronger than 5.2:

Corollary 5.3. *Let $S = \{\mathbf{s}_i\}_{i=1}^r$ be a finite sequence of points in $\mathbf{R}^d$, $\mathbf{q}$ be the mean location of the points of S, and Y be any locally compact subset of $\mathbf{R}^d$. Then the member(s) $\mathbf{y}$ of Y closest to $\mathbf{q}$ minimize(s) $\sum_{j=1}^r \|\mathbf{y} - \mathbf{s}_j\|^2$ among members of Y.*

Most, if not all, of the applications of Huyghens' Theorem to the mathematical theory of voting (in [2], [30], [31], and [4]) are actually applications of Corollary 5.3 for finite Y. We turn now to the proof of Huyghens' Theorem.

Proof. Given any sequence $U = \{\mathbf{u}_i\}_{i=1}^r$ of r points in $\mathbf{R}^d$ (with $\mathbf{u}_j = (u_{j1}, \ldots, u_{jd})$ for each j) we will string their coordinates together, forming a single point

$$\mathbf{u}_U = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r) = (u_{11}, \ldots, u_{1d}, u_{21}, \ldots, u_{2d}, \ldots, u_{r1}, \ldots, u_{rd})$$

in $\mathbf{R}^{rd}$. For any $\mathbf{x} \in \mathbf{R}^d$, $\mathbf{x} \otimes r$ will denote $(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}) \in \mathbf{R}^{rd}$. Thus $\mathbf{x} \mapsto \mathbf{x} \otimes r$ maps $\mathbf{R}^d$ to $\mathbf{R}^{rd}$; $\mathcal{D}$ will stand for the range of this function, a linear subspace of $\mathbf{R}^{rd}$.

Consider the triangle in $\mathbf{R}^{rd}$ formed by the points $\mathbf{s}_S$, $\mathbf{q} \otimes r$, and $\mathbf{y} \otimes r$, illustrated in Fig. 3 for the case $r = 3$. The angle at $\mathbf{q} \otimes r$ is right, because we know (from Proposition 5.2) that, among points of $\mathcal{D}$, $\mathbf{q} \otimes r$ is closest to $\mathbf{s}_S$. The Pythagorean Theorem then asserts that

$$\|\mathbf{y} \otimes r - \mathbf{s}_S\|^2 = \|\mathbf{y} \otimes r - \mathbf{q} \otimes r\|^2 + \|\mathbf{q} \otimes r - \mathbf{s}_S\|^2$$

($C^2 = A^2 + B^2$ in Fig. 3). But this is the same as the desired conclusion (†) of Huyghens' Theorem. □

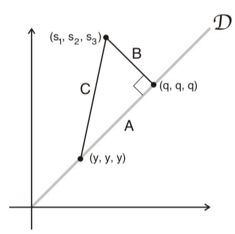

Figure 3. A right triangle in $\mathbf{R}^{rd}$.

6. Separation by aligned hyper-ellipsoids

We are interested in hyper-ellipsoids with equations having the form:

$$b_1(x_1 - c_1)^2 + b_2(x_2 - c_2)^2 + \cdots + b_n(x_n - c_n)^2 = r^2.$$

Here $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ is the ellipsoid's center, and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ determines the proportions. Such an ellipsoid is *aligned* − generated from a sphere by a series of uniform scalings along the coordinate axes. The ellipsoid is *mean-centered* with respect to $G = (N, \mathcal{W})$ if its center coincides with the mean position $\mathbf{q}$ of all characteristic vectors for winning coalitions of G. In this setting, the ellipsoid *separates the winning coalitions from the losing coalitions* if all characteristic vectors of G's winning coalitions lie strictly within the ellipsoid, while all those of losing coalitions lie strictly outside (though we show later that these roles can be reversed). When an aligned and mean-centered separating ellipsoid exists, G is *ellipsoidally separable*, and when that separating ellipsoid can be taken to be a sphere G is *spherically separable*. Our final characterization of weighted voting is then:

Theorem 6.1. *A simple game $G = (N, \mathcal{W})$ is weighted if and only if some aligned hyper-ellipsoid centered at the mean position $\mathbf{q}$ of all winning coalitions separates G's winning coalitions from its losing coalitions. In particular the possible vectors*

b *of coefficients for such separating ellipsoids are exactly those of the form* $\mathbf{b} = \frac{\mathbf{w}}{\eta}$ *where* η *is the vector of raw Penrose-Banzhaf voting powers, and* **w** *is any vector of voting weights that realizes* G *as a weighted game.*

The proportions of the separating ellipsoid thus reflect the ratios of voting weights to voting powers. In particular, an immediate corollary of Theorem 6.1 is:

Corollary 6.2. *A simple game* $G = (N, \mathcal{W})$ *is spherically separable if and only if the vector of Penrose-Banzhaf voting powers serves as a vector of voting weights in a weighted representation of the game.*

For example, the game G_{Math} (Example 1) is spherically separable.

Why place the center of the separating ellipsoid at the mean of the winning coalitions? With no restrictions on the center we could place it far away, and use an ellipsoid so large that the portion of its surface that lay within the n-cube was arbitrarily close to a section of a plane. With this approach, ellipsoidal separability would immediately follow from separability via a hyperplane, and be of little interest.

Our proof of Theorem 6.1 fits together two earlier results: Proposition 3.2, stating that Hamming distance coincides with squared Euclidean distance, and our stronger Corollary 5.3 of Huyghens' Theorem, but requires that we first modify 3.2 so that it applies to *weighted* Hamming distance. Rather than change the Euclidean metric itself, we'll stretch the n-cube into a box with unequal sides, and apply the standard Euclidean metric as measured between vertices of this box.

Recall that for each coalition $X \subseteq N$, the corresponding characteristic vector $\mathbf{x}$ of 0s and 1s represents a vertex of the n-cube. Given a vector $\mathbf{b}$ of Hamming weights, let $\mathbf{t}$ be the vector $\mathbf{t} = (t_1, \ldots, t_n)$ of square roots: $t_j = \sqrt{b_j}$ for each j. We'll write $\mathbf{t} = \sqrt{\mathbf{b}}$, for short. For $X \subseteq N$, let $\mathbf{x^t} = (x_1^t, \ldots, x_n^t)$ be defined by $x_i^t = t_i$ if $i \in X$ and $x_i = 0$ if not. These are the coordinate vectors for vertices of the t-*box*, a rectangular hyper-solid obtained by scaling the n-cube by a factor of t_i along the i^{th} axis, for each i. It follows that for $X, Y \subseteq N$, the squared Euclidean distance $\|\mathbf{x^t} - \mathbf{y^t}\|^2 = \sum_{i=1}^n (x_i^t - y_i^t)^2 = \sum_{i \in X \Delta Y} (t_i)^2 = \sum_{i \in X \Delta Y} b_i = \mathcal{H}(X, Y)$, which proves:

Proposition 6.3. *Let* $\mathbf{t} = \sqrt{\mathbf{b}}$ *be vectors in* $\mathbf{R}^n$*. Then* **b***-weighted Hamming distance between two coalitions is the same as squared Euclidean distance between corresponding vertices of the* **t***-box.*

The following extension of Lemma 4.3 is key to our ellipsoidal characterization:

Lemma 6.4. *For* $G = (N, \mathcal{W})$ *a simple game, let* $E \in \mathcal{W}$*,* $E' \notin \mathcal{W}$*,* **b** *be any vector of Hamming weights,* $\mathbf{w} = \eta \mathbf{b}$ *be the corresponding vector of voting weights,*[15] $\mathbf{t} = \sqrt{\mathbf{b}}$*, and* $\mathbf{q^t} = \frac{\sum_{F \in \mathcal{W}} \mathbf{f^t}}{|\mathcal{W}|}$ *be the mean position of* G*'s winning coalitions as located in the* t*-box. Then the following are equivalent:*

(1) $\mathbf{w}(E') < \mathbf{w}(E)$
(2) $\|\mathbf{e^t} - \mathbf{q^t}\| < \|\mathbf{e'^t} - \mathbf{q^t}\|$.

Proof. $\mathbf{w}(E') < \mathbf{w}(E)$
$\Leftrightarrow^{(a)} \mathcal{H}_\mathbf{b}(\mathcal{W}, E) < \mathcal{H}_\mathbf{b}(\mathcal{W}, E')$
$\Leftrightarrow \sum_{F \in \mathcal{W}} \mathcal{H}_\mathbf{b}(F, E) < \sum_{F \in \mathcal{W}} \mathcal{H}_\mathbf{b}(F, E')$

[15] This is an opportune time to reread footnote 9.

$\Leftrightarrow \sum_{F\in\mathcal{W}} \|\mathbf{e^t} - \mathbf{f^t}\|^2 < \sum_{F\in\mathcal{W}} \|\mathbf{e'^t} - \mathbf{f^t}\|^2$
$\Leftrightarrow^{(b)} \|\mathbf{e^t} - \mathbf{q^t}\| < \|\mathbf{e'^t} - \mathbf{q^t}\|$. Here $\Leftrightarrow^{(a)}$ follows from Lemma 4.3 and $\Leftrightarrow^{(b)}$ represents an application of Corollary 5.3 with $Y = \{\mathbf{e^t}, \mathbf{e'^t}\}$. □

Theorem 6.1 now follows readily from this lemma:

Proof. The vector $\mathbf{w}$ represents G as a weighted game
$\Leftrightarrow$ $\mathbf{w}(E') < \mathbf{w}(E)$ holds for every $E \in \mathcal{W}$ and $E' \notin \mathcal{W}$
$\Leftrightarrow$ $\|\mathbf{e^t} - \mathbf{q^t}\| < \|\mathbf{e'^t} - \mathbf{q^t}\|$ holds for every $E \in \mathcal{W}$ and $E' \notin \mathcal{W}$
$\Leftrightarrow$ some sphere centered at $\mathbf{q^t}$ contains $\mathbf{e^t}$ for every $E \in \mathcal{W}$ and omits $\mathbf{e'^t}$ for every $E' \notin \mathcal{W}$.
Next, for each i apply a uniform scaling by a factor of $\frac{1}{t_i}$ along the i^{th} coordinate axis, reversing the process that converted the n-cube into the t-box. This transform shifts each $\mathbf{e^t}$ to $\mathbf{e}$, and shifts $\mathbf{q^t}$ to $\mathbf{q}$.[16] It converts the sphere into a mean-centered and aligned separating hyper-ellipsoid for G, with equation

$$b_1(x_1 - q_1)^2 + b_2(x_2 - q_2)^2 + \cdots + b_n(x_n - q_n)^2 = r^2$$

for some r. This last step (of stretching the sphere into an ellipsoid with corresponding center) is clearly reversible, establishing the equivalence between weightedness and ellipsoidal separability. □

Finally, we explain how the separating ellipsoid of Theorem 6.1 can be taken to have center at the mean of G's losing coalitions, and to contain all losing coalitions while omitting all winning coalitions.[17] The idea is to work with the *dual* $G^d = (N, \mathcal{W}^d)$ of the original game $G = (N, \mathcal{W})$, defined as follows: $X \in \mathcal{W}^d \Leftrightarrow N \setminus X \in \mathcal{L}$. That is, a coalition is winning in the dual when it blocks approval in the original game.

For example, the dual of the weighted game $[6; 3, 3, 1, 1, 1]$ is the weighted game $[4; 3, 3, 1, 1, 1]$, because to block approval by the required minimum weight of 6 out of 9, it is enough to have the weight in opposition be at least 4 out of 9. More generally, G is weighted if and only if its dual is weighted, and the same weight vectors $\mathbf{w}$ represent G and G^d; for details, see [29]. The vector η of raw Penrose-Banzhaf voting power is also the same for G and G^d – a player i is a critical member of coalition X, according to G, if and only if i is critical as a member of $(N\setminus X)\cup\{i\}$, according to G^d.

Now, we first apply Theorem 6.1 to obtain an ellipsoid $\mathcal{E}$ for G^d and then apply the affine transform $\mathbf{x} \mapsto \mathbf{1} - \mathbf{x}$, where $\mathbf{1} = (1, 1, \ldots, 1)$. Notice that this transform carries the characteristic vector for X to that for X's complement, and thus permutes the vertices of the n-cube; in particular, it carries winning coalitions of G^d to losing coalitions of G. By these remarks, along with footnote 11, it also carries the mean location of the winning coalition of G^d to the mean of the losing coalitions of G, and carries $\mathcal{E}$ to an ellipsoid containing all of G's losing coalitions, and omitting all its winning coalitions.

[16] Any affine transform $L(\mathbf{x}) = A\mathbf{x} + B$ commutes with the mean: if S is a set of points in Euclidean space, $\mathbf{q}(S)$ denotes the mean location of the points in S, and $L[S]$ denotes the set image of S, then $L(\mathbf{q}(S)) = \mathbf{q}(L[S])$.

[17] We will no longer always be careful to distinguish a coalition from its characteristic vector.

7. Further thoughts

These results suggest several avenues for future research. In presentations of this material, the most common audience response has been to ask, "Do these characterizations have analogous versions for the Shapley Shubik voting power index?" We do not know, but either answer might tell us something about the differences between the two power indices.[18]

Because our attention has been drawn here to the class of games for which the Penrose-Banzhaf voting powers can serve as voting weights, it seems natural to ask whether there are any other nice characterizations of this class. In particular, does the class have any normative interpretation – is there any sense in which games from this class are better choices for real voting rules, because they satisfy some desirable fairness property?

Fix some point $\mathbf{y}$ in space. Next, for each winning coalition X of some weighted game G, loop one end of an *ideal rubber band*[20] about $\mathbf{x^t}$ and the other end about $\mathbf{y}$. Finally, release $\mathbf{y}$ so that it seeks a position of equilibrium under the rubber band forces. This resulting position minimizes the potential energy $P.E. = \frac{1}{2}\sum_{X\in\mathcal{W}} \|\mathbf{x^t} - \mathbf{y}\|^2$ stored in the rubber bands. Proposition 5.2 implies that the equilibrium is at $\mathbf{q^t}$, the mean position of winning coalitions as located in the t-box. These ideas have been applied in [4], in a very different voting context; there $P.E.$ may be interpreted as the total discontent of the voters. For a broad collection of multi-candidate voting rules, the winning candidate can then be seen to be the minimizer of total discontent. It seems plausible that $P.E.$ have some related interpretation for the *yes-no* type of voting we consider here.

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[18] To modify "$\mathbf{w} = \eta\mathbf{b}$" in the simplest way, one might replace η with the vector of Shapley-Shubik indices. But an obstacle appears. The marginal contribution of a coalition to the Shapley-Shubik index of voter i depends on the coalition's cardinality. That seems to force a corresponding dependence on at least one of the other two terms in the relationship. But then either $\mathbf{w}$ or η would no longer depend on i alone.

[20] A rubber band is *ideal* if its tension is perfectly proportional to the amount by which it is stretched from its resting length, and that resting length is zero.

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