

Convex Interval Games

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Abstract

In this paper, convex interval games are introduced and characterizations are given. Some economic situations leading to convex interval games are discussed. The Weber set and the Shapley value are defined for a suitable class of interval games and their relations with the interval core for convex interval games are established. The notion of population monotonic allocation scheme (pmas) in the interval setting is introduced and it is proved that each element of the Weber set of a convex interval game is extendable to such a pmas. A square operator is introduced which allows us to obtain interval solutions starting from the corresponding classical cooperative game theory solutions. It turns out that on the class of convex interval games the square Weber set coincides with the interval core.

Keywords: cooperative games, interval data, convex games, the core, the Weber set, the Shapley value

1 Introduction

In classical cooperative game theory payoffs to coalitions of players are known with certainty. A classical cooperative game is a pair $\langle N, v \rangle$ where $N =$

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$\{1, 2, \dots, n\}$ is the set of players and $v : 2^N \rightarrow \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$. Often, we also refer to such a game as a TU (transferable utility) game. We denote by G^N the family of all classical cooperative games with player set N . The class of convex games (Shapley (1971)) is one of the most interesting classes of cooperative games from theoretical point of view as well as regarding its applications in real-life situations. A game $v \in G^N$ is convex (or supermodular) if and only if the supermodularity condition $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for each $S, T \in 2^N$ holds true. Many characterizations of classical convex games are available in the literature (Driessen (1988), Biswas et al. (1999), Branzei, Dimitrov and Tijs (2008), Martinez-Legaz (1997, 2006)). On the class CG^N of classical convex games solution concepts have nice properties; for details we refer the reader to Branzei, Dimitrov and Tijs (2008). Classical convex games have many applications in economic and real-life situations. It is well known that classical public good situations (Moulin (1988)), sequencing situations (Curiel, Pederzoli and Tijs (1989)) and bankruptcy situations (O'Neill (1982), Aumann and Maschler (1985), Curiel, Maschler and Tijs (1987)) lead to convex games.

However, there are many real-life situations in which people or businesses are uncertain about their coalition payoffs. Situations with uncertain payoffs in which the agents cannot await the realizations of their coalition payoffs cannot be modelled according to classical game theory. Several models that are useful to handle uncertain payoffs exist in the game theory literature. We refer here to chance-constrained games (Charnes and Granot (1973)), cooperative games with stochastic payoffs (Suijs et al. (1999)), cooperative games with random payoffs (Timmer, Borm and Tijs (2005)). In all these models probability and stochastic theory plays an important role.

This paper deals with a model of cooperative games where only bounds for payoffs of coalitions are known with certainty. Such games are called cooperative interval games. Formally, a *cooperative interval game* in coalitional form (Alparslan Gök, Miquel and Tijs (2008)) is an ordered pair $\langle N, w \rangle$ where $N = \{1, 2, \dots, n\}$ is the set of players, and $w : 2^N \rightarrow I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$, where $I(\mathbb{R})$ is the set of all nonempty, compact intervals in \mathbb{R} . For each $S \in 2^N$, the worth set (or worth interval) $w(S)$ of the coalition S in the interval game $\langle N, w \rangle$ is of the form $[\underline{w}(S), \bar{w}(S)]$. We denote by IG^N the family of all interval games with player set N . Note that if all the worth intervals are degenerate intervals, i.e. $\underline{w}(S) = \bar{w}(S)$ for each $S \in 2^N$, then the interval game $\langle N, w \rangle$ corre-

sponds in a natural way to the classical cooperative game $\langle N, v \rangle$ where $v(S) = \underline{w}(S)$ for all $S \in 2^N$. Some classical TU -games associated with an interval game $w \in IG^N$ will play a key role, namely the *border* games $\langle N, \underline{w} \rangle$, $\langle N, \bar{w} \rangle$ and the *length* game $\langle N, |w| \rangle$, where $|w|(S) = \bar{w}(S) - \underline{w}(S)$ for each $S \in 2^N$. Note that $\bar{w} = \underline{w} + |w|$. An interval solution concept \mathcal{F} on IG^N is a map assigning to each interval game $w \in IG^N$ a set of n -dimensional vectors whose components belong to $I(\mathbb{R})$. We denote by $I(\mathbb{R})^N$ the set of all such interval payoff vectors. Cooperative interval games are very suitable to describe real-life situations in which people or firms that consider cooperation have to sign a contract when they cannot pin down the attainable coalition payoffs, knowing with certainty only their lower and upper bounds. The contract should specify how the players' payoff shares will be obtained when the uncertainty of the worth of the grand coalition is removed at an ex post stage. In the following we briefly explain how interval solutions for cooperative interval games are useful to support decision making regarding cooperation and related binding contracts. A vector interval allocation obtained by an agreed upon solution concept offers at the ex ante stage an estimation of what individual players may receive, between two bounds, when the uncertainty on the reward of the grand coalition is removed in the ex post stage. We notice that the agreement on a particular interval allocation (I_1, I_2, \dots, I_n) based on an interval solution concept merely says that the payoff x_i that player i will receive in the interim or ex post stage is in the interval I_i . This is a very weak contract to settle cooperation. Therefore, writing down in the contract the protocol to be used when the uncertainty on $w(N)$ is removed at the ex post stage, is compulsory. Such protocols are described in Branzei, Tijs and Alparslan Gök (2008b).

In this paper, we introduce the class of convex interval games and extend classical results regarding characterizations of convex games and properties of solution concepts to the interval setting.

The paper is organized as follows. In Section 2 we recall basic notions and facts from the theory of cooperative interval games. In Section 3 we introduce supermodular and convex interval games and give basic characterizations of convex interval games. Economic situations leading to convex interval games are briefly discussed. In Section 4 we introduce for size monotonic interval games the notions of marginal operators, the Shapley value and the Weber set and study their properties for convex interval games. Moreover, we introduce the notion of population monotonic allocation scheme in the interval setting and prove that each element of the Weber set of a convex interval game is

extendable to such a pmas. In Section 5 we introduce the square operator and describe some interval solutions for interval games that have close relations with existing solutions from the classical cooperative game theory. It turns out that on the class of convex interval games the interval core and the square Weber set coincide. Finally, in Section 6 we conclude with some remarks on further research.

2 Preliminaries on interval calculus and interval games

In this section some preliminaries from interval calculus and some useful results from the theory of cooperative interval games are given (Alparslan Gök, Branzei and Tijss (2008a)).

Let $I, J \in I(\mathbb{R})$ with $I = [\underline{I}, \bar{I}]$, $J = [\underline{J}, \bar{J}]$, $|I| = \bar{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then,

- (i) $I + J = [\underline{I}, \bar{I}] + [\underline{J}, \bar{J}] = [\underline{I} + \underline{J}, \bar{I} + \bar{J}]$;
- (ii) $\alpha I = \alpha [\underline{I}, \bar{I}] = [\alpha \underline{I}, \alpha \bar{I}]$.

By (i) and (ii) we see that $I(\mathbb{R})$ has a cone structure.

In this paper we also need a partial subtraction operator. We define $I - J$, only if $|I| \geq |J|$, by $I - J = [\underline{I}, \bar{I}] - [\underline{J}, \bar{J}] = [\underline{I} - \underline{J}, \bar{I} - \bar{J}]$. Note that $\underline{I} - \underline{J} \leq \bar{I} - \bar{J}$. We recall that I is weakly better than J , which we denote by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\bar{I} \geq \bar{J}$. We also use the reverse notation $I \preccurlyeq J$, if and only if $\underline{I} \leq \underline{J}$ and $\bar{I} \leq \bar{J}$. We say that I is better than J , which we denote by $I \succ J$, if and only if $I \succcurlyeq J$ and $I \neq J$.

For $w_1, w_2 \in IG^N$ we say that $w_1 \preccurlyeq w_2$ if $w_1(S) \preccurlyeq w_2(S)$, for each $S \in 2^N$. For $w_1, w_2 \in IG^N$ and $\lambda \in \mathbb{R}_+$ we define $\langle N, w_1 + w_2 \rangle$ and $\langle N, \lambda w \rangle$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\lambda w)(S) = \lambda \cdot w(S)$ for each $S \in 2^N$. So, we conclude that IG^N endowed with \preccurlyeq is a partially ordered set and has a cone structure with respect to addition and multiplication with non-negative scalars described above. For $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $\langle N, w_1 - w_2 \rangle$ is defined by $(w_1 - w_2)(S) = w_1(S) - w_2(S)$.

Now, we recall that the interval imputation set $\mathcal{I}(w)$ of the interval game w , is defined by

$$\mathcal{I}(w) = \left\{ (I_1, \dots, I_n) \in I(\mathbb{R})^N \mid \sum_{i \in N} I_i = w(N), w(i) \preccurlyeq I_i, \text{ for all } i \in N \right\},$$

and the interval core $\mathcal{C}(w)$ of the interval game w , is defined by

$$\mathcal{C}(w) = \left\{ (I_1, \dots, I_n) \in \mathcal{I}(w) \mid \sum_{i \in S} I_i \succcurlyeq w(S), \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

A game $w \in IG^N$ is called \mathcal{I} -balanced if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}_+$ we have $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)w(S) \preccurlyeq w(N)$. We recall that a map $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}_+$ is called a balanced map (Tijs (2003)) if $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)e^S = e^N$. Here, $e^N = (1, \dots, 1)$, and for each $S \in 2^N$, $(e^S)_i = 1$ if $i \in S$ and $(e^S)_i = 0$ otherwise. It is easy to prove that if $\langle N, w \rangle$ is \mathcal{I} -balanced then the border games $\langle N, \underline{w} \rangle$ and $\langle N, \bar{w} \rangle$ are balanced. A game $w \in IG^N$ is \mathcal{I} -balanced if and only if $\mathcal{C}(w) \neq \emptyset$ (Theorem 3.1 in Alparslan Gök, Branzei and Tijs (2008a)). We denote by \mathcal{IBIG}^N the class of \mathcal{I} -balanced interval games with player set N .

Let $w \in IG^N$, $I = (I_1, \dots, I_n), J = (J_1, \dots, J_n) \in \mathcal{I}(w)$ and $S \in 2^N \setminus \{\emptyset\}$. We say that I dominates J via coalition S , denoted by $I \text{ dom}_S J$, if

- (i) $I_i \succ J_i$ for all $i \in S$,
- (ii) $\sum_{i \in S} I_i \preccurlyeq w(S)$.

For $S \in 2^N \setminus \{\emptyset\}$ we denote by $D(S)$ the set of those elements of $\mathcal{I}(w)$ which are dominated via S . I is called undominated if there does not exist J and a coalition S such that $J \text{ dom}_S I$. The *interval dominance core* $\mathcal{DC}(w)$ of $w \in IG^N$ consists of all undominated elements in $\mathcal{I}(w)$, i.e. it is the complement in $\mathcal{I}(w)$ of $\cup \{D(S) \mid S \in 2^N \setminus \{\emptyset\}\}$. It holds $\mathcal{C}(w) \subset \mathcal{DC}(w) \subset A$ for all $w \in IG^N$ and A a stable set of w .

3 Supermodular and convex interval games

We say that a game $\langle N, w \rangle$ is *supermodular* if

$$w(S) + w(T) \preccurlyeq w(S \cup T) + w(S \cap T) \text{ for all } S, T \in 2^N. \quad (1)$$

From formula (1) it follows that a game $\langle N, w \rangle$ is supermodular if and only if its border games $\langle N, \underline{w} \rangle$ and $\langle N, \bar{w} \rangle$ are supermodular (convex). We introduce the notion of convex interval game and denote by CIG^N the class of convex interval games with player set N . We call a game $w \in IG^N$

convex if $\langle N, w \rangle$ is supermodular and its length game $\langle N, |w| \rangle$ is also supermodular. We straightforwardly obtain characterizations of games $w \in CIG^N$ in terms of \underline{w} , \bar{w} and $|w| \in G^N$.

PROPOSITION 3.1. *Let $w \in IG^N$ and its related games $|w|, \underline{w}, \bar{w} \in G^N$. Then the following assertions hold:*

- (i) *A game $\langle N, w \rangle$ is convex if and only if its length game $\langle N, |w| \rangle$ and its border games $\langle N, \underline{w} \rangle, \langle N, \bar{w} \rangle$ are convex;*
- (ii) *A game $\langle N, w \rangle$ is convex if and only if its border game $\langle N, \underline{w} \rangle$ and the game $\langle N, \bar{w} - \underline{w} \rangle$ are convex.*

We notice that the nonempty set CIG^N is a subcone of IG^N and traditional convex games can be embedded in a natural way in the class of convex interval games because if $v \in G^N$ is convex then the corresponding game $w \in IG^N$ which is defined by $w(S) = [v(S), v(S)]$ for each $S \in 2^N$ is also convex. The next example shows that a supermodular interval game is not necessarily convex.

EXAMPLE 3.1. *Let $\langle N, w \rangle$ be the two-person interval game with $w(\emptyset) = [0, 0]$, $w(1) = w(2) = [0, 1]$ and $w(1, 2) = [3, 4]$. Here, $\langle N, w \rangle$ is supermodular, but $|w|(1) + |w|(2) = 2 > 1 = |w|(1, 2) + |w|(\emptyset)$. Hence, $\langle N, w \rangle$ is not convex.*

The next example shows that an interval game whose length game is supermodular is not necessarily convex.

EXAMPLE 3.2. *Let $\langle N, w \rangle$ be the three-person interval game with $w(i) = [1, 1]$ for each $i \in N$, $w(N) = w(1, 3) = w(1, 2) = w(2, 3) = [2, 2]$ and $w(\emptyset) = [0, 0]$. Here, $\langle N, w \rangle$ is not convex, but $\langle N, |w| \rangle$ is supermodular, since $|w|(S) = 0$, for each $S \in 2^N$.*

Interesting examples of convex interval games are unanimity interval games. First, we recall the definition of such games. Let $J \in I(\mathbb{R})$ with $J \succcurlyeq [0, 0]$ and let $T \in 2^N \setminus \{\emptyset\}$. The unanimity interval game based on J and T is defined by

$$u_{T,J}(S) = \begin{cases} J, & T \subset S \\ [0, 0], & \text{otherwise,} \end{cases}$$

for each $S \in 2^N$.

Clearly, $\langle N, |u_{T,J}| \rangle$ is supermodular. The supermodularity of $\langle N, u_{T,J} \rangle$ can be checked by considering the following case study:

	$u_{T,J}(A \cup B)$	$u_{T,J}(A \cap B)$	$u_{T,J}(A)$	$u_{T,J}(B)$
$T \subset A, T \subset B$	J	J	J	J
$T \subset A, T \not\subset B$	J	$[0, 0]$	J	$[0, 0]$
$T \not\subset A, T \subset B$	J	$[0, 0]$	$[0, 0]$	J
$T \not\subset A, T \not\subset B$	J or $[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$

For convex TU-games various characterizations are known. In the next theorem we give some characterizations of convex interval games inspired by Shapley (1971).

THEOREM 3.1. *Let $w \in IG^N$ be such that $|w| \in G^N$ is supermodular. Then, the following three assertions are equivalent:*

- (i) $w \in IG^N$ is convex;
- (ii) For all $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$ we have

$$w(S_1 \cup U) - w(S_1) \preceq w(S_2 \cup U) - w(S_2); \quad (2)$$

- (iii) For all $S_1, S_2 \in 2^N$ and $i \in N$ such that $S_1 \subset S_2 \subset N \setminus \{i\}$ we have

$$w(S_1 \cup \{i\}) - w(S_1) \preceq w(S_2 \cup \{i\}) - w(S_2).$$

Proof. We show (i) \Rightarrow (ii), (ii) \Rightarrow (iii), (iii) \Rightarrow (i).

Suppose that (i) holds. To prove (ii) take $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$. From (1) with $S_1 \cup U$ in the role of S and S_2 in the role of T we obtain (2) by noting that $S \cup T = S_2 \cup U$, $S \cap T = S_1$. Hence, (i) implies (ii).

That (ii) implies (iii) is straightforward (take $U = \{i\}$).

Now, suppose that (iii) holds. To prove (i) take $S, T \in 2^N$. Clearly, (1) holds if $S \subset T$. Suppose that $T \setminus S$ consists of the elements i_1, \dots, i_k and let

$D = S \cap T$. Then, from (iii) follows that

$$\begin{aligned}
w(S) - w(S \cap T) &= w(D \cup \{i_1\}) - w(D) \\
&+ \sum_{s=2}^k (w(D \cup \{i_1, \dots, i_s\}) - w(D \cup \{i_1, \dots, i_{s-1}\})) \\
&\preceq w(T \cup \{i_1\}) - w(T) \\
&+ \sum_{s=2}^k (w(T \cup \{i_1, \dots, i_s\}) - w(T \cup \{i_1, \dots, i_{s-1}\})) \\
&= w(S \cup T) - w(T), \text{ for each } S \in 2^N.
\end{aligned}$$

□

Next we give as a motivating example a situation with an economic flavour leading to a convex interval game.

EXAMPLE 3.3. Let $N = \{1, 2, \dots, n\}$ and let $f : [0, n] \rightarrow I(\mathbb{R})$ be such that $f(x) = [f_1(x), f_2(x)]$ for each $x \in [0, n]$ and $f(0) = [0, 0]$. Suppose that $f_1 : [0, n] \rightarrow \mathbb{R}$, $f_2 : [0, n] \rightarrow \mathbb{R}$ and $(f_2 - f_1) : [0, n] \rightarrow \mathbb{R}$ are convex monotonic increasing functions. Then, we can construct a corresponding interval game $w : 2^N \rightarrow I(\mathbb{R})$ such that $w(S) = f(|S|) = [f_1(|S|), f_2(|S|)]$ for each $S \in 2^N$. It is easy to show that w is a convex interval game with the symmetry property $w(S) = w(T)$ for each $S, T \in 2^N$ with $|S| = |T|$.

We can see $\langle N, w \rangle$ as a production game if we interpret $f(s)$ for $s \in N$ as the interval reward which s players in N can produce by working together.

Before closing this section we indicate some other economic situations related to supermodular and convex interval games. In case the parameters determining sequencing situations are not numbers but intervals, under certain conditions also convex interval games appear (Alparslan Gök et al. (2008)). Bankruptcy situations when the estate of the bankrupt firm and the claims are intervals, under restricting conditions, give rise in a natural way to supermodular interval games which are not necessarily convex (Branzei and Alparslan Gök (2008)). Airport situations (Littlechild and Owen (1977)) with interval data lead to concave interval games (Alparslan Gök, Branzei and Tijs (2008b)). An interval game $\langle N, w \rangle$ is called *concave* if $\langle N, w \rangle$ and $\langle N, |w| \rangle$ are submodular, i.e. $w(S) + w(T) \preceq w(S \cup T) + w(S \cap T)$ and $|w|(S) + |w|(T) \geq |w|(S \cup T) + |w|(S \cap T)$, for all $S, T \in 2^N$.

4 The Shapley value, the Weber set and population monotonic allocation schemes

We call a game $\langle N, w \rangle$ *size monotonic* if $\langle N, |w| \rangle$ is monotonic, i.e. $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SMIG^N$ the class of size monotonic interval games with player set N . We notice that size monotonic games may have an empty interval core. In this section we introduce marginal operators on the class of size monotonic interval games, define the Shapley value and the Weber set on this class of games, and study their properties on the class of convex interval games.

Denote by $\Pi(N)$ the set of permutations $\sigma : N \rightarrow N$. Let $w \in SMIG^N$. We introduce the notions of *interval marginal operator* corresponding to σ , denoted by m^σ , and of *interval marginal vector* of w with respect to σ , denoted by $m^\sigma(w)$. The marginal vector $m^\sigma(w)$ corresponds to a situation, where the players enter a room one by one in the order $\sigma(1), \sigma(2), \dots, \sigma(n)$ and each player is given the marginal contribution he/she creates by entering. If we denote the set of predecessors of i in σ by $P_\sigma(i) = \{r \in N \mid \sigma^{-1}(r) < \sigma^{-1}(i)\}$, where $\sigma^{-1}(i)$ denotes the entrance number of player i , then $m_{\sigma(k)}^\sigma(w) = w(P_\sigma(\sigma(k)) \cup \{\sigma(k)\}) - w(P_\sigma(\sigma(k)))$, or $m_i^\sigma(w) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i))$. We notice that $m^\sigma(w)$ is an efficient interval payoff vector for each $\sigma \in \Pi(N)$. For size monotonic games $\langle N, w \rangle$, $w(T) - w(S)$ is well defined for all $S, T \in 2^N$ with $S \subset T$ since $|w(T)| = |w|(T) \geq |w|(S) = |w(S)|$. Now, we notice that for each $w \in SMIG^N$ the interval marginal vectors $m^\sigma(w)$ are defined for each $\sigma \in \Pi(N)$, because the monotonicity of $|w|$ implies $\overline{w}(S \cup \{i\}) - \underline{w}(S \cup \{i\}) \geq \overline{w}(S) - \underline{w}(S)$, which can be rewritten as $\overline{w}(S \cup \{i\}) - \overline{w}(S) \geq \underline{w}(S \cup \{i\}) - \underline{w}(S)$. So, $w(S \cup \{i\}) - w(S)$ is defined for each $S \subset N$ and $i \notin S$.

The following example illustrates that for interval games which are not size monotonic it might happen that some interval marginal vectors do not exist.

EXAMPLE 4.1. *Let $\langle N, w \rangle$ be the interval game with $N = \{1, 2\}$, $w(1) = [1, 3]$, $w(2) = [0, 0]$ and $w(1, 2) = [2, 3\frac{1}{2}]$. This game is not size monotonic. Note that $m^{(12)}(w)$ is not defined because $w(1, 2) - w(1)$ is undefined since $|w(1, 2)| < |w(1)|$.*

A characterization of convex interval games with the aid of interval marginal vectors is given in the following theorem.

THEOREM 4.1. Let $w \in IG^N$. Then, the following assertions are equivalent:

(i) w is convex;

(ii) $|w|$ is supermodular and $m^\sigma(w) \in \mathcal{C}(w)$ for all $\sigma \in \Pi(N)$.

Proof. (i) \Rightarrow (ii) Let $w \in CIG^N$, let $\sigma \in \Pi(N)$ and take $m^\sigma(w)$. Clearly, we have $\sum_{k \in N} m_k^\sigma(w) = w(N)$. To prove that $m^\sigma(w) \in \mathcal{C}(w)$ we have to show that for $S \in 2^N$, $\sum_{k \in S} m_k^\sigma(w) \succcurlyeq w(S)$. Let $S = \{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)\}$ with $i_1 < i_2 < \dots < i_k$. Then,

$$\begin{aligned} w(S) &= w(\sigma(i_1)) - w(\emptyset) \\ &+ \sum_{r=2}^k (w(\sigma(i_1), \sigma(i_2), \dots, \sigma(i_r)) - w(\sigma(i_1), \sigma(i_2), \dots, \sigma(i_{r-1}))) \\ &\preccurlyeq w(\sigma(1), \dots, \sigma(i_1)) - w(\sigma(1), \dots, \sigma(i_1 - 1)) \\ &+ \sum_{r=2}^k (w(\sigma(1), \sigma(2), \dots, \sigma(i_r)) - w(\sigma(1), \sigma(2), \dots, \sigma(i_r - 1))) \\ &= \sum_{r=1}^k m_{\sigma(i_r)}^\sigma(w) = \sum_{k \in S} m_k^\sigma(w), \end{aligned}$$

where the inequality follows from Theorem 3.1 (iii) applied to $i = \sigma(i_r)$ and

$$S_1 = \{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_{r-1})\} \subset S_2 = \{\sigma(1), \sigma(2), \dots, \sigma(i_{r-1})\}$$

for $r \in \{1, 2, \dots, k\}$. Further, by convexity of w , $|w|$ is supermodular.

(ii) \Rightarrow (i) From $m^\sigma(w) \in \mathcal{C}(w)$ for all $\sigma \in \Pi(N)$ follows that $m^\sigma(\underline{w}) \in C(\underline{w})$ and $m^\sigma(\overline{w}) \in C(\overline{w})$ for all $\sigma \in \Pi(N)$. Now, by the well known characterization of classical convex games with the aid of marginal vectors we obtain that $\langle N, \underline{w} \rangle$ and $\langle N, \overline{w} \rangle$ are convex games. Since $\langle N, |w| \rangle$ is convex by hypothesis, we obtain by Proposition 3.1 (i) that $\langle N, w \rangle$ is convex. \square

Now, we straightforwardly extend for size monotonic interval games two important solution concepts in cooperative game theory which are based on marginal worth vectors: the Weber set (Weber (1988)) and the Shapley value (Shapley (1953)).

The *interval Weber set* \mathcal{W} on the class of size monotonic interval games is defined by $\mathcal{W}(w) = \text{conv} \{m^\sigma(w) | \sigma \in \Pi(N)\}$ for each $w \in SMIG^N$. We notice that for traditional TU-games we have $W(v) \neq \emptyset$ for all $v \in G^N$,

while for interval games it might happen that $\mathcal{W}(w) = \emptyset$ (in case none of the interval marginal vectors $m^\sigma(w)$ is defined). Clearly, $\mathcal{W}(w) \neq \emptyset$ for all $w \in SMIG^N$. Further, it is well known that $\mathcal{C}(v) = \mathcal{W}(v)$ if and only if $v \in G^N$ is convex. However, this result can not be extended to convex interval games as we prove in the following proposition.

PROPOSITION 4.1. *Let $w \in CIG^N$. Then, $\mathcal{W}(w) \subset \mathcal{C}(w)$.*

Proof. By Theorem 4.1 we have $m^\sigma(w) \in \mathcal{C}(w)$ for each $\sigma \in \Pi(N)$. Now, we use the convexity of $\mathcal{C}(w)$. \square

The following example shows that the inclusion in Proposition 4.1 might be strict.

EXAMPLE 4.2. *Let $N = \{1, 2\}$ and let $w : 2^N \rightarrow I(\mathbb{R})$ be defined by $w(1) = w(2) = [0, 1]$ and $w(1, 2) = [2, 4]$. This game is convex. Further, $m^{(1,2)}(w) = ([0, 1], [2, 3])$ and $m^{(2,1)}(w) = ([2, 3], [0, 1])$, belong to the interval core $\mathcal{C}(w)$ and $\mathcal{W}(w) = \text{conv} \{m^{(1,2)}(w), m^{(2,1)}(w)\}$. Notice that $([\frac{1}{2}, 1\frac{3}{4}], [1\frac{1}{2}, 2\frac{1}{4}]) \in \mathcal{C}(w)$ and there is no $\alpha \in [0, 1]$ such that $\alpha m^{(1,2)}(w) + (1 - \alpha)m^{(2,1)}(w) = ([\frac{1}{2}, 1\frac{3}{4}], [1\frac{1}{2}, 2\frac{1}{4}])$. Hence, $\mathcal{W}(w) \subset \mathcal{C}(w)$ and $\mathcal{W}(w) \neq \mathcal{C}(w)$.*

In Section 5 we introduce a new notion of Weber set and show that the equality between the interval core and that Weber set still holds on the class of convex interval games.

The *interval Shapley value* $\Phi : SMIG^N \rightarrow I(\mathbb{R})^N$ is defined by

$$\Phi(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N. \quad (3)$$

Since $\Phi(w) \in \mathcal{W}(w)$ for each $w \in SMIG^N$, by Proposition 4.1 we have $\Phi(w) \in \mathcal{C}(w)$ for each $w \in CIG^N$. Without going into details we note here that the Shapley value Φ on the class of size monotonic interval games, and consequently on CIG^N , satisfies the properties of additivity, efficiency, symmetry and dummy player. In the next two propositions we show that on the class of interval games $w \in IG^N$ whose length games are supermodular, which we denote for further use by $SLIG^N$, marginal vectors and the interval Shapley value have simple expressions. Clearly, $CIG^N \subset SLIG^N$. Note that the fact that $\langle N, |w| \rangle$ is supermodular implies that $\langle N, |w| \rangle$ is monotonic because for each $S, T \in 2^N$ with $S \subset T$ we have

$$|w|(T) + |w|(\emptyset) \geq |w|(S) + |w|(T \setminus S),$$

and from this inequality follows $|w|(S) \leq |w|(T)$ since $|w|(T \setminus S) \geq 0$. So, $SLIG^N$ is a subclass of $SMIG^N$ implying that $CIG^N \subset SMIG^N$.

PROPOSITION 4.2. *Let $w \in SLIG^N$ and let $\sigma \in \Pi(N)$. Then, $m_i^\sigma(w) = [m_i^\sigma(\underline{w}), m_i^\sigma(\bar{w})]$ for all $i \in N$.*

Proof. By definition,

$$m^\sigma(\underline{w}) = (\underline{w}(\sigma(1)), \underline{w}(\sigma(1), \sigma(2)) - \underline{w}(\sigma(1)), \dots, \underline{w}(\sigma(1), \dots, \sigma(n)) - \underline{w}(\sigma(1), \dots, \sigma(n-1))),$$

and

$$m^\sigma(\bar{w}) = (\bar{w}(\sigma(1)), \bar{w}(\sigma(1), \sigma(2)) - \bar{w}(\sigma(1)), \dots, \bar{w}(\sigma(1), \dots, \sigma(n)) - \bar{w}(\sigma(1), \dots, \sigma(n-1))).$$

Now, we prove that $m^\sigma(\bar{w}) - m^\sigma(\underline{w}) \geq 0$. Since $|w| = \bar{w} - \underline{w}$ is a classical convex game we have for each $k \in N$

$$\begin{aligned} m_{\sigma(k)}^\sigma(\bar{w}) - m_{\sigma(k)}^\sigma(\underline{w}) &= (\bar{w} - \underline{w})(\sigma(1), \dots, \sigma(k)) - (\bar{w} - \underline{w})(\sigma(1), \dots, \sigma(k-1)) \\ &= |w|(\sigma(1), \dots, \sigma(k)) - |w|(\sigma(1), \dots, \sigma(k-1)) \\ &\geq |w|(\sigma(k)) - |w|(\emptyset) = |w|(\sigma(k)) \geq 0, \end{aligned}$$

where the first inequality follows from the properties of classical convex games. So, $m_i^\sigma(\underline{w}) \leq m_i^\sigma(\bar{w})$ for all $i \in N$, and

$$([m_i^\sigma(\underline{w}), m_i^\sigma(\bar{w})])_{i \in N} = (w(\sigma(1)), \dots, w(\sigma(1), \dots, \sigma(n)) - w(\sigma(1), \dots, \sigma(n-1))) = m^\sigma(w).$$

□

Since $CIG^N \subset SLIG^N$ we obtain from Proposition 4.2 that $m_i^\sigma(w) = [m_i^\sigma(\underline{w}), m_i^\sigma(\bar{w})]$ for each $w \in CIG^N$, $\sigma \in \Pi(N)$ and for all $i \in N$.

PROPOSITION 4.3. *Let $w \in SLIG^N$ and let $\sigma \in \Pi(N)$. Then, $\Phi_i(w) = [\phi_i(\underline{w}), \phi_i(\bar{w})]$ for all $i \in N$.*

Proof. From (3) and Proposition 4.2 we have for all $i \in N$,

$$\begin{aligned} \Phi_i(w) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^\sigma(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} [m_i^\sigma(\underline{w}), m_i^\sigma(\bar{w})] = \\ &\left[\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^\sigma(\underline{w}), \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^\sigma(\bar{w}) \right] = [\phi_i(\underline{w}), \phi_i(\bar{w})]. \end{aligned}$$

□

From Proposition 4.3 we obtain that for each $w \in CIG^N$ we have $\Phi_i(w) = [\phi_i(\underline{w}), \phi_i(\overline{w})]$ for all $i \in N$.

In the sequel we introduce the notion of (interval) *population monotonic allocation scheme* (pmas) for totally \mathcal{I} -balanced interval games, which is a direct extension of pmas for classical cooperative games (Sprumont (1990)). A game $w \in IG^N$ is called totally \mathcal{I} -balanced if the game itself and all its subgames are \mathcal{I} -balanced.

We say that for a game $w \in TLBIG^N$ a scheme $A = (A_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ with $A_{iS} \in I(\mathbb{R})^N$ is a pmas of w if:

- (i) $\sum_{i \in S} A_{iS} = w(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,
- (ii) $A_{iS} \preceq A_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$.

Notice that the total \mathcal{I} -balancedness of an interval game is a necessary condition for the existence of a pmas for that game. A sufficient condition is the convexity of the interval game. We notice that all subgames of a convex interval game are also convex. In what follows we focus on pmas on the class of convex interval games.

We say that for a game $w \in CIG^N$ an imputation $I = (I_1, \dots, I_n) \in \mathcal{I}(w)$ is *pmas extendable* if there exist a pmas $A = (A_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ such that $A_{iN} = I_i$ for each $i \in N$.

THEOREM 4.2. *Let $w \in CIG^N$. Then, each element I of $\mathcal{W}(w)$ is extendable to a pmas of w .*

Proof. Let $w \in CIG^N$. First, we show that for each $\sigma \in \Pi(N)$, $m^\sigma(w)$ is extendable to a pmas. We know that the interval marginal operator $m^\sigma : SMIG^N \rightarrow I(\mathbb{R})^N$ is efficient for each $\sigma \in \Pi(N)$. Then, for each $S \in 2^N$, $\sum_{i \in S} m_i^\sigma(w) = \sum_{k \in S} m_{\sigma(k)}^\sigma(w) = w(S)$ holds, where (S, w_S) is the corresponding (convex) subgame.

Further, by convexity, $m_i^\sigma(w_S) \preceq m_i^\sigma(w_T)$ for each $i \in S \subset T \subset N$, where (S, w_S) and (T, w_T) are the corresponding subgames.

Second, each $I \in \mathcal{W}(w)$ is a convex combination of $m^\sigma(w)$, $\sigma \in \Pi(N)$, i.e. $I = \sum \alpha_\sigma m^\sigma(w)$ with $\alpha_\sigma \in [0, 1]$ and $\sum_{\sigma \in \Pi(N)} \alpha_\sigma = 1$. Now, since all $m^\sigma(w)$ are pmas extendable, we obtain that I is pmas extendable as well. \square

From Theorem 4.2 we obtain that the total interval Shapley value generates a pmas for each convex interval game. We illustrate this in Example 4.3, where the calculations are based on Proposition 4.3.

EXAMPLE 4.3. Let $w \in CIG^N$ with $w(\emptyset) = [0, 0]$, $w(1) = w(2) = w(3) = [0, 0]$, $w(1, 2) = w(1, 3) = w(2, 3) = [2, 4]$ and $w(1, 2, 3) = [9, 15]$. It is easy to check that the interval Shapley value generates for this game the pmas depicted as

$$\begin{array}{l}
 N \\
 \{1, 2\} \\
 \{1, 3\} \\
 \{2, 3\} \\
 \{1\} \\
 \{2\} \\
 \{3\}
 \end{array}
 \begin{bmatrix}
 & 1 & 2 & 3 \\
 & [3, 5] & [3, 5] & [3, 5] \\
 [1, 2] & [1, 2] & [1, 2] & * \\
 [1, 3] & [1, 2] & * & [1, 2] \\
 [2, 3] & * & [1, 2] & [1, 2] \\
 \{1\} & [0, 0] & * & * \\
 \{2\} & * & [0, 0] & * \\
 \{3\} & * & * & [0, 0]
 \end{bmatrix}
 .$$

5 Interval solutions obtained with the square operator

Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ with $a \leq b$. Then, we denote by $a \square b$ the vector $([a_1, b_1], \dots, [a_n, b_n]) \in I(\mathbb{R})^N$ generated by the pair $(a, b) \in \mathbb{R}^N$. Let $A, B \subset \mathbb{R}^N$. Then, we denote by $A \square B$ the subset of $I(\mathbb{R})^N$ defined by $A \square B = \{a \square b \mid a \in A, b \in B, a \leq b\}$.

Now, with the use of the \square operator, we give a procedure to extend classical multi-solutions on G^N to interval multi-solutions on IG^N .

For a multi-solution $\mathcal{F} : G^N \rightarrow \mathbb{R}^N$ we define $\mathcal{F}^\square : IG^N \rightarrow I(\mathbb{R})^N$ by $\mathcal{F}^\square = \mathcal{F}(\underline{w}) \square \mathcal{F}(\bar{w})$ for each $w \in IG^N$.

Now, we focus on this procedure for multi-solutions such as the core and the Weber set on interval games. We define the square interval core $\mathcal{C}^\square : IG^N \rightarrow I(\mathbb{R})^N$ by $\mathcal{C}^\square(w) = C(\underline{w}) \square C(\bar{w})$ for each $w \in IG^N$. We notice that a necessary condition for the non-emptiness of the square interval core is the balancedness of the border games.

PROPOSITION 5.1. Let $w \in \mathcal{IBIG}^N$. Then, $\mathcal{C}(w) = \mathcal{C}^\square(w)$.

Proof. $(I_1, \dots, I_n) \in \mathcal{C}(w)$ if and only if $(\underline{I}_1, \dots, \underline{I}_n) \in C(\underline{w})$ and $(\bar{I}_1, \dots, \bar{I}_n) \in C(\bar{w})$ if and only if $(I_1, \dots, I_n) = (\underline{I}_1, \dots, \underline{I}_n) \square (\bar{I}_1, \dots, \bar{I}_n) \in \mathcal{C}^\square(w)$. \square

Since $CIG^N \subset \mathcal{IBIG}^N$ we obtain that $\mathcal{C}(w) = C(\underline{w}) \square C(\bar{w})$ for each $w \in CIG^N$.

We define the square Weber set $\mathcal{W}^\square : IG^N \rightarrow I(\mathbb{R})^N$ by $\mathcal{W}^\square(w) = W(\underline{w}) \square W(\bar{w})$ for each $w \in IG^N$. Note that $\mathcal{C}^\square(w) = \mathcal{W}^\square(w)$ if $w \in CIG^N$.

The next two theorems are very interesting because they extend for interval games, with the square interval Weber set in the role of the Weber set, the well known results in classical cooperative game theory that $C(v) \subset W(v)$ for each $v \in G^N$ (Weber (1988)) and $C(v) = W(v)$ if and only if v is convex (Ichiishi (1981)).

THEOREM 5.1. *Let $w \in IG^N$. Then, $\mathcal{C}(w) \subset \mathcal{W}^\square(w)$.*

Proof. If $\mathcal{C}(w) = \emptyset$ the inclusion holds true. Suppose $\mathcal{C}(w) \neq \emptyset$ and let $(I_1, \dots, I_n) \in \mathcal{C}(w)$. Then, by Proposition 5.2, $(\underline{I}_1, \dots, \underline{I}_n) \in C(\underline{w})$ and $(\bar{I}_1, \dots, \bar{I}_n) \in C(\bar{w})$, and, because $C(v) \subset W(v)$ for each $v \in G^N$, we obtain $(\underline{I}_1, \dots, \underline{I}_n) \in W(\underline{w})$ and $(\bar{I}_1, \dots, \bar{I}_n) \in W(\bar{w})$. Hence, we obtain $(I_1, \dots, I_n) \in \mathcal{W}^\square(w)$. \square

From Theorem 5.1 and Proposition 4.1 we obtain that $\mathcal{W}(w) \subset \mathcal{W}^\square(w)$ for each $w \in CIG^N$. This inclusion might be strict as Example 4.2 illustrates.

THEOREM 5.2. *Let $w \in \mathcal{IBIG}^N$. Then, the following assertions are equivalent:*

- (i) w is convex;
- (ii) $|w|$ is supermodular and $\mathcal{C}(w) = \mathcal{W}^\square(w)$.

Proof. By Proposition 3.1 (i), w is convex if and only if $|w|, \underline{w}$ and \bar{w} are convex. Clearly, the convexity of $|w|$ is equivalent with its supermodularity. Further, \underline{w} and \bar{w} are convex if and only if $W(\underline{w}) = C(\underline{w})$ and $W(\bar{w}) = C(\bar{w})$. These equalities are equivalent with $\mathcal{W}^\square(w) = \mathcal{C}^\square(w)$. Finally, since w is \mathcal{I} -balanced by hypothesis, we have by Proposition 5.1 that $\mathcal{C}(w) = \mathcal{W}^\square(w)$. \square

With the aid of Theorem 5.2 we will show that the interval core is additive on the class of convex interval games, which is inspired by Dragan, Potters and Tijs (1989).

PROPOSITION 5.2. *The interval core $\mathcal{C} : CIG^N \rightarrow I(\mathbb{R})^N$ is an additive map.*

Proof. The interval core is a superadditive solution concept for all interval games (Alparslan Gök, Branzei and Tijs (2008a)). We need to show the subadditivity of the interval core. We have to prove that $\mathcal{C}(w_1 + w_2) \subset$

$\mathcal{C}(w_1) + \mathcal{C}(w_2)$. Note that $m^\sigma(w_1 + w_2) = m^\sigma(w_1) + m^\sigma(w_2)$ for each $w_1, w_2 \in CIG^N$. By definition of the square interval Weber set we have $\mathcal{W}^\square(w_1 + w_2) = W(\underline{w}_1 + \underline{w}_2) \square W(\bar{w}_1 + \bar{w}_2)$. By Theorem 5.2 we obtain

$$\mathcal{C}(w_1 + w_2) = \mathcal{W}^\square(w_1 + w_2) \subset \mathcal{W}^\square(w_1) + \mathcal{W}^\square(w_2) = \mathcal{C}(w_1) + \mathcal{C}(w_2).$$

□

Finally, we define $\mathcal{DC}^\square(w) = DC(\underline{w}) \square DC(\bar{w})$ for each $w \in IG^N$ and notice that for convex interval games we have $\mathcal{DC}^\square(w) = DC(\underline{w}) \square DC(\bar{w}) = C(\underline{w}) \square C(\bar{w}) = \mathcal{C}^\square(w) = \mathcal{C}(w)$, where the second equality follows from the well known result in the theory of TU-games that for convex games the core and the dominance core coincide, and the last equality follows from Proposition 5.1.

6 Concluding remarks

In this paper we define and study convex interval games. We note that the combination of Theorems 3.1, 4.1 and 5.2 can be seen as an interval version of Theorem 96 in Branzei, Dimitrov and Tijs (2008). In fact these theorems imply Theorem 96 in Branzei, Dimitrov and Tijs (2008) for the embedded class of classical TU-games. Extensions to convex interval games of the characterizations of classical convex games where exactness of subgames and superadditivity of marginal (or remainder) games play a role (Biswas et al. (1999), Branzei, Dimitrov and Tijs (2004) and Martinez-Legaz (1997, 2006)) can be found in Branzei, Tijs and Alparslan Gök (2008a).

There are still many interesting open questions. For further research it is interesting to study whether one can extend to interval games the well known result in the traditional cooperative game theory that the core of a convex game is the unique stable set (Shapley (1971)). It is also interesting to find an axiomatization of the interval Shapley value on the class of convex interval games. Other topics for further research could be related to introducing new models in cooperative game theory by generalizing cooperative interval games. For example, the concepts and results on (convex) cooperative interval games could be extended to cooperative games in which the coalition values $w(S)$ are ordered intervals of the form $[u, v]$ of an (infinite dimensional) ordered vector space. Such generalization could give more applications to the interval game theory. Also to establish relations between

convex interval games and convex games in other existing models of cooperative games could be interesting. One candidate for such study could be convex games in cooperative set game theory (Sun (2003)).

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