Egalitarianism Under Earmark Constraints

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Abstract

We consider a model in which a homogeneous commodity (the resource) is shared by several agents with single-peaked preferences and capacity constraints, and the resource is coming from different suppliers under arbitrary bilateral feasibility constraints: each supplier can only deliver to a certain subset of agents. Examples include balancing the workload of machines, sharing earmarked funds between different projects, and distributing utilities under geographic constraints.

Unlike in the one supplier model (Sprumont [21]), that we generalize, in a Pareto Optimal allocation agents who get more than their peak typically coexist with agents who get less. A variant of the Gallai-Edmonds decomposition identifies these two subsets of agents, that we call respectively the over-demanded and the under-demanded side of the market. Like in the one supplier model, there is a Lorenz dominant Pareto optimal allocation. We call it the Egalitarian solution, and characterize it, in the case of identical capacity constraints, by the combination of Strategyproofness (truthful revelation of peaks), Pareto Optimality, and a variant of Equal Treatment of Equals. The analysis relies on submodular optimization techniques as in Dutta and Ray [9].

Keywords: Bipartite graph, egalitarianism, Lorenz dominance, single-peaked preferences.

JEL codes: C72, D63, D61, C78, D71.

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1 Introduction

Egalitarianism, the central principle of fair division, may conflict with incentives, feasibility or efficiency constraints. Maximizing the leximin ordering over profiles of relevant characteristics (a.k.a. the Rawlsian approach) is the most common implementation of egalitarianism under constraints. It is however a controversial method. Indeed, it recommends to take arbitrarily large amounts of resources from the “rich” if this allows to raise by even a tiny amount the lot of the “poor”. The only case where egalitarianism eschews this critique is when we can find a Lorenz dominant distribution of welfare, or resources: at the Lorenz dominant outcome, we simultaneously maximize the share of the $k$ poorest individuals, for any number $k$ of agents.\footnote{References on the Lorenz optimum: [20], [13], [11].} Unlike the leximin ordering that always reaches a unique maximum in any closed convex set, a Lorenz dominant outcome may not exist. We know very few fair division models admitting Lorenz dominating solutions over a reasonably rich domain of problems. The two main instances follow.

Dutta and Ray ([9]) observed that the core of a supermodular (convex) cooperative game is one general instance where a Lorenz dominant solution exists; this solution has been known after their work as the egalitarian selection in the core. The second model, due to Sprumont, is the fair division of a single commodity under single-peaked preferences and no free disposal ([3], [21]). The uniform solution selects for each agent either his peak, or a common share in such a way that the resource is fully distributed. Although the original motivation of the uniform solution was its incentive properties ([3]), its most compelling fairness property, and its shortest definition, is to be Lorenz dominant among all Pareto Optimal allocations of the resource ([8]). While the Lorenz dominance property of the Uniform solution is central in our paper, the solution satisfies many other compelling fairness criteria, see for instance ([19]) for an interesting characterization in terms of distributions of shares among agents.

Our Contribution: We study a considerable generalization of the Sprumont model, where a homogeneous commodity (the resource) is still shared by several agents with single-peaked preferences, but the resource is coming from any number of different suppliers, under arbitrary bilateral feasibility constraints: each supplier can only deliver to a certain subset of agents. Examples include earmark constraints: the resource is money, and there are several funds (the suppliers) to support different projects (the agents).\footnote{In United States politics, an earmark is a congressional provision that directs approved funds to be spent on specific projects.} Each fund must spend a certain budget, but the earmarks limit the set of projects that a given fund can support. The resource can be water available from several sources, and geographical constraints limit the set of customers (the agents) that a given source can serve. Or the suppliers can be different jobs, each one with a given size in work-hours, to be completed by a set of workers (the agents) with different skills, so that each worker can only perform certain jobs. And so on.

The resources coming from different suppliers are, strictly speaking, different commodities, but any two commodities are perfect substitutes for an agent who can consume both. Thus we speak of a single commodity, like in the original model, but of different resources (the suppliers in the above examples). Our agents have single-peaked preferences over the total amount of commodity they consume. We explore the implications of efficiency (Pareto Optimality), incentive-compatibility, and fairness in our bipartite model.
The set of Pareto Optimal allocations has a much more complicated structure than in the one-resource model. There everyone consumes at most his peak if total demand exceeds the available resource, while everyone consumes at least his peak if total demand is smaller than the available resource. Here a Pareto Optimal allocation involves typically agents consuming more than their peak, as well as agents consuming less. More precisely, the Pareto set is described by a three-components partition of the agents and resources: the first set of overdemanded resources are consumed exclusively by the first set of agents, who each receive at most their peak allocation; the second set of underdemanded resources are consumed exclusively by the second set of agents, who each receive at least their peak allocation; and the third set of balanced resources is allocated to the third set of agents, who each receive exactly their peak allocation.

We take Strategyproofness (truthful report of one’s preferences is a dominant strategy) as our incentive compatibility design constraint. We identify a canonical Egalitarian solution that generalizes the familiar Uniform solution of the one-resource model. As in that model, this solution selects a Pareto Optimal allocation, defines a Strategyproof direct revelation mechanism, and is fair in the strong sense that it selects the Lorenz dominant Pareto Optimal allocation.

Our axiomatic characterization resembles closely that of the Uniform solution in [21], [7], provided we formulate the two equity tests No Envy and Equal Treatment of Equals with more care than in the one-resource model. Given the bilateral constraints, a transfer of resources between two given agents may require to alter the share of a third one. We postulate that Ann’s envy of Bob’s share is legitimate only if it is feasible to improve her share at the expense only of Bob, i.e., while preserving the shares of every agent other than Bob. Similarly Equal Treatment of Equals is violated only if we can bring Ann’s and Bob’s shares closer together without altering any other share. Our solution meets both properties. It is characterized by the combination of Strategyproofness, Pareto Optimality and Equal Treatment of Equals, just like the Uniform solution in [7].

Related literature: 1). In a recent paper, Kibris and Kar ([12]) also consider the division of a single commodity coming from multiple suppliers. Every agent can consume from any supplier, but must receive his entire allocation from a single supplier: in effect, they form coalitions to consume different private goods. It turns out that efficiency is incompatible with using a simple division rule, such as the Uniform rule, for every supplier.

2). The mathematical result driving the structure of the Pareto set is a variant of the Gallai-Edmonds (henceforth, GE) decomposition for bipartite flow graphs,(see [16] for a formal treatment, or [5] and [17] for applications in matching).

3). The GE decomposition appears also in our companion paper [4], where we develop a model of bipartite trade in which both suppliers and demanders are active agents. Each supplier (resp. demander) has single peaked preferences over the amount of commodity he wants to supply (resp. receive); the homogenous commodity can only be transferred across certain bilateral edges. There is a close formal analogy with the current paper, in the sense that the set of relevant Pareto Optimal allocations is described by the same GE decomposition. There is again a Lorenz dominant allocation in the Pareto set, and it defines a Strategyproof direct revelation mechanism. The difference is that, in order to guarantee Voluntary Participation in the mechanism, no supplier (or demander) ever gets to supply (or receive) more than her ideal level. This implies that some suppliers and some demanders consume their peak allocation, and to compute the solution we need only one of the algorithms in Section 5 (the one corresponding to \((M_-, Q_+)\)).
To see why neither one of the model in [4] and the one here is a special case of the other, consider the impact of destroying one edge in the bipartite graph of compatibility constraints. In [4] this is (weakly) detrimental to the agents at both ends of the edge. By contrast, in our model, dropping an edge can be good or bad news for the agent at one end of this edge, as well as to other agents; this is explained in the concluding section 8. This implies that if the edges are not verifiable (agents can freely claim to be incompatible with some suppliers) our solution is vulnerable to manipulation, while the rule analyzed in [4] is not.

We use in [4] the techniques of flows on graphs, in particular the max-flow min-cut theorem, instead of the GE decomposition. The flow approach simplifies some proofs there, but in the current model it does not appear to work well. The main reason for this is the asymmetric treatment of the two sides in this paper where the supply must be depleted completely by the agents. This hinders the use of flow models where nodes (except the sink and the source) play an identical role.

4. We conclude with three follow up papers. Chandramoulin and Sethuraman ([6]) establish that our Egalitarian solution is actually group-strategyproof, thus answering an open question in an earlier version of this paper (see Proposition 4 in Section 6).

Szwagrzak ([22]) obtains an alternative characterization of the Egalitarian solution where instead of Strategyproofness he uses a bipartite version of the Preference Replacement property used in [24] to characterize the Uniform rule in the one-resource model.

Finally, Moulin and Sethuraman ([15]) discuss the extension to our bipartite model of other solutions of the one-resource rationing problem, such as the Proportional and Equal Losses solutions. Their results depend critically on a Consistency property that plays no role here (but see comment 4 in section 8).

Contents: After a numerical example in the next section, we introduce the model in Section 3, and characterize Pareto optimal allocations in Section 4. The Egalitarian solution is defined in Section 5 and its fairness and incentives properties are the subject of Section 6. Section 7 is the characterization result, and Section 8 collects final comments about variants and possible extensions of our model.

2 A numerical example

We have four resources \( r, s, t, u \), and four agents \( A, B, C, D \). Figure 1 shows the compatibility constraints: e.g., agent \( A \) can consume from any resource, while agent \( D \) can only consume from resource \( u \). Total supply \( 11 + 15 + 8 + 6 = 40 \) equals total demand \( 10 + 15 + 10 + 5 = 40 \). In the one-resource model, this would allow to give every agent her peak allocation. The bipartite constraints do not allow this: \( A \), for instance, must consume at least 11 units, because no one else can consume the first resource. On the other hand, \( D \) should not consume more than 5 units in a Pareto optimal allocation: for instance if we give him 6 units, \( C \) can get at most 8 units, and transferring one \( t \)-unit from \( D \) to \( C \) is a Pareto improvement.

The three-component partition identifies \( r \) as the underdemanded resource, and \( A \) as the agent consuming exclusively the \( r \)-resources. There are two overdemanded resources \( t \) and \( u \), whose resources go exclusively to \( C \) and \( D \). Finally resource \( s \) is balanced with agent \( B \). By Proposition 1 below, an allocation is Pareto optimal if and only if: \( A \) gets 11 units from \( r \); \( B \) gets 15 from \( s \); \( D \) gets \( \theta \) units from \( u \), where \( 4 \leq \theta \leq 5 \), and \( C \) gets all of \( t \) plus \( (6 - \theta) \) from \( u \). This guarantees that each one of \( C, D \) gets no more than his peak.
Within the Pareto set, our Egalitarian solution picks the most egalitarian set of shares, corresponding to $\theta = 5$: $C$ gets 9 units and $D$ gets 5.

3 The model

We have a set $M$ of agents with generic elements $i, j, k, \ldots$, and $m = |M|$; a set $Q$ of resources with generic element $r, s, \ldots$, and $q = |Q|$. Resource $r$ is of size $\omega_r$, with $\omega_r > 0$.

All resources must be allocated between the agents, but each resource can only be assigned to some of the agents. The bipartite graph $G$, a subset of $M \times Q$, represents the compatibility constraints between resources and agents: $ir \in G$ means that it is possible to transfer resource $r$ to agent $i$. We assume throughout that the graph $G$ is connected, else we can treat each connected component of $G$ as a separate problem.

We use the following notation: for any subsets $S \subseteq M, T \subseteq Q$ the restriction of $G$ is $G(S, T) = G \cap \{S \times T\}$ (not necessarily connected); the set of resources compatible with agents in $S$ is $f(S) = \{r \in Q | G(S, \{r\}) \neq \emptyset\}$, the set of agents compatible with resources in $T$ is $g(T) = \{i \in M | G(\{i\}, T) \neq \emptyset\}$.

A transfer of resources from $Q$ to $M$ is described by a $G$-flow $\varphi$, i.e., a vector $\varphi \in \mathbb{R}_+^G$ such that $\varphi_{ir} > 0 \Rightarrow ir \in G$. We call a $G$-flow $\varphi$ feasible if it allocates all the resources and we write $x(\varphi)$ for the allocation it realizes:

$$\text{for all } r \in Q: \sum_{i \in g(r)} \varphi_{ir} = \omega_r; \text{ for all } i \in M: x_i(\varphi) = \sum_{r \in f(i)} \varphi_{ir}$$  \hspace{1cm} (1)$$

We write $\mathcal{F}(G; \omega)$ for the set of feasible $G$-flows, and $\mathcal{A}(G, \omega) = x(\mathcal{F}(G; \omega))$ for the set of allocations achieved by some feasible $G$-flow. Both sets are obviously non empty.

We use the notation $x_S = \sum_{i \in S} x_i$, $\omega_T = \sum_{r \in T} \omega_r$ etc.. The allocation $x \in \mathbb{R}^M$ is in the lower (resp. upper) core of the cooperative game $(M, v)$ if $x_M = w(M)$, and $x_S \geq w(S)$ (resp.
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We write \( \mathcal{PO}(G, \omega, R) \), for the set of Pareto Optimal allocations of the economy \((G, \omega, R)\). The structure of this set is given by a three-component partition of \( M \cup Q \) that we derive as a variant of the classic Gallai-Edmonds decomposition for bipartite graphs ([16]). This partition, as well as the set \( \mathcal{PO}(G, \omega, R) \) itself, depend only upon the profile of peaks \( p[R] \), but not on the way \( R \) compares allocations across \( p \).

We call a triple \((G, \omega, p)\) a problem, keeping in mind it represents all economies where \( p \) is the profile of peaks in \( R \). Until the end of section 5 it is enough to work with problems rather than economies. We define three properties of a problem:

- \((G, \omega, p)\) is balanced if \( p \in \mathcal{A}(G, \omega) \);
- \((G, \omega, p)\) exhibits under-demand if for all \( S \subseteq M \), \( p_S < \omega_f(S) \);
- \((G, \omega, p)\) exhibits over-demand if for all \( T \subseteq Q \), \( \omega_T < p_g(T) \);

In a balanced problem we can give exactly his peak allocation to every agent. By Lemma 1 in a problem with under-demand we can give each agent at least his peak, and must give strictly more to at least one agent; in a problem with over-demand we can give each agent at most his peak, and must give strictly less to at least one.

We show now that any allocation problem \((G, \omega, p)\) can be decomposed in three subproblems, one of each type above, and at most two types may be absent. When we speak of the subproblem restricted to \( S \times T \subseteq M \times Q \), we mean that the resources in \( T \) must be assigned to the agents in \( S \) along the restricted graph \( G(S, T) \) (that may not be connected), and for simplicity we write this subproblem as \((G(S, T), \omega, p)\) when in fact only the \( S \)-coordinates of \( p \) and the \( T \)-coordinates of \( \omega \) matter. In particular the set \( \mathcal{A}(G(S, T), \omega) \) of feasible allocations is in \( \mathbb{R}^3_\omega \).

Lemma 2: For any problem \((G, \omega, p)\) where \( G \) is connected, and \( p \geq 0, \omega \gg 0 \), there exists unique partitions \( M_+, M_0, M_- \) of \( M \), and \( Q_+, Q_0, Q_- \) of \( Q \) such that

i) \( G(M_-, Q_0) = G(M_-, Q_-) = G(M_0, Q_-) = \emptyset \)

ii) \( (G(M_+, Q_-), \omega, p) \) exhibits under-demand;
iii) \( (G(M_0, Q_0), \omega, p) \) is balanced;
iv) \( (G(M_-, Q_+), \omega, p) \) exhibits over-demand.

We repeat that up to two of the pairs \((M_+, Q_-), (M_0, Q_0)\), or \((M_-, Q_+)\) may be empty. For instance, if there are no bilateral constraints \((G = M \times Q)\), our model is a generalization of Sprumont’s, where the GE decomposition reduces to a single component: if \( \omega_Q < p_M \) we have over-demand, \( M = M_-, Q = Q_+ \); if \( \omega_Q > p_M \) underdemand and \( M = M_+, Q = Q_- \); if \( \omega_Q = p_M \) the problem is balanced and \( M = M_0, Q = Q_0 \). See the discussion of this special case after Proposition 2 in section 5.

Proof of Lemma 2:

Fix \((G(M, Q), \omega, p)\) and define the partitions with the help of two simple maximization problems. Set \( D = \arg \max_{S \subseteq M} \{ p_S - \omega_f(S) \} \) if there is at least one \( S \) such that \( p_S > \omega_f(S) \), \( D = \emptyset \) otherwise. As \( S \to p_S - \omega_f(S) \) is supermodular, \( D \) is stable by intersection and union. If \( D \neq \emptyset \), we define \( M_- \) as its smallest element, and \( M_- \cup M_0 \) as its largest element. Set similarly \( B = \arg \max_{T \subseteq Q} \{ \omega_T - p_g(T) \} \) if there is at least one \( T \) such that \( \omega_T > p_g(T) \), \( B = \emptyset \) else. Then if \( B \neq \emptyset \), it is stable by intersection and union, and \( Q_- \) is its smallest element, while \( Q_- \cup Q_0 \) is its largest element.
Suppose first that $D$ and $B$ are both non empty, and check $G(M_-, Q_- \cup Q_0) = \emptyset$ by contradiction. If this set is non empty, we define $A = g(Q_- \cup Q_0) \cap M_-$, and $B = f(M_-) \cap \{Q_- \cup Q_0\}$, so that $A = g(B) \cap M_-$ and $B = f(A) \cap \{Q_- \cup Q_0\}$. Consider the equality
\[ (p_{M_- \setminus A} - \omega f(M_- \setminus B)) + (p_A - \omega_B) = p_{M_-} - \omega f(M_-) \]
By construction $f(M_- \setminus A) \subseteq f(M_-) \setminus B$, therefore the inequality $p_{M_- \setminus A} - \omega f(M_- \setminus B) \geq p_{M_-} - \omega f(M_-)$ would imply that $M_- \setminus A$ is a smaller element of $D$ than $M_-$ Thus we must have $p_A - \omega_B > 0$. Consider now the equality
\[ (\omega(Q_- \cup Q_0) \setminus B - p g(Q_- \cup Q_0) \setminus A) + (\omega_B - p_A) = \omega_{Q_- \cup Q_0} - p g(Q_- \cup Q_0) \]
By construction $g((Q_- \cup Q_0) \setminus B) \subseteq g(Q_- \cup Q_0) \setminus A$; combining this with $\omega_B - p_A < 0$, we see that $T = \{Q_- \cup Q_0\} \setminus B$ gives a higher $\omega_T - g(T)$ than $Q_- \cup Q_0$, the desired contradiction.

A symmetrical argument, omitted for brevity, establishes $G(M_- \cup M_0, Q_-) = \emptyset$.

Next we define $Q_+ = f(M_-)$, and check that $Q_+, Q_0, Q_-$ partition $Q$. We already know that these sets are disjoint. If they are not a partition, the set $T = Q \setminus \{Q_+ \cup Q_- \cup Q_0\}$ is non empty. Because $T \cup \{Q_- \cup Q_0\}$ is not in $B$, we have and observe that $p g(T) \setminus (Q_- \cup Q_0) > \omega_T$. Therefore the set $S = g(T) \setminus g(Q_- \cup Q_0)$ is non empty. Moreover $S$ and $M_-$ are disjoint and $f(M_- \cup S) \subseteq Q_+ \cup T$. This gives a contradiction of the definition of $M_+$:
\[ p_{M_- \cup S} - f(M_- \cup S) \geq p_{M_- \cup S} - \omega_{Q_+ \cup T} = p_{M_-} - \omega_Q + p_S - \omega_T > p_{M_-} - \omega Q_+ \]

We omit the symmetrical argument establishing that $M_+ = g(Q_-)$ completes the partition of $M$ as $M_+, M_0, M_-$. To check the overdemand in $(G(M_-, Q_+), \omega, p)$ recall that $\omega_Q < p_{M_-}$ because $D$ is non empty. Then fix a proper subset $T$ of $Q_+$ and assume $\omega_T \geq p_{g(T) \cap M_-}$. Then the equality
\[ (p_{M_- \setminus g(T)} - \omega_Q \setminus T) + (p_{g(T) \cap M_-} - \omega_T) = p_{M_-} - \omega Q_+ \]
implies that $M_- \setminus g(T)$ is a smaller element of $D$ than $M_-$, contradiction.

The proof that $(G(M_0, Q_0), \omega, p)$ is balanced proceeds along similar lines. If there is $S \subseteq M_0$ such that $p_S > f(S) \cap Q_0$, it follows that $M_- \cup S$ gives a higher $p_S - \omega f(S')$ than $M_-$. We omit the details.

We also omit for brevity the symmetrical proof that $(G(M_+, Q_-), \omega, p)$ is underdemanded, and the treatment of the remaining cases where at least one of $D$ or $B$ is empty. For instance if they are both empty we have $p_S \leq f(S) \cap Q_0$ and $\omega_T \leq p_{g(T)}$ for all $S \subseteq M, T \subseteq Q$, and Lemma 1 implies that $(G, \omega, p)$ is balanced; so $M = M_0$ and $Q = Q_0$.

The above proof shows that the canonical partition obtains by maximizing two supermodular set functions, one over the subsets of $M$, the other those of $Q$. This can be done by the standard greedy algorithm\(^3\), of polynomial complexity in the number of nodes $|M| + |Q|$. We already described the partition in the example of section 2: $M_+ = A, M_0 = B; M_- = C, D; Q_- = r; Q_0 = s; Q_+ = t, u$. For another example consider a variant of Figure 1 in which the only change is that the resource $s$ is of size 17 instead of 15. See Figure 2. Now the canonical partition has only two components: $r, s$ are the underdemanded resources, while $t, u$ remain overdemand, i.e., $M_+ = A, B; M_- = C, D; Q_- = r, s; Q_+ = t, u$.

\(^3\)If $S \rightarrow v(S)$ is supermodular over $2^M$, we solve first $\max_{i \in M} v(i)$, keep a winner $i^*$, then solve $\max_{i \in M \setminus \{i^*\}} v(\{i, i^*\})$, and so on.
Figures 1 and 2 illustrate a general property, an immediate consequence of Lemma 2: for any pair of agents $i, j$ such that $f(i) \subseteq f(j)$, we have $\{j \in M_- \Rightarrow i \in M_-\}$ and $\{i \in M_+ \Rightarrow j \in M_+\}$, and if two resources $r, s$ satisfy $g(r) \subseteq g(s)$, then $\{s \in Q_- \Rightarrow r \in Q_+\}$ and $\{r \in M_+ \Rightarrow s \in M_+\}$.

We are ready to describe the set $\mathcal{PO}(G, \omega, R)$ of Pareto Optimal allocations in terms of the canonical decomposition of Lemma 2.

**Proposition 1:** Fix $G$ and $\omega$.

i) For any profile $R \in R^M$ with peaks $p$, the set $\mathcal{PO}(G, \omega, R)$ is non empty, convex and compact;

ii) The property of Pareto Optimality is Peak-Only: for all $R, R'$ with $p[R] = p[R'] \Rightarrow \mathcal{PO}(G, \omega, R) = \mathcal{PO}(G, \omega, R')$;

iii) A flow implementing a Pareto Optimal allocation is null on $G(M_+, Q_+)$, $G(M_+, Q_0)$, and $G(M_0, Q_+)$;

iv) The allocation $x$ is Pareto Optimal if and only if

\[
\begin{align*}
&\text{in } M_+: x \in A(G(M_+, Q_-), \omega), \text{ and } p \leq x \tag{4} \\
&\text{on } M_0: x = p \tag{5} \\
&\text{in } M_-: x \in A(G(M_-, Q_+), \omega), \text{ and } x \leq p \tag{6}
\end{align*}
\]

(recall that $A(G(S, T), \omega)$ is a subset of $\mathbb{R}^S$ that only depends upon the $S$ coordinates of $\omega$)

In words, agents in $M_+$ consume precisely all the resources in $Q_-$, each one gets at least his peak, and at least one, strictly more (Lemma 2); those in $M_-$ share the resources in $Q_+$, consume no more than their peak, and at least one gets strictly less; those in $M_0$ consume the resources in $Q_0$ and each gets precisely his peak.

The Peak-Only property allows us to write the Pareto Optimal set simply as $\mathcal{PO}(G, \omega, p)$. From statements iv) and v) in Lemma 1, we can also describe $\mathcal{PO}(G, \omega, p)$ as the cartesian product of three sets: on $M_+$, the subset of the upper core of $(M_+, v^+)$ such that $x \geq p$, where $v^+$ is the game (3) for the restricted problem $(G(M_+, Q_-), \omega)$; on $M_-$ the subset of the lower core of $(M_-, w^-)$ such that $x \leq p$, where $w^-$ is the game (2) for $(G(M_-, Q_+), \omega)$; and $p$ on $M_0$. 

Proof of Proposition 1:

**Step 1** We first prove the “if” part of Statement iv). By statement 2 in Lemma 1, and the fact that \((G(M_+, Q_-), \omega, p)\) exhibits under-demand (Lemma 2), the set \(A(G(M_+, Q_-), \omega)\) is non empty; the set \(A(G(M_-, Q_+), \omega)\) is similarly non empty because \((G(M_-, Q_+), \omega, p)\) exhibits over-demand. Finally, \(A(G(M_0, Q_0), \omega, p)\) is non empty because \((G(M_0, Q_0), \omega, p)\) is balanced. Suppose now that an allocation \(x\) satisfying (4),(5),(6) is Pareto dominated by some \(y \in A(G, \omega)\). Clearly \(y = x\) on \(M_0\). Because \(G(M_- \cup M_0, Q_-) = \emptyset\) we have \(y_{M_+} \geq \omega_{Q_-} = x_{M_+}\); on the other hand if \(y_i > x_i\) for some \(i \in M_+\), this agent with peak \(p_i \leq x_i\) strictly prefers \(x_i\) to \(y_i\) which our assumption precludes. We conclude \(y = x\) on \(M_+\). The argument establishing \(y = x\) on \(M_-\) is entirely similar.

**Step 2** We prove next both the “only if” part of Statement iv), and Statement iii). Note that statements i, ii) are then clear because (4),(5),(6) together define a peak-only, convex and compact set of allocations.

We fix throughout Step 2 an economy \((G, \omega, R)\), a Pareto optimal allocation \(x\), and a flow \(\varphi\) implementing \(x\). We color agent \(i\) in green if \(x_i < p_i\), in red if \(x_i > p_i\), and in black if \(x_i = p_i\). We also construct a directed graph \(G^\varphi\) as follows: all edges in \(G\) are oriented from \(M\) to \(Q\); if \(\varphi_{ij} > 0\), and only then, we add a “backward” edge from resource \(j\) to agent \(i\).

We claim there is no directed path\(^4\) in \(G^\varphi\) from a green agent to a red one. If there was such a path from \(i\) to \(i'\), we could increase a little the flow along that path (with the convention that increasing the flow on a backward edge amounts to decrease by the same amount the flow \(\varphi_{ij}\) in \(G\)), and obtain a new allocation where \(i\) consumes a little more, \(i'\) a little less, and everyone else as before; this would contradict Pareto optimality.

Define now \(X\) as the set of all green nodes in \(M\) together with the nodes in \(M \cup Q\) that one can reach in \(G^\varphi\) from a green node; \(Y\) as the set of nodes in \(M \cup Q\) that are either a red agent, or a node from which one can reach a red node in \(G^\varphi\); and \(Z\) as the remaining subset of \(M \cup Q\). Thus \(X, Y, Z\) partition \(M \cup Q\), and every agent in \(X \cap M\) (resp. \(Y \cap M\), resp. \(Z \cap M\)) is green or black (resp. red or black, resp. black). Also, there is no path in \(G^\varphi\) from \(X\) to \(Z\) or \(Y\), or from \(Z\) to \(Y\).

**Step 2.1** In this substep we focus on \(M_-\) and \(Q_+\) and show \(M_-, Q_+ \subseteq X\), in particular there is no red agent in \(M_-,\) and \(x \leq p\) on \(M_+\).

Assume to the contrary \((Y \cup Z) \cap M_- \neq \emptyset\). Then \(x_{(Y \cup Z) \cap M_-} \geq p_{(Y \cup Z) \cap M_-}\), because all such agents are red or black. We also have \(x_{(Y \cup Z) \cap M_-} \leq \omega_{(Y \cup Z) \cap Q_+}\), because the only positive flow out of \((Y \cup Z) \cap M_-\) goes to \((Y \cup Z) \cap Q_+\): it cannot go to \(X\) without creating a path in \(G^\varphi\) from \(X\) to \(Y \cup Z\), and there is no edge from \(M_-\) to \(Q_- \cup Q_0\). If \((Y \cup Z) \cap Q_+ = \emptyset\), then \((Y \cup Z) \cap M_-\) must be empty as well, contradiction. If \((Y \cup Z) \cap Q_+ \neq \emptyset\), apply statement iv) in Lemma 2:

\[
\omega_{(Y \cup Z) \cap Q_+} < p_{g((Y \cup Z) \cap Q_+)} \leq p_{(Y \cup Z) \cap M_-}
\]

(7)

where the second inequality follows from the fact that there is no edge between an agent in \(X\) and a resource in \(Y \cup Z\). This is the desired contradiction.

**Step 2.2** We focus now on \(M_+\) and \(Q_-\), and show \(M_+, Q_- \subseteq Y\), in particular, there is no green agent in \(M_+,\) and \(x \geq p\) on \(M_+\).

Assume to the contrary \((X \cup Z) \cap M_+ \neq \emptyset\). Then we have

\[
x_{(X \cup Z) \cap M_+} \leq p_{(X \cup Z) \cap M_+} < \omega_{f((X \cup Z) \cap M_+)} \cap Q_- \leq \omega_{(X \cup Z) \cap Q_-}
\]

(8)

\(^4\)When we speak below of a path, we always mean a directed path.
where the first inequality is because the nodes in \((X \cup Z) \cap M_+\) are green or black, the second is from statement \(ii\) in Lemma 2, and the third because there is no edge from \(X \cup Z\) to \(Y\). Similarly we have

\[
x_{(X \cup Z) \cap M_0} \leq p_{(X \cup Z) \cap M_0} \leq \omega_f((X \cup Z) \cap M_0) \cap Q_0 \leq \omega((X \cup Z) \cap Q_0) \tag{9}
\]

where the only differences are that the middle inequality, from statement \(iii\) in Lemma 2, is not strict, and the fact that \((X \cup Z) \cap M_0\) could be empty. On the other hand

\[
x_{(X \cup Z) \cap (M_+ \cup M_0)} \geq \omega((X \cup Z) \cap (Q_- \cup Q_0)) \tag{10}
\]

because the only edges in \(G\) to \(Q_- \cup Q_0\) are from \(M_+ \cup M_0\), and a resource in \(X \cup Z\) can receive a positive flow only from one in \(X \cup Z\).

Summing up inequalities (8, 9, 10), gives a contradiction. Hence \((X \cup Z) \cap M_+\) must be empty after all, i.e., \(M_+ \subseteq Y\). Now (10) becomes \(x_{(X \cup Z) \cap M_0} \geq \omega((X \cup Z) \cap (Q_- \cup Q_0))\), whereas (9) gives \(x_{(X \cup Z) \cap M_0} \leq \omega((X \cup Z) \cap Q_0)\). Whether \((X \cup Z) \cap M_0\) is empty or not, this implies \(\omega((X \cup Z) \cap Q_-) = 0\), i.e., \(Q_- \subseteq Y\) as announced.

We derive a few more facts. First, all inequalities in (9) must be equalities, so \(x = p\) in \((X \cup Z) \cap M_0\). Second, a positive flow to \((X \cup Z) \cap Q_0\) can’t come from \(M_-\) (statement \(i\) Lemma 2), or from \(M_+\) (because \(M_+ \subseteq Y\)), therefore it only comes from \((X \cup Z) \cap M_0\). We showed in the previous paragraph \(x_{(X \cup Z) \cap M_0} = \omega((X \cup Z) \cap Q_0)\). Therefore the entire flow from \((X \cup Z) \cap M_0\) goes to \((X \cup Z) \cap Q_0\).

**Step 2.3** We focus finally on \((Y \cup Z) \cap M_0\). The flow from \((Y \cup Z) \cap M_0\) cannot go to \(Q_-\) \((G(M_0, Q_-) = \emptyset)\), or to \(Q_+\) (contained in \(X\)), or to \(X \cap M_0\), so it must end up in \((Y \cup Z) \cap Q_0\). Together with the last statement in step 2.2, this shows that the entire flow from \(M_0\) goes to \(Q_0\). Assume for a moment that \(x = p\) on \(M_0\): because \((G(M_0, Q_0), \omega, p)\) is balanced, this implies that there is no other flow coming to \(Q_0\), in particular there is no flow on \(G(M_+, Q_0)\), or \(G(M_0, Q_+)\). Finally, as \(Q_+ \subseteq X\) and \(M_+ \subseteq Y\), there is no flow on \(G(M_+, Q_+)\) either, which completes the proof of statements \(iii\) and \(iv\).

It remains to show \(x = p\) on \((Y \cup Z) \cap M_0\) (we already know it is true on \((X \cup Z) \cap M_0\) from step 2.2). We have

\[
p_{(Y \cup Z) \cap M_0} \leq x_{(Y \cup Z) \cap M_0} \leq \omega((Y \cup Z) \cap Q_0) \tag{11}
\]

The first inequality because the nodes in \(Y \cup Z\) are red or black, the second one because we saw above that the flow from \((Y \cup Z) \cap M_0\) goes to \((Y \cup Z) \cap Q_0\). Next we have

\[
\omega((Y \cup Z) \cap Q_0) \leq p_{(Y \cup Z) \cap Q_0} \cap M_0 \leq p_{(Y \cup Z) \cap M_0} \tag{12}
\]

the first one because \((G(M_0, Q_0), \omega, p)\) is balanced, the second one because there is no edge from \(X \cap M_0\) to \(Y \cup Z\). Together, (11) and (12) imply \(x = p\) on \((Y \cup Z) \cap M_0\), as desired.

We illustrate Proposition 1 by two examples, each with four agents and four resources.

In Figure 3, the peaks are \(p = (10, 10, 5, 10)\) and the resources \(\omega = (11, 10, 8, 9)\). The two dashedline boxes show the GE decomposition: resources \(r, s, t\), are under-demanded by \(A, B, C\), \(u\) is overdemanded by \(D\) \((M_0, Q_0)\) is absent). Most of the inequalities in the system (4),(5),(6) are redundant, and the Pareto Optimal set is given by

\[
x_A + x_B + x_C = 29, x_D = 9
\]
In Figure 4, we use the same profile of peaks and resources as in the previous example, but the feasibility constraints have changed. The GE decomposition has now agent resources \( r, s \) underdemanded by \( A \), while \( t, u \) are overdemanded by \( B, C, D \) (\( (M_0, Q_0) \) is absent). Note that the graph \( G(M_-, Q_+) \) is disconnected. The system (4),(5),(6) reduces to

\[
x_A = 21, \quad x_B + x_C = 8, \quad x_C \leq 5, \quad x_D = 9
\]

5 The Egalitarian solution

In this section we give an algorithmic definition of the Egalitarian solution, then characterize it as the Lorenz dominant element of the Pareto set.

Given an economy \((G, \omega, p)\), we write our solution as \( E(G, \omega, p) \) and define it separately on \( M_+ \) and on \( M_- \). It is a selection from, respectively \( A(G(M_+, Q_-), \omega) \) (see (4)), and \( A(G(M_-, Q_+), \omega) \) (see (6)). By Pareto Optimality ((5)) \( E(G, \omega, p) = p \) on \( M_0 \).

**Computing** \( E(G, \omega, p) \) **in** \( M_+ \): We use an *ascending* algorithm based on the following system \( \Theta(\lambda) \) of inequalities, where \( \lambda \) is a non negative parameter:

\[
\Theta(\lambda) : \gamma_S(\lambda) \leq \omega_f(S) \cap Q_- \text{ for all } S \subseteq M_+
\]

where for all \( i \in M_+ \), \( \gamma_i(\lambda) = \max\{\lambda, p_i\} \), so that \( p \leq \gamma(\lambda) \) for all \( \lambda \).

For \( \lambda = 0, \gamma(0) = p \) and \( \Theta(0) \) holds true, even strictly, because there is underdemand in \( (G(M_+, Q_-), \omega, p) \) (Lemma 2). For \( \lambda = \infty, \gamma(\infty) = \infty \). Therefore there is a smallest number \( \lambda^1 \), strictly positive, such that one of the inequalities in \( \Theta(\lambda^1) \) is tight. As \( S \to \omega_f(S) \cap Q_- - \gamma_S(\lambda^1) \) is
submodular, the equality \( \gamma_S(\lambda^1) = \omega_{f(S) \cap Q_-} \) is stable by union and intersection of the sets \( S \). We call \( S^1 \) the largest such subset. By statement \( ii \) in Lemma 1 applied to \( G(S^1, f(S^1) \cap Q_-) \), the (restricted) allocation \( \gamma_i(\lambda^1) \) for the agents in \( S^1 \) is feasible by using all the resources in \( f(S^1) \cap Q_- \) and no more.

In the restricted problem \( (G(M_+ \setminus S^1, Q_- \setminus f(S^1)), \omega) \) the bilateral graph is described by \( f^1(S) = (f(S) \setminus f(S^1)) \cap Q_- \). We claim \( \gamma_S(\lambda^1) < \omega_{f^1(S)} \) for all non-empty \( S \subseteq M_+ \setminus S^1 \). Indeed, \( \Theta(\lambda^1) \) is true and \( S^1 \) is the largest set such that the corresponding inequality is tight, therefore \( \gamma_{S \cup S^1}(\lambda^1) < \omega_{f(S \cup S^1) \cap Q_-} \iff \gamma_S(\lambda^1) + \gamma_{S^1}(\lambda^1) < \omega_{f^1(S)} + \omega_{f(S^1) \cap Q_-} \).

Repeating the argument above, there is a smallest number \( \lambda^2 \), strictly above \( \lambda^1 \), at which one of the inequalities \( \gamma_S(\lambda) < \omega_{f^1(S)} \), \( S \subseteq M_+ \setminus S^1 \), becomes an equality. We call \( S^2 \) the largest such subset of \( M_+ \setminus S^1 \). The allocation \( \gamma_i(\lambda^2) \) for the agents in \( S^2 \) is achievable by using precisely all the resources in \( f(S^2) \setminus f(S^1) \) (Lemma 1).

Continuing in this fashion, we obtain a partition \( S^1, S^2, \ldots, S^K \), of \( M_+ \), and a strictly increasing sequence \( \lambda^1 < \lambda^2 < \cdots < \lambda^K \), such that for all \( k, 1 \leq k \leq K \), the allocation \( \gamma_i(\lambda^k) \) to the agents in \( S^k \) is feasible by assigning the resources in \( f(S^k) \setminus f(S^1 \cup \cdots \cup S^{k-1}) \) to these agents. By construction this allocation is bounded below by \( p \).

Computing \( \mathcal{E}(G, \omega, p) \) in \( M_- \): Turning to the agents in \( M_- \), we use a descending algorithm based on the system \( \Xi(\mu) \) with non-negative parameter \( \mu \):

\[
\Xi(\mu) : \quad \omega_T \leq \delta_{g(T) \cap M_-}(\mu) \quad \text{for all} \quad T \subseteq Q_+ \tag{14}
\]

where for all \( i \in M_- \), \( \delta_i(\mu) = \min\{\mu, p_i\} \), so that \( \delta(\mu) \leq p \) for all \( \mu \).

We have \( \delta(\infty) = p \), so \( \Xi(\infty) \) is true, even strictly, because there is overdemand in \( (G(M_-, Q_+), \omega, p) \) (Lemma 2). Also, \( \delta(0) = 0 \), therefore there is a largest number \( \mu^1 \) such that one of the inequalities in \( \Xi(\mu^1) \) is tight. We let \( T^1 \) be the largest subset of \( Q_+ \) for which we have an equality (its existence guaranteed by the submodularity of \( T \to \delta_{g(T) \cap M_-}(\mu^1) - \omega_T \)). The allocation \( \delta_i(\mu^1) \) to the agents of \( g(T^1) \cap M_- \) is feasible by using exactly the resources in \( T^1 \) (statement \( iii \) in

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Lemma 1 applied to $G(g(T^1) \cap M_-, T^1))$. We repeat this construction in the restricted problem $(G(M_- \setminus g(T^1), Q_+ \setminus T^1), \omega)$, etc.

We end up with a partition $T^1, \ldots, T^L$ of $Q_+$, and a strictly decreasing sequence $\mu^1 > \cdots > \mu^L$, such that for all $l, 1 \leq l \leq L$, the allocation $\delta_l(\mu^l)$ to the agents in $g(T^l) \setminus g(T^1 \cup \cdots \cup T^{l-1})$ is feasible by assigning exactly the resources in $T^l$ to these agents. By construction this allocation is bounded above by $p$. This concludes the definition of $E(G, \omega, p)$.

By Proposition 1, $E(G, \omega, p)$ is Pareto Optimal. It is also Peak-Only.

The main normative property of our solution relies on the concept of Lorenz dominance. For any finite set $N$ and any $z \in \mathbb{R}^N$, denote by $z^*$ the order statistics of $z$, obtained by rearranging the coordinates of $z$ in increasing order: $z^{*1} \leq z^{*2} \leq \cdots \leq z^{*n}$. Say that $z$ Lorenz dominates $w$, written $z LD w$, if for all $k, 1 \leq k \leq n$

$$\sum_{t=1}^{k} z^{*t} \geq \sum_{t=1}^{k} w^{*t}$$

Finally $z$ is Lorenz dominant in the set $A$ if $z LD z'$ for all $z' \in A$. Lorenz dominance is a partial ordering: not every set, even convex and compact, admits a Lorenz dominant element. On the other hand, in a convex set $A$ there can be at most one Lorenz dominant element.

**Proposition 2:** For any economy $(G, \omega, R)$ with peaks $p$, the Egalitarian solution $E(G, \omega, p)$ is the Lorenz dominant Pareto Optimal allocation:

$$E(G, \omega, p) \text{ LD } x \text{ for all } x \in \mathcal{PO}(G, \omega, p)$$

**Proof of Proposition 2:** We set $x = E(G, \omega, p)$. By Proposition 1, we need to prove two statements: the restriction of $x$ to $M_+$ (resp. $M_-$) is Lorenz dominant within $A(G(M_+, Q_-), \omega)$ (resp. $A(G(M_-, Q_+), \omega)$). We prove each statement in turn.

**Step 1** We write $x^+$ for the restriction of $x$ to $M_+$, and show it is Lorenz dominant in $A(G(M_+, Q_-), \omega)$. Recall that $M_+$ is partitioned by $S^1, \ldots, S^K$, such that for all $k$, $\gamma_k(\lambda^k) = \omega_{f(S^k)\setminus f(S^1\cup \cdots \cup S^{k-1})}$, and in $S^k$ we have $x^+_i = \max\{\lambda^k, p_i\}$. Moreover $\lambda^k$ is strictly increasing in $k$. We further partition $S^k$ as follows

$$A^k = \{i \in S^k | x^+_i > p_i\}; \quad B^k = \{i \in S^k | x^+_i = p_i\}$$

Note that agent $i \in S^k$ is in $A^k$ iff $\lambda^k > p_i$; $i$ is in $B^k$ iff $\lambda^k \leq p_i$. We check first that $A^k$ is non empty for all $k$. By Lemma 2 ii)

$$p_{S^1} < \omega_{f(S^1)} = \sum_{S^1} \max\{\lambda^1, p_i\}$$

so $A^1$ is non empty. Next

$$p_{S^2} \leq \gamma_{S^2}^+(\lambda^1) < \omega_{f(S^2)\setminus f(S^1)} = \sum_{S^2} \max\{\lambda^2, p_i\}$$

where the strict inequality is explained in the construction of $E(p)$. So $A^2$ is non empty. And so on.

Now we label the agents in $M_+$ as $\{1, 2, \cdots, m_+\}$ in such a way that $x^+_i$ is weakly increasing in $i$, and moreover
• the first $|A^1|$ terms cover $A^1$
• the next terms cover a possibly empty subset $\tilde{B}^1$ of $B^1$
• the next $|A^2|$ terms cover $A^2$
• the next terms cover a possibly empty subset $\tilde{B}^2$ of $B^1 \cup B^2$

and so on. This is possible because in $A^k$ agent $i$ gets $\lambda^k$, so in $A^1 \cup \cdots \cup A^{k-1}$ no one receives more than $\lambda^{k-1}$; on the other hand, in $B^{k'}$, $k' \geq k$, every agent receives no less than $\lambda^k$; and the sequence $\lambda^k$ increases strictly.

We fix $y \in A(G(M_+, Q_-), \omega)$ and check that it is Lorenz dominated by $x^+$. We use the notation $y^*(i) = \sum_{j=1}^i y^j$. We have $y_S \geq y^*(|S|)$ for all $S$, and if $S \subseteq M_+$ is such that $y_S = y^*(|S|)$ we say that $S$ is a $y$-tail. Note that our labeling of $M_+$ implies that any subset of the form $\{1, \cdots, i\}$ is an $x^+$-tail.

By feasibility (Lemma 1) $y_{|S|} \leq \omega_f(\{S\}) = x^+_S$ and on the other hand $y \geq x^+$ in $B^1$. Therefore

$$y_S \leq x^+_S \text{ for all } S \text{ such that } A^1 \subseteq S \subseteq A^1 \cup \tilde{B}^1$$

When the above $S$ takes the form $\{1, \cdots, i\}$, it is an $x^+$-tail, hence we have $x^+ = y_S \geq y^*(|S|)$ Next we note that $\frac{y^*(i)}{i}$ increases weakly in $i$, so that for $i \leq |A^1|$ we have

$$\frac{y^*(i)}{i} \leq \frac{y^*(|A^1|)}{|A^1|} \leq \frac{y_{A^1}}{|A^1|} \leq \frac{x^+|A^1|}{i}$$

where the equality is because $x^+$ is egalitarian in $A^1$. We have proved the desired inequality $y^*(i) = x^+(i)$ up to $i = |A^1 \cup \tilde{B}^1|$.

Now consider $S^2$: feasibility implies $y_{S^1 \cup S^2} \leq \omega_f(\{S^1 \cup S^2\}) = x^+_S$ and on the other hand $y \geq x^+$ in $B^1 \cup B^2$. Therefore $y_S \leq x^+_S$ for any $S$ such that $S \subseteq S^1 \cup S^2$ and $S^1 \cup S^2 \setminus S \subseteq B^1 \cup B^2$. In particular

$$y_S \leq x^+_S \text{ for all } S \text{ such that } A^1 \cup \tilde{B}^1 \cup A^2 \subseteq S \subseteq A^1 \cup \tilde{B}^1 \cup A^2 \cup \tilde{B}^2$$

Again such a set $S$ is an $x^+$-tail if of the form $\{1, \cdots, i\}$, so the inequality $y^*(i) \leq x^+(i)$ follows as above for $i$ such that $|A^1 \cup \tilde{B}^1 \cup A^2| \leq i \leq |A^1 \cup \tilde{B}^1 \cup A^2 \cup B^2|$. For $i = |A^1 \cup \tilde{B}^1| + a \leq |A^1 \cup \tilde{B}^1 \cup A^2|$, we pick $S$ such that $A^1 \cup \tilde{B}^1 \subseteq S \subseteq A^1 \cup \tilde{B}^1 \cup A^2$, with $|S| = i$. Because $x^+$ is egalitarian in $A^2$, we have

$$x^+(i) = x^+_{A^1 \cup \tilde{B}^1} + \frac{a}{|A^2|} x^+_{A^1 \cup \tilde{B}^1} = (1 - \frac{a}{|A^2|}) x^+_{A^1 \cup \tilde{B}^1} + \frac{a}{|A^2|} x^+_{A^1 \cup \tilde{B}^1 \cup A^2}$$

We claim

$$y^*(i) \leq y_{A^1 \cup \tilde{B}^1} + \frac{a}{|A^2|} y_{A^1 \cup \tilde{B}^1} = (1 - \frac{a}{|A^2|}) y_{A^1 \cup \tilde{B}^1} + \frac{a}{|A^2|} y_{A^1 \cup \tilde{B}^1 \cup A^2}$$

which will imply $y^*(i) \leq x^+(i)$ because $y_S \leq x^+_S$ is true both for $A^1 \cup \tilde{B}^1$ and $A^1 \cup \tilde{B}^1 \cup A^2$. Observe that if $X, Y, Z$ are three disjoint subsets, we have

$$y^*(|X| + |Y|) \leq y_X + \frac{|Y|}{|X| + |Z|} y_{Y \cup Z}$$

Indeed $\frac{|Y|}{|X| + |Z|} y_{Y \cup Z}$ is no less than the sum of the $|Y|$ lowest terms in the $Y \cup Z$-coordinates of $y$, and $y_X$ is no less than the sum of the $|X|$ lowest terms in its $X$-coordinates. Applying the claim to $X = A^1 \cup \tilde{B}^1, Y = S \setminus (A^1 \cup \tilde{B}^1)$ and $Z = (A^1 \cup \tilde{B}^1 \cup A^2) \setminus S$ gives (17).
Step 2 We show that $x^-$, the restriction of $x$ to $M_-$, is Lorenz dominant in $A(G(M_-, Q_+), \omega)$. Recall that $Q_+$ is partitioned as $T^1, \ldots, T^L$, such that the resources of $T^1$ are entirely assigned to agents in $S^1 = g(T^1) \setminus g(T^1 \cup \cdots \cup T^{l-1})$, and $\omega_T = \delta_S(\mu^l)$ for all $k$, where $\mu^l$ is strictly decreasing; moreover $x^-_i = \min\{\mu_i, p_i\}$ for $i \in S^l$. As in Step 1 we partition $S^l$ as follows

$$A^l = \{i \in S^l | x^-_i < p_i\}; \quad B^l = \{i \in S^l | x^-_i = p_i\}$$

so that an agent $i \in S^l$ is in $A^l$ iff $\mu_i < p_i$, while $i$ is in $B^l$ iff $\mu_i \geq p_i$.

The set $A^1$ is non empty because $\sum_{S^1} \min\{\mu_i, p_i\} = \omega_{T^1} < p_{S^1}$; $A^2$ is non empty because $p_{S^2} \geq \delta_{S^2}(\mu^1) > \omega_{T^2} = \sum_{S^2} \min\{\mu^2, p_i\}$ and so on. We label the agents in $M_-$ so that $x^-_1$ is weakly increasing in $i$, and moreover in the sequence $\{m_-, m_--1, \ldots, 1\}$ (with corresponding allocations weakly decreasing), we have

- the first $|A^1|$ terms cover $A^1$
- the next terms cover a possibly empty subset $B^1$ of $B^1$
- the next $|A^2|$ terms cover $A^2$
- the next terms cover a possibly empty subset $B^2$ of $B^1 \cup B^2$

and so on. This is possible because a coordinate in $A^l$ receives $\mu^l$ and one in $B^l$ no more than $\mu^l$; thus before $A^l$ (in $A^1 \cup \cdots \cup A^{l-1}$) no one gets less than $\mu^{l-1}$ while in $B^l$, $l' \geq l$, everyone gets at most $\mu^{l'}$.

For an arbitrary $y \in A(G(M_-, Q_+), \omega)$ we use the notation $y^s(i) = \sum_{j=m_-}^{m_- - i + 1} y^s$, so that $y_S \leq y^s(|S|)$ for all $S$. The end of the proof that $y$ is Lorenz dominated by $x^-$ is entirely similar to the one in step 1, upon reversing the direction of inequalities. That is, the feasibility constraints $\omega_T \leq y_{g(T)}$ imply now $y_{S^1} \geq \omega_{T^1} = x^-_{S^1}$; on the other hand $y \leq x^-$ in $B^1$, and so (15) follows (up to a change of sign). Similarly the inequality $y_{S^1 \cup S^2} \geq \omega_{T^1 \cup T^2} = x^-_{S^1 \cup S^2}$ holds by feasibility of $y$.

We omit the details.

We illustrate the computation of the Egalitarian solution with a few examples.

No bilateral constraints: Here $G$ is the complete graph $G = M \times Q$. As mentioned already after Lemma 2, the model is equivalent to Sprumont’s one-resource model. Suppose first we have overdemand, $\omega_Q < p_M$. Then $M = M_-, Q = Q_+$, and $g(T) = M$ for all $T$. Therefore system (14) reads

$$\omega_T \leq \sum_{i \in M} \min\{\mu_i, p_i\} \text{ for all } T$$

When $\mu$ is the largest number such that this is an equality for some $T$, it must be an equality for $Q$, therefore $\mu^1$ solves $\sum_{i \in M} \min\{\mu_i, p_i\} = \omega_Q$ and the algorithm stops after one step.

When the resources are underdemanded, $\omega_Q > p_M$, the algorithm stops similarly in one step: the number $\lambda^1$ solves $\sum_{i \in M} \max\{\lambda, p_i\} = \omega_Q$, and $S^1 = M$.

Want all or nothing: In the one-resource model, the Egalitarian solution when all peaks are zero (resources are all “bad”), is the same as when all are infinite (resources are all “good”), namely it
divides the resource equally. This is still true in the bipartite model. By Proposition 1, when \( p = 0 \) we have \( M = M_+, Q = Q_- \), and \( \mathcal{PO}(G, \omega, 0) = \mathcal{A}(G, \omega) \). Similarly \( M = M_-, Q = Q_+ \) when \( p = \infty \), and \( \mathcal{PO}(G, \omega, \infty) = \mathcal{A}(G, \omega) \). By Proposition 2, the egalitarian solution picks the Lorenz dominant feasible allocation in both problems. Note that it is also the Dutta-Ray egalitarian solution ([9]) of the supermodular game \((M, w)\), and of its dual game \((M, v)\).

In the example of Figure 2 (just before Proposition 1), Pareto Optimal allocations assign the 14 units of overdemanded resources \( t, u \) to agents \( C, D \), and the 28 units of underdemanded \( r, s \) to \( A, B \). The Egalitarian solution is \( x_A = 13, x_B = 15 \), \( x_C = 9, x_D = 5 \).

In the example of Figure 3 (just after Proposition 1), Pareto Optimal allocations distribute 29 units to \( A, B, C \), with peaks 11, 10, 5. This amounts to a one-resource problem, and the Egalitarian solution is \( x_A = 11, x_B = 10, x_C = 8 \); and \( x_D = \)

In the example of Figure 4, we only need to divide 8 units between \( B, C \). Full equality \( x_B = x_C = 4 \) is feasible and stays below both peaks, so it is the Egalitarian solution.

6 Other properties of the egalitarian rule

**Definition:** Given the problem \((G, \omega)\), a rule selects for every preference profile \( R \in \mathcal{R}^M \) a feasible allocation \( \psi(R) \in \mathcal{A}(G, \omega) \). We write \( \mathcal{E} \) for the Egalitarian rule.

We start with the familiar equity test of No Envy, that must be adapted to our model because of the feasibility constraints. In a classic fair division problem, individual shares can always be exchanged between two agents, say agents 1 and 2, without affecting other agents’ shares. This is not true in our model. First, the bilateral constraints may prevent us from exchanging \( x_1 \) and \( x_2 \). But more importantly, even if this exchange is possible, it may require to alter the allocation of agents other than 1 and 2. We postulate that agent 1’s envy of agent 2’s allocation is legitimate only if it is feasible to improve upon agent 1’s allocation without altering the allocation of anyone other than agent 2.

**No Envy:** A rule \( \psi \) satisfies No Envy if for each \( R \in \mathcal{R}^M \) and \( i, j \in M \) such that \( \psi_j(R)P_i\psi_i(R) \), there exists no \( x \in \mathcal{A}(G, \omega) \) such that

\[
\psi_k(R) = x_k \text{ for each } k \neq i, j \text{ and } x_iP_i\psi_i(R)
\] (18)

In the example of Figure 3, consider the Pareto Optimal allocation \( (x_A, x_B, x_C) = (11, 11, 7) \), where the burden of overdemand is shared by all 3 agents in \( M_+ \). If \( 7P_B11 \) (recall that the peak of \( R_B \) is at 10), \( B \) envies \( C \), and this is legitimate because \( (x_B, x_C) = (10, 8) \) is feasible and improves upon \( B \) while affecting only \( C \). At the Egalitarian allocation \( (x_A, x_B, x_C) = (11, 10, 8) \), \( A \) envies \( B \) but no feasible allocation allows a lower load for \( A \).

The basic horizontal equity property Equal Treatment of Equals must be similarly adapted to take feasibility constraints into account.

**Equal Treatment of Equals:** A rule \( \psi \) treats equals equally if for each \( R \in \mathcal{R}^M \) and \( \{i, j\} \subseteq M \) such that \( R_i = R_j \), if \( \psi_j(R) \neq \psi_i(R) \) there exists no \( x \in \mathcal{A}(G, \omega) \) such that

\[
\psi_k(R) = x_k \text{ for each } k \neq i, j \text{ and } |x_i - x_j| < |\psi_i(R) - \psi_j(R)|
\] (19)
The property says that equalizing transfers among agents with identical preferences are legitimate only if they do not disrupt others agents’ allocations. This egalitarian requirement is restricted to agents with identical preferences, so it is as weak as in the one-resource mode. For instance consider the rule operating as a serial dictatorship where agent 1 is served first, agent 2 next and so on, at each preference profile where the preferences of all agents differ; and selecting the Egalitarian allocation whenever some two agents exhibit the same preferences. This rule treats equals equally (and is also Pareto optimal).

**Proposition 3:**

i) Pareto Optimality and No Envy imply Equal Treatment of Equals.

ii) The Egalitarian rule $E$ satisfies No Envy.

**Proof:** We prove both statements in turn.

*Statement i*) Suppose the rule $\psi$ satisfies (18), contradiction. Write $y = \psi(R)$ and assume without loss of generality $y_1 < y_2$. Distinguish two cases. If 1 and 2’s common peak $p$ is in $[y_1, y_2]$, then for $\varepsilon$ small enough the allocation $x = (1 - \varepsilon)y + \varepsilon x$, feasible by convexity of $\mathcal{A}(G, \omega)$, is Pareto superior to $y$. If $p \leq y_1 < y_2$, then 2 envies 1, and the allocation $x$ satisfies (18), contradiction.

*Statement ii*) Let $R$ be a profile at which 1 envies 2 via allocation $x$. From $x_1 + x_2 = \mathcal{E}_2(R) + \mathcal{E}_1(R)$ and the fact that $\mathcal{E}(R)$ Lorenz dominates $x$ we must have $|x_1 - x_2| > |\mathcal{E}_2(R) - \mathcal{E}_1(R)| > 0$. If $x_1 - x_2$ and $\mathcal{E}_2(R) - \mathcal{E}_1(R)$ have the same sign, then we have $x_1 < \mathcal{E}_1(R) < \mathcal{E}_2(R) < x_2$ (or a symmetric condition by exchanging 1 and 2). Now $\mathcal{E}_2(R) P_1 \mathcal{E}_1(R)$ implies $p_1 > \mathcal{E}_1(R)$, hence $\mathcal{E}_1(R) P_1 x_1$, contradiction. If $x_1 - x_2$ and $\mathcal{E}_2(R) - \mathcal{E}_1(R)$ have opposite signs, convexity of $\mathcal{A}(G, \omega, c)$ implies that the allocation $x’$, $x’_1 = x’_2 = \frac{1}{2}(x_1 + x_2)$, $x’_k = \mathcal{E}_k(R)$ otherwise, is feasible. But $x’$ Lorenz dominates $\mathcal{E}(R)$, a contradiction.$\blacksquare$

We turn to Strategyproofness. We decompose it into a monotonicity and an invariance condition, as in [7]. For clarity, we go back to the notation $p[R_i]$ for the peak of $R_i$.

**Monotonicity:** A rule $\psi$ is monotonic if for all $R \in \mathcal{R}^M$, $i \in M$, and $R_i' \in \mathcal{R}_i$

$$p[R_i] \leq p[R_i] \Rightarrow \psi_i(R_i', R_{-i}) \leq \psi_i(R)$$

Note that Monotonicity implies own-peak-only, namely $p[R_i'] = p[R_i] \Rightarrow \psi_i(R_i', R_{-i}) = \psi_i(R)$: my allocation depends only upon the peak of my preferences.

**Invariance:** A rule $\psi$ is invariant if for all $R \in \mathcal{R}^M$, $i \in M$, and $R_i' \in \mathcal{R}_i$

$$\{p[R_i] < \psi_i(R) \text{ and } p[R_i'] \leq \psi_i(R)\} \Rightarrow \psi_i(R_i', R_{-i}) = \psi_i(R) \quad (20)$$

$$\{p[R_i] > \psi_i(R) \text{ and } p[R_i'] \geq \psi_i(R)\} \Rightarrow \psi_i(R_i', R_{-i}) = \psi_i(R) \quad (21)$$

**Strategyproofness:** A rule $\psi$ is Strategyproof if for all $R \in \mathcal{R}^M$, $i \in M$, and $R_i' \in \mathcal{R}_i$

$$\psi_i(R) R_i \psi_i(R_i', R_{-i})$$

The next Lemma connects these three properties and Pareto Optimality.

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Lemma 3:
i) If a rule is Monotonic and Invariant, it is Strategyproof;
ii) A Pareto Optimal and Strategyproof rule is Monotonic and Invariant.

Proof: Statement i) is proven just as in the one-resource model (see [7]).

Statement ii) Fix a Pareto Optimal and Strategyproof rule \( \psi \). We show first that the mapping \( R_i \rightarrow \psi_i(R_i, R_{-i}) \) is peak-only. Fix \( R_{-i} \) and consider two preferences \( R_i, R'_i \) such that \( p[R_i] = p[R'_i] \). The GE decomposition (Lemma 2) is the same in \( R \) and \( (R'_i, R_{-i}) \), so by Pareto Optimality agent \( i \)'s allocations \( x_i \) and \( x'_i \) are on the same side of \( p[R_i] \). Now Strategyproofness implies Peak-Only.

Next we prove Monotonicity. We fix \( i, R \) and \( R'_i \) such that \( p'_i = p[R'_i] \leq p[R_i] = p_i \), and let \( p, p' \) be the profile of peaks at \( R \) and \( (R'_i, R_{-i}) \) respectively. We also set \( x_i = \psi_i(R), x'_i = \psi_i(R'_i, R_{-i}) \). We distinguish two cases.

Case 1: \( i \in M_-(p) \). Assume first \( p'_i > x_i \). Then the decomposition is unchanged, in particular \( M_-(p) = M_-(p') \), so by Pareto Optimality \( x'_i \leq p'_i \). Assume \( x_i < x'_i \); then we have \( x_i < x'_i \leq p'_i \leq p_i \), and we get a contradiction of Strategyproofness for agent \( i \) at profile \( R \). Assume next \( p'_i \leq x_i \). Then \( x_i < x'_i \) would give \( p'_i \leq x_i < x'_i \), a violation of Strategyproofness for agent \( i \) with preference \( R'_i \).

Case 2: \( i \in (M_0 \cup M_+)(p) \). Then \( p_i \leq x_i \), so \( x_i < x'_i \) would give \( p'_i \leq p_i \leq x_i < x'_i \), hence a violation of Strategyproofness for agent \( i \) at \( R'_i \).

We show finally that \( \psi \) is Invariant. Under the premises of property (20), if \( \psi_i(R'_i, R_{-i}) > \psi_i(R) \) we have \( p[R'_i] \leq \psi_i(R) < \psi_i(R'_i, R_{-i}) \), hence a violation of Strategyproofness for agent \( i \) at \( R'_i \). If \( \psi_i(R'_i, R_{-i}) < \psi_i(R) \) we can find a preference \( R'^*_i \) such that \( p[R'^*_i] = p_i[R_i] \) and \( \psi_i(R'_i, R_{-i})P^*_i\psi_i(R) \). By Peak-Only, \( \psi_i(R'_i, R_{-i}) = \psi_i(R) \), so agent \( i \) with preferences \( R'^*_i \) benefits by reporting \( R'_i \). The proof of the second property is identical.

Proposition 4: The Egalitarian rule is Monotonic and Invariant, hence Strategyproof as well.

Proof: We fix \( (G, \omega, c) \), an agent \( i \) and a benchmark profile of peaks \( p \), with corresponding Egalitarian allocation \( x \). We consider a change of peak by agent \( i \) only, to \( p'_i \), and we write \( p'_j = p_j \) for all \( j \neq i \), so that \( p' = (p'_i, p_{-i}) \). The notation \( M_+(p), M_-(p') \) stands for this component of the GE decomposition (Lemma 2) at \( p, p' \), etc...

Step 1 We prove Monotonicity for shifts in \( p_i \) inside \( M_+(p) \) or inside \( M_-(p) \).

Step 1a Consider a change of peak from \( p_i \) to \( p'_i \) such that \( i \in M_+(p) = M_+(p') \).

Suppose first \( p'_i < p_i \). We show \( x'_i \leq x_i \) by distinguishing two cases. Write in both cases \( S^k, \lambda^k \) for the partition and corresponding parameters of the ascending algorithm at \( p \), and let \( i \in S^k, x_i = \lambda^k \cup p_i \).

First case: \( p_i < \lambda^k \). Then the partition and corresponding parameters are unchanged at \( p' \) so that \( x'_i = x_i \).

Second case: \( p_i = x_i \geq \lambda^k \). Then \( S^k, \lambda^k \) are unchanged for \( 1 \leq k \leq \ell - 1 \), but \( S^\ell, \lambda^\ell \) may change. However for \( \lambda = p_i \) we have

\[
\sum_{S^\ell} \lambda \cup p'_j \geq \sum_{S^\ell} \lambda^\ell \cup p'_j = \omega T^i
\]

where we set \( T^i = f(S^i) \setminus f(S^1 \cup \cdots \cup S^{i-1}) \). Therefore if \( S^\ell \) changes, the new set \( \tilde{T}^i \) contains \( i \) and \( \lambda^\ell \leq p_i \), hence \( x'_i \leq p_i = x_i \).

Suppose next, until the end of step 1a, \( p_i < p'_i \). If \( p'_i \geq x_i \) notice that \( i \in M_+(p') \) implies \( x'_i \geq p'_i \), so we are done. We are left with the case \( p_i < p'_i < x_i = \lambda^\ell \), that requires more work. We
prove by induction on \( \ell \) that the first \( \ell \) terms \( S_k^k, 1 \leq k \leq \ell \), of the partition and corresponding parameters are unchanged at \( p' \). We write \( \bar{S}_k^k, \bar{\lambda}_k^k \) for the latter.

Suppose \( \ell = 1 \), then \( \sum_{i \in S} \mu_1 \lor p_j = \sum_{j \in S} \mu_1 \lor p'_j \) for all \( S \subseteq M_+(p) \), so the claim holds.

Next suppose \( \ell \geq 2 \). Assume \( S^1 \neq \bar{S}^1 \) and derive a contradiction. This implies there exists a coalition \( S \subseteq M_+(p) \) such that \( S \not\subseteq S^1 \) and

\[
\sum_{j \in S} \lambda^1 \lor p_j \geq \omega_f(S) \tag{22}
\]

Indeed suppose (22) fails for all \( S \not\subseteq S^1 \): as \( p \) and \( p' \) coincide inside \( S^1 \), we would get \( S^1 = \bar{S}^1 \). Fix a coalition \( S \) as in (22), that must contain \( i \), hence \( S \cap S^\ell \) is non empty. By definition of the ascending algorithm, the sets \( T^1 = f(S \cap S^1), \ldots, T^k = f(S \cap S^k) \setminus (T^1 \cup \cdots \cup T^{k-1}), \ldots \), are pairwise disjoint and \( \sum_k S \cap S^k \lambda^k \lor p_j \leq \omega_Tk \) for all \( k \), therefore

\[
\sum_{1 \leq k \leq K} \left[ \sum_{S \cap S^k} \lambda^k \lor p_j \right] \leq \sum_{j \in S} \lambda^1 \lor p'_j \]

In view of (22), we get

\[
\sum_{1 \leq k \leq K} \left[ \sum_{S \cap S^k} \lambda^k \lor p_j \right] \leq \sum_{j \in S} \lambda^1 \lor p'_j
\]

For all \( k \neq \ell \), we have \( \lambda^k \geq \lambda^1 \) and \( p = p' \in S \cap S^k \), implying \( \sum_{S \cap S^k} \lambda^k \lor p_j \geq \sum_{S \cap S^k} \lambda^1 \lor p'_j \). As \( \lambda^k \) is larger than \( \lambda^1, p'_i \), and \( p_i \), and \( S \cap S^\ell \) is non emty, we get \( \sum_{S \cap S^\ell} \mu^\ell \lor p_j \geq \sum_{S \cap S^\ell} \mu^1 \lor p'_j \).

The desired contradiction follows and we conclude \( S^1 = \bar{S}^1 \).

To show next \( S^2 = \bar{S}^2 \), we replicate the above argument as follows. If \( \ell = 2 \), then \( \sum_{j \in S} \lambda^2 \lor p_j = \sum_{j \in S} \lambda^2 \lor p'_j \) for all \( S \subseteq M_+(p) \setminus S^1 \), because \( p_i, p'_i < \lambda^2 \), and the claim holds. If \( \ell \geq 3 \) and \( S^2 \neq \bar{S}^2 \), we can pick a coalition \( S \subseteq M_+(p) \setminus S^1 \) such that \( S \not\subseteq S^2 \) and

\[
\sum_{j \in S} \lambda^2 \lor p'_j \geq \omega_f(S) \setminus S^1
\]

and proceed as above by decomposing \( S \) along \( S^k \), \( 2 \leq k \leq K \). The induction step is now clear.

**Step 1b** For a change of peak from \( p_i \) to \( p'_i \) such that \( i \in M_-(p) = M_-(p') \), the parallel argument is omitted for brevity.

**Step 2** We examine the critical peaks at which the GE decomposition change.

**Step 2a** Suppose \( i \in M_+(p) \). If \( p'_i < p_i \), the decomposition does not change, so \( i \in M_+(p') \). Consider the critical report \( p^*_i, p^*_i > p_i \), if any, at which the GE decomposition and the status of agent \( i \) change. By Lemma 2 ii), \( M_+(p) = M_+(p') \) as long as \( p_S^S < \omega_f(S) \cap Q_-(p) \) for all \( S \subseteq M_+(p) \).

Thus \( p^*_i \) is the smallest number such that

\[
p_{S \setminus i}^S + p^*_i = \omega_f(S \setminus Q_-(p)) \tag{23}
\]

for some subset \( S \) of \( M_+(p) \) containing \( i \). Let \( S^* \) be the largest \( S \) satisfying (23) (well defined by the usual submodularity argument). Recall from the proof of Lemma 2 that \( (M_\cup M_0)(p) \) is the largest solution of \( \arg \max_{S \subseteq M} \{ p_S - \omega_f(S) \} \). At \( p^* \) we have \( \max_S \{ p_S - \omega_f(S) \} = \max_S \{ p_S - \omega_f(S) \} \)
and the largest solution of \( \arg\max_{S \subseteq M} \{ p^*_S - \omega_f(S) \} \) is now \((M_- \cup M_0) \cup S^* = (M_- \cup M_0)(p^*)\); moreover \(M_-(p)\) is still a solution of \( \arg\max_{S \subseteq M} \{ p^*_S - \omega_f(S) \} \), therefore it is the smallest. So \(i \in M_0(p^*)\).

Now the restriction of \(x\) to \(M_+\) is in \(A(G(M_+,Q_-),\omega)\) and \(S^* \subseteq M_+(p)\); these two facts imply

\[
x_{S^*} \leq \omega_f(S^*) \cap Q_- (p)
\]

In \(M_+\) we have \(p \leq x\), so \(p_i^* < x_i\), would give \(p_{S^*}^* < x_{S^*}\), and a contradiction of (23) for \(S^*\). Therefore

\[
p_i^* \geq x_i \geq p_i
\]

(24)

**Step 2b** Suppose \(i \in M_-(p)\). By entirely symmetric arguments we can show that one of two cases arises. There is a critical peak \(p_i^*\) below \(p_i\) at which the decomposition changes for the first time. The details of the decomposition at \(p^*\) are similar and they only matter to prove \(i \in M_0(p^*)\) and

\[
p_i^* \leq x_i^* \leq p_i
\]

(25)

**Step 3** We consider now a move from \(p_i\) to \(p_i^*\) when \(i \in M_0(p)\). If \(p_i^* > p_i\), we have \(i \in M_-(p')\). Then in the downward shift starting at \(p_i^*\), \(p_i\) is the critical value (described in step 2b) at which the status of \(i\) changes, so by (25) \(x_i^* \geq p_i = x_i\) as desired. Symmetrically \(p_i^* < p_i\) gives \(i \in M_+(p')\) and \(p_i\) is the critical value starting from \(p_i^*\) described in step 1a, so (24) gives \(x_i^* \leq p_i = x_i\).

It remains to look at a shift from \(p_i\) to \(p_i^*\) such that \(i \in M_+(p)\) and \(i \in M_-(p')\). This requires \(p_i^* > p_i\); clearly the critical value \(p_i^*\) for \(p_i\) described in step 2a is the same as the critical value for \(p_i^*\) in step 2b. Therefore (24) and (25) imply

\[
p_i \leq x_i \leq p_i^* \leq x_i^* \leq p_i
\]

**Step 4** The invariance property is clear from (24) and (25) and the arguments of steps 1a and 1b.

### 7 Characterization result

Our characterization of the Egalitarian rule generalizes Ching’s characterization ([7]) of the Uniform rule in the one-resource model.

**Theorem 1:** The Egalitarian rule \(E\) is characterized by Pareto Optimality, Strategyproofness and Equal Treatment of Equals.

**Proof:** We fix \(G, \omega\) and a rule \(\psi\) meeting the three properties. We proceed in steps.

**Step 1** We fix in this step two partitions \(M_{+,\emptyset}\) of \(M\) and \(Q_{+,\emptyset}\) of \(Q\), that coincide with the GE decomposition for some profile of peaks \(p^5\). Then we choose a profile of preferences \(\tilde{R}\) with peaks \(\tilde{p}\) such that

\[
\tilde{R}_i = \tilde{R}_j \text{ if } i, j \in M_+ \text{ and if } i, j \in M_-
\]

\[
\tilde{p}_i = 0 \text{ if } i \in M_+; \quad \tilde{p}_j > \omega Q \text{ if } j \in M_-; \quad (G(M_0, Q_0), \omega, \tilde{p}) \text{ is balanced}
\]

\(^5\)It is easy to check from Lemma 2 that this is possible iff \(f(M_-) = Q_+, g(Q_-) = M_+, \text{ and } f^{-1}(Q_0) \supseteq M_0, g^{-1}(M_0) \supseteq Q_0\).
We show that \( \psi(\widetilde{R}) = \mathcal{E}(\widetilde{R}) \). Setting \( y = \psi(\widetilde{R}), x = \mathcal{E}(\widetilde{R}) \), by Proposition 2 it is enough to check that the \( M_+ \)-component of \( y \) (resp. its \( M_- \)-component) is Lorenz dominant in the corresponding component of \( \mathcal{PO}(G, \omega, \widetilde{R}) \).

As explained immediately after the statement of Proposition 1, the \( M_+ \)-component of \( \mathcal{PO}(G, \omega, \widetilde{R}) \) contains \( z \geq 0 \) iff \( z \) is in the upper core of the submodular game \((M_+, v^+)\), where

\[ v^+(S) = \omega(f(S) \cap Q_+ \right) \text{ for all } S \subseteq M_+ \right); \quad v^+(M_+) = \omega_{Q_+} \]

Similarly the \( M_- \)-component of \( \mathcal{PO}(G, \omega, \widetilde{R}) \) contains \( z \) iff \( z \) is in the lower core of the supermodular game \((M_-, w^-)\) where

\[ w^-(S) = \max\{\omega_T | T \subseteq Q_+ \right), g(T) \cap M_- \subseteq S \right) \text{ for all } S \subseteq M_- \right); \quad w^-(M_-) = \omega_{Q_-} \]

(note that, by our choice of \( \widetilde{p} \), the constraints \( z \leq \widetilde{p} \) are not binding).

We use only Equal Treatment of Equals and Pareto Optimality to show \( y = x \) on \( M_+ \). We omit for brevity the similar argument establishing this equality on \( M_- \). Set \( m_+ = |M_+| \right) and recall that \( y^{m_+} \geq y^{(m_+)-1} \geq \ldots \geq y^1 \) is the order statistics of \( y \).

**Claim 1** Fix an agent \( i_1 \in M_+ \right) , such that \( y_{i_1} = y^{m_+} \right) ; \text{ then} \]

\[ y_{i_1} = x_{i_1} = y^{m_+} = x^{m_+} \right) (26) \]

Because \( x \) Lorenz dominates \( y \), we have \( y^{m_+} \geq x^{m_+} \). If \( y_{i_1} = y^{m_+} \) for all \( i \in M_+ \right) \text{ then } y = x \right) \text{ at once and we are done} \right). \text{ If } y_{i_1} = 0 \right) \text{ (recall in } M_+ \right) \text{ all peaks are } 0 \right) \text{ then } x_{i_1} \geq 0 \right) \text{ implies } x_{i_1} = y_{i_1} \right) \text{ and} \right) (26) \right) \text{ is again proven} \right). \text{ From now on we assume } y_{i_1} \right) > 0 \right) , \text{ and that there is at least one agent such that} \right) \text{ that } y_i \right) < y^{m_+} \right). \text{ We show there exists a subset } S(i) \subseteq M_+ \right) \text{ such that} \right)

\[ i_1 \notin S(i) \right), \quad i \in S(i) \right), \text{ and } y_{S(i)} = v^+(S(i)) \right) (27) \]

Otherwise \( y_S < v^+(S) \) for all \( S \in M_+ \right) \text{ containing } i \right) \text{ but not } i_1 \right). \text{ Choosing } \varepsilon > 0 \right) \text{ smaller than the smallest such difference } v^+(S) - y_S \right) , \text{ we see that an } \varepsilon \text{-transfer from agent } i_1 \right) \text{ to agent } i \right) \text{ (a Pigou-Dalton transfer) preserves the core property (inequalities } y_S \leq v^+(S) \right) \text{ for } S \right) \text{ containing } i \right) \text{ are automatically satisfied}, \right) \text{ and } y_{i_1} \right) > 0 \right) \text{ ensures the new allocation is non negative. This contradiction of } (19) \right) \text{ proves } (27) \right).

Set \( S^* = \cup_{i:y_i < y^+1} S(i) \right). \text{ Submodularity of } v^+ \right) \text{ implies } y_S^* = v^+(S^*) \right), \text{ so} \right)

\[ x_{S^*} \leq v^+(S^*) = y_{S^*} \Rightarrow x_{M_+ \setminus S^*} \geq y_{M_+ \setminus S^*} \right) \]

But by construction \( y_j = y^{m_+} \geq x_j \) for all \( j \in M_+ \setminus S^* \right), \text{ therefore } x_j = y^{m_+} \right) \text{ for all } j \in M_+ \setminus S^* \right). \text{ Combining this with } y^{m_+} \geq x^{m_+} \right) \text{ proves } (26) \right).

**Claim 2** Fix an agent \( i_2 \in M_+ \right), \text{ such that } i_2 \neq i_1 \right) \text{ and } y_{i_2} = y^{(m_+)-1} \right), \text{ then} \right)

\[ y_{i_2} = x_{i_2} = y^{(m_+)-1} = x^{(m_+)-1} \right) (28) \]

As \( x \) Lorenz dominates \( y \), we have \( y^{m_+} + y^{(m_+)-1} \geq x^{m_+} + x^{(m_+)-1} \Rightarrow y^{(m_+)-1} \geq x^{(m_+)-1} \) (by Claim 1) \right). \text{ If } y_i = y^{(m_+)-1} \right) \text{ for all } i \in M_+ \setminus i_1 \right) \text{ then } y \geq x \right) \text{ so } y = x \right) \text{ by } y_{M_+} = x_{M_+} \right) \text{ and we are done} \right). \text{ If } y_{i_2} = 0 \right) \text{ then } x_{i_2} \geq 0 \right) \text{ implies } x_{i_2} = y_{i_2} \right) \text{ and } (28) \right) \text{ is again proven} \right). \text{ From now on we assume } y_{i_2} \right) > 0 \right) \right), \text{ and that there is at least one agent } i \in M_+ \right) \text{ such that } y_i \right) < y^{(m_+)-1} \right) \right) \text{ For any such agent, we claim there is a subset } S(i) \subseteq M_+ \right) \text{ such that} \right)

\[ i_2 \notin S(i) \right), \quad i \in S(i) \right), \text{ and } y_{S(i)} = v^+(S(i)) \right)
Otherwise, we can construct as above a Pigou-Dalton transfer from agent \( i_2 \) to agent \( i \), in contradiction of (19). Set \( S^* = \cup_{i:y_i < y^*} S(i) \), then submodularity of \( v^+ \) gives \( y_{S^*} = v^+(S^*) \), hence

\[ x_{S^*} \leq v^+(S^*) = y_{S^*} \Rightarrow x_{M_+ \setminus S^*} \geq y_{M_+ \setminus S^*} \Rightarrow x_{M_+ \setminus (S^* \cup \{i_1\})} \geq y_{M_+ \setminus (S^* \cup \{i_1\})}. \]

But by construction \( y_j \geq x_i \) for all \( j \in M_+ \setminus (S^* \cup \{i_1\}) \) (as \( y_j \geq y^2 \)), and \( M_+ \setminus (S^* \cup \{i_1\}) \) contains \( i_2 \). Combining this with \( y^{*(m+1)} \geq x^{*(m+1)} \) proves (26).

The inductive establishing argument \( y = x \) is now clear.

**Step 2** (as in [7]) We fix an arbitrary profile \( R^* \in \mathcal{R}^M \) with peaks \( p \), and associated GE decomposition \( M_{+,0}, Q_{+,0} \) of \( (G, \omega, p^*) \). We choose \( \tilde{R} \) with peaks \( \tilde{p} \) as in step 1, and the additional requirement \( p^* \leq \tilde{p} \) on \( M_- \) and \( p^* \geq \tilde{p} \) on \( M_0 \); we also have \( \tilde{p} = 0 \leq p^* \) on \( M_+ \).

Given \( S \subset M \), we write \( (R^*_S, \tilde{R}(M_\setminus S)) \) for the profile equal to \( R^* \) for agents in \( S \) and to \( \tilde{R} \) for agents in \( M_\setminus S \). For any integer \( n, 0 \leq n \leq m \), consider the following subset of preference profiles

\[ D_n = \{(R^*_S, \tilde{R}(M_\setminus S)) \mid \text{for some } S \subset M : |S| \leq n\} \]

We prove by induction on \( n \) the property \( \mathcal{H}^+(n): \psi = \mathcal{E} \) on \( D_n \). This is enough because step 1 establishes \( \mathcal{H}^+(0) \), and \( \mathcal{H}^+(m) \) means \( \psi(R^*) = \mathcal{E}(R^*) \) for an arbitrary \( R^* \).

Assume \( \mathcal{H}^+(n-1) \) is true, and fix \( R = (R^*_S, \tilde{R}(M_\setminus S)) \) with \( |S| = n \). We pick an agent \( i \in S \cap M_+ \), so by Pareto optimality \( p^*_i \leq \psi_i(R), \mathcal{E}_i(R) \). To prove \( \psi_i(R) = \mathcal{E}_i(R) \) we consider the profile \( R' = (R^*_{S \setminus \{i\}}, \tilde{R}(M_\setminus S \cup \{i\})) \in D_{n-1} \) where the inductive assumption gives \( \psi_i(R') = \mathcal{E}_i(R') = z_i \). We compare \( \psi_i(R), \mathcal{E}_i(R) \) and \( z_i \) by distinguishing two cases.

If \( p^*_i \leq \psi_i(R) < \mathcal{E}_i(R) \) then \( z_i \leq \psi_i(R) \) by Monotonicity of \( \psi \) (Lemma 3) and \( \tilde{p}_i \leq p^*_i \) in \( M_+ \). As \( \mathcal{E} \) is Invariant (Proposition 4) and \( \tilde{p}_i, p^*_i \leq \mathcal{E}(p^*_i, p_{-i}) \), we have \( \mathcal{E}(\tilde{p}_i, p_{-i}) = \mathcal{E}(p^*_i, p_{-i}) \), i.e., \( z_i = \mathcal{E}_i(R) \). This is a contradiction. If \( p^*_i \leq \mathcal{E}_i(R) < \psi_i(R) \) the same contradiction obtains by exchanging the role of \( \psi \) and \( \mathcal{E} \).

We just proved \( \psi_i(R) = \mathcal{E}_i(R) \) for \( i \in S \cap M_+ \). We check it next for \( M_\setminus S \). Write \( \psi(R) = y, \mathcal{E}(R) = x \), both vectors in \( M_+ \) and \( \hat{y}, \hat{x} \) their restrictions to \( M_\setminus S \), and \( \pi \) their common restriction to \( S \cap M_+ \). Consider the set

\[ \mathcal{C}(R) = \{ z \in \mathbb{R}^{M_+ \setminus S} \mid (z, x_{S \cap M_+}) \text{ in the upper core of } (M_+, v^+) \} \]

that contains \( y \). Clearly \( \hat{x} \) is still Lorenz dominant in \( \mathcal{C}(R) \), hence we can mimic the proof of Step 1 to show that Equal Treatment of Equals and Pareto Optimality imply the desired equality of \( \hat{y} \) and \( \hat{x} \). The key is that the restriction of the profile \( \tilde{R} \) to \( M_\setminus S \) consists of pairwise identical preferences, therefore we can apply Equal Treatment of Equals to any pair of agents in \( M_\setminus S \). To copy the proof of Step 1, observe that \( \mathcal{C}(R) \) is defined, besides the constraints \( z \geq 0 \), by the inequalities

\[ z_A \leq \hat{v}^+(A) = v^+(A \cup (S \cap M_+)) - x_{S \cap M_+} \text{ for all } A \subset M_+ \setminus S \]

and the equality \( z_{M_+ \setminus S} = x_{M_+ \setminus S} \). Thus \( \mathcal{C}(R) \) is the upper core of the submodular game \( (M_\setminus S, \hat{v}^+) \) and the proof proceeds exactly as in Step 1. We omit the details.

We also omit the entirely similar proof that \( \psi_i(R) = \mathcal{E}_i(R) \) on \( M_- \) .
8 Concluding comments

We list first five more natural normative requirements in our model, three of which are satisfied by the Egalitarian rule.

1) Group-Strategyproofness strengthens Strategyproofness by ruling out profitable joint misreports by arbitrary subsets of agents (see e.g. [2]). It is well known that the Uniform rule in the one-resource model is Group-Strategyproof. Chandranmouli and Sethuraman ([6]) recently established that the present Egalitarian rule is Group-Strategyproof as well.

2) Resource Monotonicity requires that the share of every agent increases weakly when the amount of one of the resources (one of the numbers ωr) increases([23]). For instance we go from Figure 1 (section 2) to Figure 2 (section 4) by adding 2 units to resource s, and the Egalitarian solution goes from (11,15,9,5) in Figure 1 to (13,15,9,5) in Figure 2. Our Egalitarian rule is Resource Monotonic. The proof mimicks that of the analog “Peak Monotonicity” property in the companion paper([4]); it is omitted for brevity.

3) Consistency plays a central role in characterizing the parametric rules of the one-resource model (these include the Uniform rule and many more): see [26] and [25]. It can be adapted to our model in two ways, by dropping an agent or dropping a resource. Consider a rule ψ, a problem (M,G,ω,R), and a G-flow ϕ implementing ψ(M,G,ω,R). If agent i leaves with her share of ϕ, we delete from G all edges between i and Q, and reduce the endowment of resource r to ωr(i) = ωr − ϕri. Agent-Consistency of ψ requires that ϕ(i) implements the allocation ψ(M\i,G(−i),ω(−i),R(−i)). From its Lorenz dominance property, it is clear that the Egalitarian solution is agent-consistent.

Symmetrically, Resource-Consistency considers dropping a resource r, deleting from G all edges between M and r, and shifting the preferences of agent i by ϕri (so her peak becomes pi − ϕri). Resource- Consistency requires that ϕ(−r) implements the solution ψ(M,G(−r),ω−r,R(−r)). The Egalitarian rule fails this property, as one sees in the examples of Figure 1 and Figure 2, by dropping resources r,s,t. The reduced one-resource problem has C with peak 2, and D with peak 5, sharing 6 units: the Uniform solution is ̅xC = 2, ̅xD = 4, whereas the Egalitarian solution of the initial problem gives only one unit of resource u to C, and xD = 5.

As mentioned in the introduction, the follow up paper [15] focuses on rules meeting both versions of Consistency. In particular the one-resource Uniform rule admits infinitely many consistent extensions to the bipartite model.

4) Edge Monotonicity: it is natural to think of an edge in G as a resource, and to require that the addition of an edge from agent i to some resource always weakly benefits this agent. But our Egalitarian rule violates this property. Recall that in the example of Figure 2 the Egalitarian solution is (xA, xB, xC, xD) = (13,15,9,5). Now add an edge from agent C to resource r; in the new economy, all resources are underdemanded (M = M+, Q = Q−), and the new Egalitarian allocation is (10,5,15,10,5,6). Agent A strictly benefits in the change, while D is hurt. If 9PC10,5, agent C is hurt as well.

Interestingly in the model of our companion paper [4], with agents on both sides of the edges, we find that the analog Egalitarian rule satisfies Edge Monotonicity. It is unclear which reasonable rules in our model satisfy either Edge Monotonicity or Edge Solidarity (adding an edge affects the welfare of all agents in the same direction).

We discuss finally three extensions of our model.

5) We can add capacity constraints to the total allocation of each agent. Fix c−, c+ ∈ R+M such that c− ≤ c+. Combining
this with the bipartite constraints, the set of feasible allocations becomes $A(G, \omega) \cap [c^-, c^+]$. This set is non empty if and only if

\[
\text{for all } S \subseteq M, c^+_S \leq \omega_f(S); \text{ and for all } T \subseteq Q, \omega_T \leq c^+_f(T)
\] (29)

an assumption we must maintain. Then the "core" representations of feasible allocations, statements $iv$) and $v$), in Lemma 1, are easily adapted\(^6\). Preferences of agent $i$ bear only on $[c^-_i, c^+_i]$, so the profile of peaks $p$ is in $[c^-, c^+]$. The GE decomposition (Lemma 2) is unchanged, and so is the description of Pareto optimal allocations (Proposition 1), except for the addition of the capacity constraints. For instance in $M_+$, the constraints become $x \in A(G(M_+, Q_-), \omega)$, and $p \leq x \leq c^+; they are compatible because the system (29) is true when $p$ replaces $c^-$. To define the egalitarian solution in $M_+$, we use the same system $\Theta(\lambda)$ but with the median function $\gamma_1(\lambda) = med\{\lambda, p_i, c^+_i\}$ guaranteeing that $\gamma_1(\lambda)$ remains below $c^+_i$. Similarly in system $\Xi(\mu)$ we set $\delta_1(\mu) = med\{\mu, c^-_i, p_i\}$. The Lorenz dominant position of $E(G, \omega, p, c)$ in the Pareto set (Proposition 2) follows from the same argument, and so does the proof that the Egalitarian rule is strategyproof (Proposition 4).

On the other hand, the properties of Equal Treatment of Equals, and No Envy have much less bite in the presence of arbitrary capacity constraints. Two agents with disjoint capacity ranges $([c^-_i, c^+_i] \cap [c^-_j, c^+_j] = \emptyset$) cannot envy one another, nor can we talk of their preferences being equal. So our Egalitarian rule passes the version of these two axioms applying only to overlapping (or even equal) capacity ranges, but this is not enough to extend the characterization result.

6) There is a "discrete" variant of our model where indivisible units have to be distributed. Two papers ([18], [10]) characterize in this case the randomized Uniform rule for the one-resource model. It is possible that their result could be adapted to include bilateral constraints.

7) We have considered only symmetric rules. In the one-resource model, the rich family of lotment rules ([2]) preserves the incentive properties of the Uniform rule while allowing a very different treatment of the agents. Similarly the family of fixed paths rules ([14]) is characterized by the combination of Pareto Optimality, Strategyproofness, Resource Monotonicity and Consistency. Both families can naturally be extended to our model, though the corresponding characterization results, if any, would require further research.

References


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\(^6\) $x \in A(G, \omega) \cap [c^-, c^+] \Leftrightarrow \{x \leq c^+ \text{ and } x \text{ is in the lower core of the supermodular game } (M, w), \text{ where } w(S) = \max_{T: S \subseteq T} \{\omega_T + c^+_S \} \}\Leftrightarrow \{x \geq c^- \text{ and } x \text{ is in the upper core of the submodular game } (M, v), \text{ where } v(S) = \min_{S' \subseteq S} \{\omega_f(S') + c^+_S \}\}$


